Quantum quenches

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Main references:

1. J. Dziarmaga, Advances in Physics, **59** 1063 (2010).

2. A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).

3. A. Dutta, U. Divakaran, D. Sen, B. K. Chakrabarti, T. F. Rosenbaum and G. Aeppli, arxiv: 1012:0653 v2. (To appear from CUP)

4. Diptiman Sen, ICTS lecture notes

5. *Quantum quenching, annealing and optimization*, edited by Anjan Chandra, Arnab Das and B. K. Chakrabarti, springer (2010).

Phase Transitions

Driven by thermal fluctuations; 2nd order phase transition

Paradigmatic example: Ferromagnet to paramagnet

Classical Ising model: $H = -J \sum_i S_i S_{i+1}$

Order parameter: spontaneous magnetization

Thermal phase transition at a critical temperature T_c

A diverging length scale: correlation length

Critical slowing down: Diverging relaxation time

Quantum Phase Transitions

Phase transitions are driven by quantum fluctuations

• Zero temperature transition due to non-commuting terms in the Hamiltonian; Driven by quantum fluctuations

Paradigmatic model: Transverse Ising Model

$$H = -\sum_{<ij>} \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

The Hamltonian has Z_2 symmetry

Exactly solved in one dimension

For $h \gg 1$, spins are all aligned in the x

$\langle \sigma_i^z \rangle = 0$; Paramagnetic

For $h < 1$, cooperative term dominates

$\langle \sigma_i^z \rangle \neq 0$; Ferromagnetic

• Quantum phase transitions at $\lambda = h - 1 = 0$.

Quantum Phase Transitions: Critical Exponents

Notion of Universality:

Exponents depend on Symmetry, dimensionality and the nature of the fixed point

- $d \to (d+1)$
- Diverging length Scale: $\xi \sim \lambda^{-\nu}$
- Diverging time Scale: $\xi_\tau \sim \xi^z$
- Energy gap scales as $\lambda^{\nu z}$
- At the quantum critical point gap scales k^z.

Quantum critical exponents

Two spin correlation function:
$< \sigma_i^z \sigma_{i+j}^z > - < \sigma_i^z >< \sigma_j^z > \sim \exp(-|j|/\xi)$

correlation length diverges as $\xi \sim \lambda^{-\nu}$

Low lying excitations: $\omega(k)$;

At the QCP $\omega(k)$ vanishes for the critical mode $\omega(k) \sim |k - k_c|^z$

Close to the QCP: the gap between the ground state and the excited state,

$$\Delta E = \omega(k_c) \sim \Delta E \sim \lambda^{\nu z}$$

Quantum critical exponent ν and z

To be shown later:

Transverse Ising chain: $\omega(k) = 2\sqrt{1 + h^2 + 2h\cos k}$

$\lambda = h - 1 = 0$, critical mode $k = \pi$,

$\Delta E \sim |h - 1| \sim |\lambda|$; yields $\nu z = 1$

$\omega(h = 1) \sim |k - \pi|$; yields $z = 1$

Points to note

- ν how one moves away from the critical point
- The dynamical exponent z associated with the critical point.

The scaling of $\omega(k)$

$$\omega(k, \lambda) = \epsilon_k - \epsilon_0 = \lambda^{z\nu} f\left(\frac{k}{\lambda^\nu}\right); \quad k_c = 0$$

Away from the QCP, $x \to 0$, $f(x) \to$ constant

$$k \ll \lambda^\nu \text{ or } (L \gg \xi); \text{ gap} \sim \lambda^{\nu z}$$

Close to the QCP: $x \to \infty$; $f(x) \to x^z$; gap $\sim k^z$

Transverse Ising chain

$$\epsilon_k = \sqrt{k^2 + \lambda^2}$$

crossover to the linear function of momentum $k^* \sim \lambda$; $\xi \sim 1/k^*$.

Critical exponent ν

Consider the transverse Ising chain at the critical point

$$H = -\sum_i \left(\sigma_i^z \sigma_{i+1}^z + \sigma_i^x\right)$$

Perturbation by the transverse field: $\nu = 1$.

Perturbation by a longitudinal field: $h_z \sum_i \sigma_i^z$; $\nu = 8/15$.

critical exponent $\beta = 1/8$; $[\sigma_z] = 1/8$

Additional contribution: $\delta S = \int dx dt h_z \sigma^z(x,t)$ should be dimensionless; $[h_z] = 15/8$

$h_z \sim 1/\xi_z^{15/8}$ implies $\xi_z \sim h_z^{-8/15}$; $\nu = 8/15$.
F. Pollmann, S. Mukerjee, A. M. Turner, and J. E. Moore, Phys. Rev. E **81**, 020101 (2010).

Adibatic Dynamics: A tale of two time Scales

- Two time-scales

(i) Time scale (rate) of driving

(ii) Inherent time scale of the system: Relaxation time

Dynamics is adiabatic as long as relaxation time $<<$ rate of driving

Can one have an adiabatic dynamics if a quantum system driven slowly across a classical/quantum critical point?

No. The relaxation time diverges close to the critical point

Dynamics can not be adiabatic close to the critical point. Defect Generation

Classical Kibble Zurek Scaling

Critical slowing down: dynamics across a 2nd order phase transition is not adiabatic

Consider a linear variation $T - T_c = t/\tau$

A defect is formed when $T \sim T_c$; the correlation time $\xi_t \sim (T - T_c)\tau$

Scaling: $\xi_t \sim \xi^z$ and $\xi \sim |T - T_c|^{-\nu}$

this leads to $\hat{\xi}^z \sim \hat{\xi}^{-1/\nu}\tau$ yielding $\hat{\xi} \sim \tau^{-\nu/(\nu z+1)}$

volume associated with a point defect $\hat{\xi}^d$

The defect density $n \sim 1/\hat{\xi}^d \sim \tau^{-\nu d/\nu z+1}$.

Kibble-Zurek: Classical System

In the early universe, cooling may have led to formation of topological defects (like domains, strings or magnetic monopoles) due to spontaneous symmetry breaking of some scalar field.

Zeldovich, Kobzarev and Okun, JETP 40 (1974) 1; Kibble, J. Phys. A 9 (1976) 138

Classical phase transition and Kibble-Zurek Scaling:

Superfluid

Liquid Crystals

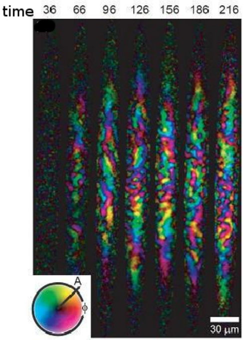
Zurek, Nature 317 (1985) 505

Review: Zurek, Phys. Rep. 276 (1996) 177

Domains in an atomic gas

Spin-1 bosonic atomic gas ^{87}Rb

The system undergoes a transition between two phases when the magnetic field is varied. If the field is quenched across the transition very quickly and the system is then allowed to relax for some time, ferromagnetic domains form and grow



Sadler et al, Nature 443 (2006) 312

We have discussed that defects are generated following a quench across a classical critical point as well as a quantum critical point. This is due to the diverging relaxation time close a critical point

Then what is special about quenching through a quantum critical point?

In a quantum system statistics and dynamics are intermingled; there is a dynamical exponent

The dynamics is Unitary

Diverging relaxation time is associated with a vanishing gap in the spectrum

$\Delta E \sim \lambda^{z\nu}$, and at $\lambda = 0$ (QCP), $\omega \sim k^z$ (assuming $k_c = 0$)

For a finite system, the gap $\sim 1/L^z$ at the QCP.

Slow Quenching Dynamics across quantum critical points

System initially in its Ground state

↓

Quench a parameter of the Hamiltonian with rate τ and fixed scheme

↓

Allow the system to pass through a Quantum Critical Point

↓

System gets excited

↓

Question: What is the density of defects??

<u>References</u>:

- T. W. B. Kibble, J. Phys. A **9**, 1387 (1976); W. H. Zurek, Nature(London) **317**,505 (1985).
- B. Damski, Phys. Rev. Lett. **95**, 035701 (2005).
- A. Polkovnikov, Phys. Rev. B **72**, 161201.
- J. Dziarmaga, Advances in Physics **59**, 1063 (2010).

Universal Scaling of density of defects: Kibble-Zurek Scaling

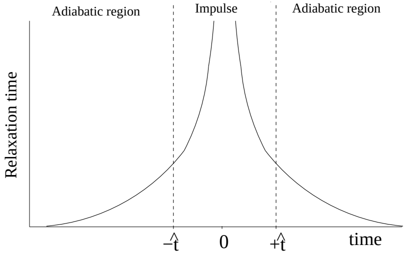
Density of defects scale as $n \sim \tau^{-\nu d/(\nu z+1)}$

For , $\nu = \nu z = 1$ and $d = 1$, $n \sim \tau^{-1/2}$

- • Assuming a linear quench across an Isolated critical point
- • Non-linear Quench $(t/\tau)^r$; $\nu \to r\nu$[1] Protocol Dependent.
- • Generalised to Multicritical point, Gapless phase, Random systems

[1] Sen, Sengupta and Mondal, Phys. Rev. Lett, **101**,016806 (2008)

A quick derivation of KZ scaling



Consider a linear driving $\lambda = t/\tau$

Non-adiabatic effect dominates when

$$\xi_\tau \sim \left(\frac{\hat{t}}{\tau}\right)^{-\nu z} = \frac{\lambda}{\dot{\lambda}} = \hat{t}$$

- • Characteristic time scale: $\hat{t} \sim \tau^{z\nu/(z\nu+1)}$; $\xi \sim \tau^{\nu/(z\nu+1)}$
- • Defect density $n \sim \frac{1}{\xi^d} \sim \tau^{-\nu d/\nu z+1}$

low energy and high energy modes

The energy gap vanishes for $k = k_c$
example of transverse Ising chain

- Low energy modes $(k \to k_c)$ are influenced by a QCP
- High energy modes evolve adiabatically; insensitive to a QCP

Alternative derivation

τ sets a momentum scale

$\hat{k} \sim \tau^{-\nu/(\nu z+1)}$; the defect density $n \sim \hat{k}^d \sim \tau^{-\nu d/(\nu z+1)}$

$\hat{k}$ vanishes as $\tau \to \infty$

When do high energy modes contribute significantly?

Phase space argument

Assume $\lambda(t) = \lambda_0(t/\tau)$; QCP at $\lambda = 0$ at $t = 0$.

The non-adiabaticity condition $\partial\Delta(\vec{k}, t)/\partial t \sim \Delta^2$,

Δ is the characteristic energy-scale: $\Delta \sim \lambda^{z\nu}$.

With $\partial\Delta(t)/\partial t = [\partial\Delta(\lambda)/\partial\lambda]\tau^{-1}$, $\tau^{-1}\lambda^{z\nu-1} \sim \Delta^2$.

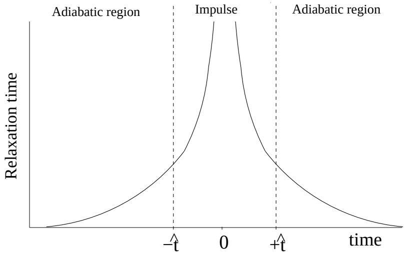
Near a QCP, $\Delta \sim k^z$ and $\lambda \sim k^{1/\nu}$; $k \sim \tau^{-\nu/(\nu z+1)}$

The scaling caling of the gap as $\Delta \sim \tau^{-z\nu/(z\nu+1)}$.

The available phase space for quasiparticle excitations (proportional to n):

$$\Omega \sim k^d \sim \Delta^{d/z} \sim \tau^{-\nu d/(\nu z+1)}$$

The limit of Sudden Quenching:



- Characteristic time scale: $\hat{t} \sim \tau^{1/(z\nu+1)}$.
- Wave function is frozen in the impulse region $-\hat{t} < t < \hat{t}$.
- $\psi(-\hat{t}) = \psi(+\hat{t})$,Meaningful for low energy modes

Start from the QCP $(t = 0)$:

- If $t_f << \hat{t}$: Limit of sudden quenching
- If $t_f >> \hat{t}$: Limit of slow quenching: Kibble-Zurek Scaling

Is there a Kibble-Zurek Scaling for a sudden quenching?

References: C De Grandi, V. Gritsev and A. Polkovnikov, Phys. Rev. B **81**, 012303 (2010).

QPT , dynamics and Quantum Information theoretic Measures:

- Concurrence, negativity, entanglement entropy ...
- The Quantum fidelity: The modulus of overlap of the wave function: Connected to....
  - Scaling of the geometric phase near a QCP
  - Sudden Quench of Small amplitude starting from a QCP
  - Quantum Critical Environment and Loschmidt Echo
- Quantum Discord

  Detects criticality in a finite size system.

  Quenching through a QCP can generate non-zero quantum correlations

A. Dutta, U. Divakaran, D. Sen, B. K. Chakrabarti, T. F. Rosenbaum and G. Aeppli, arxiv: arXiv:1012.0653.

Comments on adiabatic perturbation theory

Assume a translationally invariant system:

- first order perturbation theory; Fermi golden rule
- scaling relation of ω and the rate of change $< k|\frac{\partial}{\partial\lambda}|0 >$

leads to the scaling relation $n \sim 1/\tau^{\nu d(nuz+1)}$

For $d \geq 2(z + 1/\nu)$ which implies $\frac{\nu d}{\nu z+1} \geq 2$; $n \sim 1/\tau^2$.

All the momenta contribute, not only the small momenta

A. Polkovnikov, Phys. Rev. B **72** 161201 (2005)

Transverse Ising chain:quenching of the transverse field

$H = -\sum_i \sigma_i^z \sigma_{i+1}^z - h(t)\sum_i \sigma_i^x$; $h = t/\tau$ for $-\infty < t < \infty$,

The system starts in the ground state and crosses the two Ising critical points at $t = \pm t_0 = \pm\tau$.

The initial state $|...\downarrow\downarrow\downarrow\downarrow\downarrow...\rangle$ (i.e., $\sigma_x = -1$ for all i)

The final state should be $|...\uparrow\uparrow\uparrow\uparrow\uparrow....\rangle$

Non-adiabatic transition: the system cannot follow the instantaneous ground state close to QCP

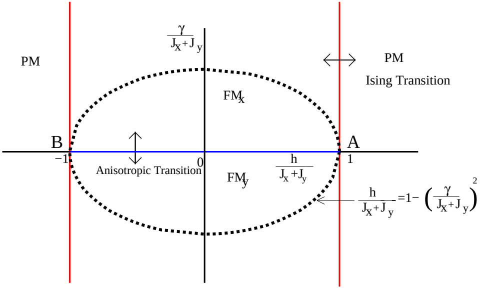
The final state therefore has a structure such as $|...\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow....\rangle$, where the down spins corresponds to defects.

In the next few slides, we shall study the quenching of the transverse XY spin chain by changing the magnetic field and show that the defect density scales as $1/\sqrt{\tau}$ as expected.

Transverse XY spin chain

Let us consider the Transverse XY spin chain

$$H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - J_y \sum_i \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z$$



- Critical Exponents for Ising transition $\nu = z = 1$
- Exponents with the Mulctcritical point $z_{mc} = 2$ and $\nu_{mc} = 1/2$.

Jordan-Wigner transformations

Consider periodic boundary condition:

$$H = -\sum_{i=1}^{N-1}\left(J_x\sigma_i^x\sigma_{i+1}^x + J_y\sum_i \sigma_i^y\sigma_{i+1}^y\right) - h\sum_{i=1}^{N}\sigma_i^z$$

spin-$\frac{1}{2}$ $\rightarrow$ spinless Fermions

$$c_i = \left[\prod_{m=-\infty}^{i-1}\sigma_m^z\right]\sigma_i^+; \quad c_i^\dagger = \left[\prod_{m=-\infty}^{i-1}\sigma_m^z\right]\sigma_i^-$$

$$\{c_m, c_n\} = 0 \text{ and } \{c_m, c_n^\dagger\} = \delta_{mn}$$

$$\sigma_n^z = 2c_n^\dagger c_n - 1; \quad \sigma_n^- = c_n \exp\left(i\pi\sum_{j=1}^{n-1}c_j^\dagger c_j\right)$$

Lieb, Schultz, Mattis, Ann. Phys. **16** 407 (1961); Kogut, Rev. Mod. Phys. **51**, 659 (1979)

Jordan-Wigner transformation

$$
\begin{aligned}
H &= \sum_{n=1}^{N-1} \left[-(J_x + J_y)\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right) \right. \\
&+ \left. (J_x - J_y)\left(c_{n+1}^\dagger c_n^\dagger + c_n c_{n+1}\right)\right] + \sum_{n=1}^{N} h\left(2c_n^\dagger c_n - 1\right),
\end{aligned}
$$

$$
c_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} c_n e^{-ikn}; \quad c_n = \frac{1}{\sqrt{N}} \sum_{-\pi<k<\pi} c_k e^{ikn}
$$

k lies in the range $[-\pi, \pi]$

Two-Level System

$$H = \sum_{k>0} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_k \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix},$$

$$H_k = 2 \begin{pmatrix} -(J_x + J_y)\cos(k) - h & i(J_x - J_y)\sin(k) \\ -i(J_x - J_y)\sin(k) & (J_x + J_y)\cos(k) + h \end{pmatrix},$$

decoupled two level systems with basis $|0>$ and $|k, -k>$

Diagonalized using Bogoliubov transformation

$$H = 2\sum_{k>0} \omega_k (d_k^\dagger d_k + d_k^\dagger d_k) - \sum \omega_k$$

$$\omega_k^2 = 2[h^2 + J_x^2 + J_y^2 + 2h(J_x + J_y)\cos k + 2J_x J_y \cos 2k]$$

Quenching: two-Level System

$$|GS>=\bigotimes_{k>0}(\cos\theta_k|0>+i\sin\theta_k|k,-k>$$

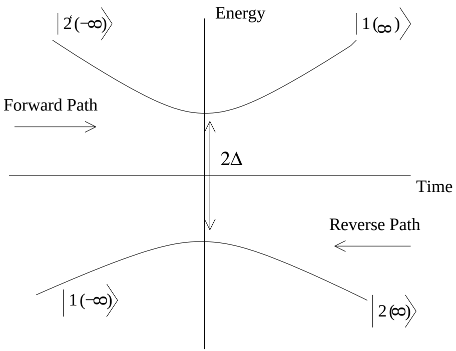
$$\tan(2\theta_k)=-\frac{(J_x-J_y)\sin k}{(J_x+J_y)\cos k+h}$$

$h\to\infty$, the spins are down; $h\to-\infty$, the spins are up

What happens when $h=t/\tau$?

Crosses critical points at $h=\pm(J_x+J_y)$; Wrongly oriented spins
Use Landau-Zener transition probability formula to show $n\sim\tau^{-1/2}$

Landau-Zener formula



two approaching levels $\pm\sqrt{\epsilon^2 + \Delta^2}$ with $\epsilon = t/\tau$.

Probability of excitation $P = \exp(-\pi\Delta^2\tau)$

Gap protects from the excitation At the QCP , the gap for the critical mode vanishes; Gap is small for other modes close it.

Zener, Proc. R. Soc. London Ser A 137 (1932) 696; Landau and Lifshitz, Quantum Mechanics

Applications of Landau-Zener formula

Call $J_x + J_y = J$ and the anisotropy $\gamma = J_x - J_y$.

$$i \begin{pmatrix} \dot{\psi}_{1k} \\ \dot{\psi}_{2k} \end{pmatrix} = \begin{pmatrix} h(t) + J\cos k & i\gamma \sin k \\ -i\gamma \sin k & -h(t) - J\cos k \end{pmatrix} \begin{pmatrix} \psi_{1k} \\ \psi_{2k} \end{pmatrix},$$

We consider a linear variation of the variation $h(t) = t/\tau$; initial condition $|\psi_{1k}(-\infty)|^2 = 1$ and $|\psi_{2k}(-\infty)|^2 = 0$
the system crosses the two Ising critical lines at $t = \pm t_0 = \pm\tau J$.
The excitation energy for the critical modes $k = 0, \pi$,

$$p_k = |\psi_{1k}(+\infty)|^2 = e^{-\pi\tau\gamma^2 \sin^2 k}.$$

$$\begin{aligned} n &= \int_{-\pi}^{\pi} \frac{dk}{2\pi}\; p_k \;=\; \frac{1}{\pi}\int_0^{\pi} dk\; e^{-\pi\tau\gamma^2 \sin^2 k} \\ &\simeq \frac{1}{\pi}\left[\int_0^{\infty} dk\; e^{-\pi\tau\gamma^2 k^2} + \int_{-\infty}^{0} dk\; e^{-\pi\tau\gamma^2 (\pi - k)^2}\right] \sim \frac{1}{\pi\gamma\sqrt{\tau}}. \end{aligned}$$

A small note on the LZ transition formula

We shall here indicate how to get a generic scaling

$$H = \begin{pmatrix} t/\tau & \Delta \\ \Delta^* & -t/\tau \end{pmatrix},$$

Δ is independent of time and linear variation of the diagonal term are necessary to get an exact solution

Initial condition $|\psi_1(-\infty)|^2 = 1$ and $|\psi_2(-\infty)|^2 = 0$

Generic state at any instatnt $\psi(t) = \psi_1|1\rangle + \psi_2(t)|2\rangle$

Non-adiabatic transition probability $P = |\psi_1(\infty)|^2$

Rescaling $t \rightarrow \frac{t}{\sqrt{\tau}}$, one can argue that $P = f\left(|\Delta|^2\tau\right)$

see Sen, Sengupta, Mondal, PRL (2008);arxiv0803.2081

LZ: Exact results

$P = f\left(|\Delta|^2\tau\right) = \exp(-\pi|\Delta|^2\tau)$ when the passage is from $t \to -\infty$ to $= \infty$

$P \sim \frac{1}{\Delta^4\tau^2}$ otherwise.

The crossover between these two results occur when initial and final time are of the order of $\Delta\tau$

see the review article by Sei Suzuki in the book edited by Chandra et al cited before

The quenching of the previous spin chain is a collection of LZ problems. However,

Δ vanishes for the critical mode and very small for modes close to it. Non-adiabatic effects are dominant for these modes

From Landau Zener to Kibble-Zurek: A hand waving argument

$$H = 2\begin{pmatrix} \Delta E \mathrm{sgn(t)} & |k|^z \\ |k|^z & -\Delta E \mathrm{sgn(t)} \end{pmatrix},$$

In the generic case, $\Delta E \sim \lambda^{\nu z}$ and $\lambda(t) = t/\tau$

In the Schrodinger equation rescale $t' = \frac{t}{\tau^{z\nu/(z\nu+1)}}$

The excitation probability $p_k = f\left(k^{2z}\tau^{2\nu z/(z\nu+1)}\right)$.

The defect density $n = \int d^d k \ f \ \left(k^{2z}\tau^{2\nu z/(z\nu+1)}\right) \sim \tau^{-\nu d/(\nu z+1)}$.

Beyond integrability; Ising model in a generic field

Consider the Hamiltonian

$$H = -\left(\sum_i \sigma_i^z \sigma_{i+1}^z + \sigma_i^x + g(\sigma_i^x \cos\phi + \sigma_i^z \sin\phi)\right)$$

The JW transformation fails for for any non-zero longitudinal field

Quench $g = t/\tau$:

$\phi = 0, \pi$; the defect density $n \sim 1/\sqrt{\tau}$

for any other ϕ; $n \sim 1/\tau^{8/23}$. (Recall $\nu = 8/15$)

F. Pollmann, S. Mukerjee, A. M. Turner, and J. E. Moore, Phys. Rev. E **81**, 020101 (2010).

Modifications of the Kibble-Zurek Scaling:non-linear quenching

A non-linear variation of the quenching parameter

$$\lambda(t) = \lambda_0 |\tau|^r \mathrm{sign}(t)$$

$$\hat{t} = \tau^{r\nu z/(1+r\nu z)}$$

$$n \sim \tau^{-r\nu d/(1+r\nu z)}$$

The non-linearity exponent r seems to renormalize the critical exponent ν to $r\nu$. An exact solution of the LZ problem is not possible, but one can find a scaling from

$\Delta E \sim \lambda^{\nu z}$ and $\lambda(t) = |t/\tau|^r \mathrm{sgn(t)}$

In the Schrodinger equation rescale $t' = \frac{t}{\tau^{zr\nu/(zr\nu+1)}}$

The excitation probability $p_k = f\left(k^{2z}\tau^{2r\nu z/(zr\nu+1)}\right)$.

Sen, Sengupta, Mondal, Phys. Rev. Lett., 2008

Optimized rate of quenching to minimize defect generation

Consider $\lambda \simeq \lambda_0 |t/T|^r \mathrm{sign}(t)$, where $-T < t < T$ and T

A large number of excitations: for both $r \to 0$ and $r \to \infty$.

Defect density $n \sim (r\delta)^{dr\nu/(rz\nu+1)}$; $\delta = 1/(T\Delta_0)$

Δ_0 is the lowest excitation energy at $\lambda = \lambda_0$.

minimum excitation occurs at $r_{\rm opt}\delta \simeq \exp(-r_{\rm opt} z\nu)$.

For a transverse Ising chain with $z = \nu = 1$, $n \simeq [A(r)\delta]^{r/(r+1)}$ where $A(r)$ scales linearly with r; hence the optimal condition gets modified to $r_{\rm opt} \simeq -\ln[(\delta/C)\ln(C/\delta)]$ with the non-universal constant $C = 14.7$

Barankov and Polkovnikov, PRL (2008)

In next few slides, we shall connect quenching to the quantum information; quenching through quantum critical points, generates defects which in turn lead to non-zero quantum correlations in the final state; we shall discuss the local entropy generation and the scaling of the decoherence factor of a qubit coupled to a driven spin chain

Generation of entropy

Following an evolution from $-T < t < T$; $\rho = \bigotimes \rho_k$

$$\begin{bmatrix} p_k & q_k \\ q_k^* & 1-p_k \end{bmatrix}.$$

In the limit $T \to \infty$, the phase of the off-diagonal term varies rapidly; as a result the integral of q_k over k vanishes.
decohered reduced density matrix

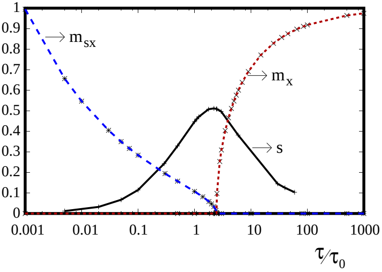
$$\rho_D = \begin{bmatrix} p_k & 0 \\ 0 & 1-p_k \end{bmatrix}. \tag{1}$$

von Neumann entropy density of the system, $s = -\mathrm{tr}(\rho_D \ln \rho_D)$

$$s \;=\; -\int_0^{\pi} \frac{dk}{\pi}\,[\; p_k \ln(p_k) \;+\; (1-p_k)\ln(1-p_k)\;].$$

Cherng and Levitov, Phys. Rev. A (2006)

generation of entropy



Mukherjee, Divakaran, Dutta and Sen, Phys. Rev. B (2007)

Non-zero quantum correlation: concurrence and discord in the final state generated through quenching scale identically as the defect density

Sengupta and Sen, Phys. Rev. A (2009); Nag, Patra and Dutta, J. Stat. Mech (2011)

Is it true for models not reducible to two-level systems?

Quantum quenching and Decoherence

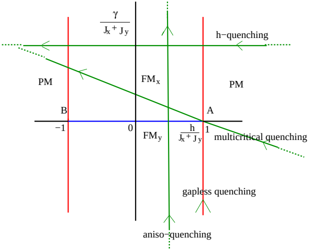
- Central Spin model and Quenching of the environment.
- Dynamics of the decoherence of the qubit.
- Is there a universal scaling of the decoherence factor?

Recall the quenching of the transverse XY spin chain:

$$H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - J_y \sum_i \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z$$

Quenching through the critical and Multi-critical points

Different Quenching paths:



Critical point $h = t/\tau$; $n \sim \tau^{-1/2}$

Multicritical point

Quench $J_x = t/\tau$ with $h = 2J_y$; cross the MCP when $J_x = J_y$

We find Defect density: $n \sim \tau^{-1/6}$

Quenching through the gapless critical line $\gamma = t/\tau$:$n \sim \tau^{-1/3}$

The result for the multi critical point is derived later and the last result is a homework

Decoherence

- Loss of phase information
- Emergence of a mixed state from a pure state.

A qubit coupled to a quantum critical many body system

Does a QCP influence the decoherence of the qubit?

The Central Spin model

- A central spin globally coupled to an environment.
- We choose the enviroment to be Transverse XY spin chain

$$H = -J_x \sum_i \sigma_i^x \sigma_{i+1}^x - J_y \sum_i \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z$$

- and a global coupling $-\delta \sum_i \sigma_i^z \sigma_S^z$
- Qubit State: $|\phi_S(t \to -\infty)\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$
- The environment is in the ground state $|\phi_E(t \to -\infty)\rangle = |\phi_g\rangle$
- Composite initial wave function:

$$|\psi(t \to -\infty)\rangle = |\phi_S(t \to -\infty)\rangle \otimes |\phi_g\rangle$$

Quan *et al*, Phys. Rev. Lett. **96**, 140604 (2006).

Coupling and Evolution of the environmental spin chain

• At a later time t, the composite wave function is given by $|\psi(t)\rangle = c_1|\uparrow\rangle \otimes |\phi_+\rangle + c_2|\downarrow\rangle \otimes |\phi_-\rangle$.

$|\phi_\pm\rangle$ are the wavefunctions evolving with the environment Hamiltonian $H_E(h \pm \delta)$ given by the Schrödinger equation

$$i\partial/\partial t|\phi_\pm\rangle = \hat{H}[h \pm \delta]|\phi_\pm\rangle.$$

• The coupling δ essentially provides two channels of evolution of the environmental wave function with the transverse field $h + \delta$ and $h - \delta$.

What happens to the central spin?

The reduced density matrix:

$$\rho_S(t) = \begin{pmatrix} |c_1|^2 & c_1 c_2^* d^*(t) \\ c_1^* c_2 d(t) & |c_2|^2 \end{pmatrix}.$$

• The decoherence factor (Loschmidt Echo)

$$D(t) = d^*(t) d(t) = |\langle \phi_+(t) | \phi_-(t) \rangle|^2$$

Overlap between two states evolved from the same initial state with different Hamiltonian

• $D(t) = 1$, pure state. $D(t) = 0$ Complete Mixing

• Coupling to the environment may lead to Complete loss of coherence

Loschmidt Echo

The generic definition:

$$D(t) = |\langle\psi_0|e^{i(H_0+V)t}e^{-iH_0t}|\psi_0\rangle|^2$$

Equilibrium Situation:

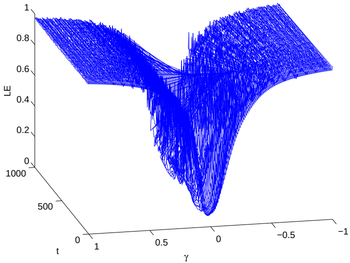
- No explicit time dependence in the Hamiltonian
- $|\psi_0\rangle$ is an eigenstate of H_0.

Non-Equilibrium Situation:

- Explicit time dependence in the Hamiltonian
- $|\psi_0\rangle$ is an not an eigenstate of H_0.

Equilibrium study: Complete mixing close to a QCP

• No explicit time dependence in the environmental Hamiltonian. Consider variation of γ through critical points:



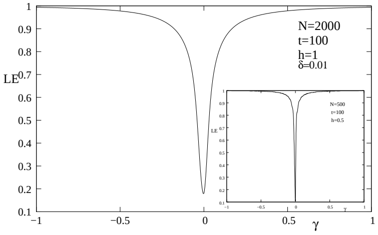
$$D(t)(\gamma, t) = \exp(-\Gamma t^2) \ \text{ and } \ \Gamma \sim \delta^2/(\gamma^2 N^2)$$

Here the anisotropy parameter $\gamma = J_x - J_y$ is being changed; QCP is at $\gamma = 0$.

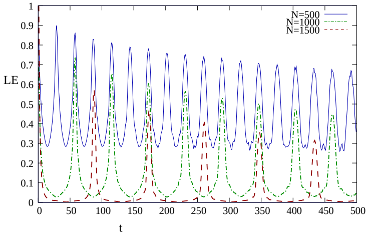
a. Complete loss of coherence b. Indicator of a QCP

Equilibrium case: h independent of time

The decay of the D(t) or Loschmidt Echo close to QCP at a fixed t

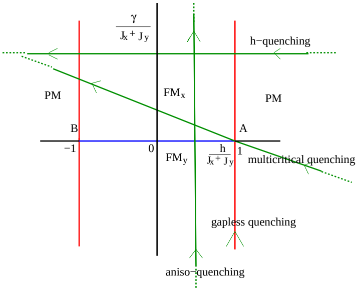


The collapse and revival at the QCP, $h + \delta = 1$



What happens when the environmental spin chain is driven?

Different Quenching paths:



Assume $h(t) = 1 - t/\tau$, driven spin chain

$$H_k^{\pm}(t) = 2\begin{pmatrix} h(t) \pm \delta + \cos k & \gamma \sin k \\ \gamma \sin k & -(h(t) \pm \delta + \cos k) \end{pmatrix}.$$

B. Damski, Quan and Zurek, Phys. Rev. A **83**, 062104 (2011).

The decoherence factor $D(t)$

$$|\phi^{\pm}(t)\rangle = \prod_k |\phi_k^{\pm}(t)\rangle = \prod_{k>0} \left[u_k^{\pm}(t)|0\rangle + v_k^{\pm}(t)|k,-k\rangle\right].$$

$$i\partial/\partial t \left(u_k^{\pm}(t), v_k^{\pm}(t)\right)^T = H_k^{\pm}(t)\left(u_k^{\pm}(t), v_k^{\pm}(t)\right)^T$$

with $\prod_k F_k(t) = \prod_k |\langle\phi_k(h(t)+\delta)|\phi_k(h(t)-\delta)\rangle|^2$,

$$D(t) = \exp\left[\frac{N}{2\pi}\int_0^{\pi} dk\ \ln F_k\right] \qquad (2)$$

where F_k can be written in terms of $u_k^{\pm}$ and $v_k^{\pm}$.

The question we address:

We assume $\delta \to 0$ and and work within the appropriate range of time;
λ is the driving parameter.

One finds

(i) $\ln D_{non-ad} \sim (-t^2 f(\tau))$, if QCP is at $\lambda = 0$

(ii) $\ln D_{non-ad} \sim \{-(t - \lambda_0 \tau)^2 f(\tau)\}$, if QCP is at λ_0

What is the scaling of this function $f(\tau)$?

How to Calculate $D(t)$?

Use the integrable two-level nature of the environmental Hamiltonian.

Far away from the QCP ($|h(t)| \gg 1$ ($t \to +\infty$))

$$\rangle\phi_k(h+\delta) = u_k\rangle 0 + v_k e^{-i\Delta^+ t}\rangle k, -k$$

$$\rangle\phi_k(h-\delta) = u_k\rangle 0 + e^{-i\Delta^- t} v_k\rangle k, -k$$

$$\Delta^+ = 4\sqrt{(h+\delta+1)^2 + \gamma^2 \sin k^2}$$
$$\Delta^- = 4\sqrt{(h-\delta+1)^2 + \gamma^2 \sin k^2},$$

are the energy of two excitations in $\rangle k, -k$ when the transverse field is equal to $h+\delta$ and $h-\delta$, respectively.

Excitations occur only in the vicinity of QCPs

F. Pollman *et al*, Phys. Rev. E **81** 020101 (**R**) (2010).

How to Calculate $D(t)$?...

How does one know u_k and v_k?

• Use the Landau-Zener transition formula:
$p_k = |u_k|^2 = \exp(-2\pi\tau\gamma^2 \sin^2 k)$

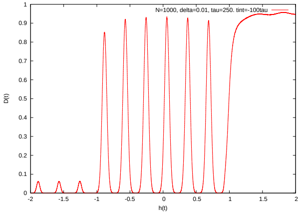
$$\begin{aligned} F_k(t) &= |\langle\phi_k(h(t)+\delta)|\phi_k(h(t)-\delta)\rangle|^2 \\ &= \left||u_k|^2 + |v_k|^2 e^{-i(\Delta^+ - \Delta^-)t}\right|^2, \end{aligned} \tag{3}$$

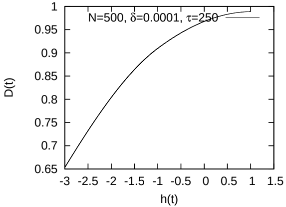
In the vicinity of the quantum critical point at $h = 1$
$\Delta = (\Delta^+ - \Delta^-)/2$,

$$\begin{aligned} F_k(t) &= 1 - 4p_k(1-p_k)\sin^2(\Delta t) \\ &= 1 - 4\left[e^{-2\pi\tau\gamma^2 k'^2} - e^{-4\pi\tau\gamma^2 k'^2}\right]\sin^2(4\delta t) \end{aligned} \tag{4}$$

$\sin k$ has been expanded near the critical modes $k = \pi$, with
$k' = \pi - k$ and we have taken the limit $\delta \to 0$.

Large δ and small δ





- Limit of large δ
- Oscillations

- Limit of small δ
- Exponential Decay

How to calculate $D(t)$?

Assume $\delta \to 0$

$$D_{non-ad}(t) = \exp \frac{N}{2\pi} \int_0^\infty dk \; \ln \left[1 - \left(e^{-2\pi\tau\gamma^2 k'^2} - e^{-4\pi\tau\gamma^2 k'^2}\right) 64\delta^2 t^2\right]$$

Finally D_{non-ad} is given by

$$D_{non-ad}(t) \sim \exp\{-8(\sqrt{2} - 1)N\delta^2 t^2/(\gamma\pi\sqrt{\tau})\}.$$

- $\ln D_{non-ad} \sim \tau^{-1/2}$

The same scaling as the defect density

Non-linear Quenching

Non-linear Quenching: $h = 1 - \mathrm{sgn(t)(t/\tau)}^{\alpha}$

The scaling form $p_k = |u_k|^2 = G(k^2\tau^{2\alpha/(\alpha+1)})$

$$D_{non-ad}(t) = \exp(-CN\delta^2 t^2/\tau^{\alpha/(\alpha+1)})$$

- $\ln D_{non-ad}(t) \sim \tau^{-\alpha/(\alpha+1)}$

<u>Quenching through a MCP</u>

$$\ln D_{non-ad}(t) \sim (t - J_y\tau)^2/\tau^{1/6} \sim (J_x - J_y)\tau^{11/6}$$

- Quenching through Isolated critical points: $\ln D_{non-ad}(\tau) \sim n$

Is this scenario true in general?

Quenching through a critical line

Change $\gamma = t/\tau$ with $h = 1$. Quenched through the MCP

Modified CSM with interaction:

$$H_{SE} = -(\delta/2)\sum_i (\sigma_i^x \sigma_{i+1}^x - \sigma_i^y \sigma_{i+1}^y)\sigma_S^z$$

The coupling δ provides two channels of the temporal evolution of the environmental ground state with anisotropy $\gamma + \delta$ and $\gamma - \delta$. The appropriate two-level Hamiltonain

$$H_k^{\pm}(t) = 2\begin{pmatrix} (\gamma \pm \delta)\sin k & h + \cos k \\ h + \cos k & -(\gamma \pm \delta)\sin k \end{pmatrix}.$$

- The defect density in the final state $n \sim \tau^{-1/3*}$

Does that mean ln $D_{non-ad} \sim \tau^{-1/3}$?

* U. Divakaran *et al*, Phys. Rev. B **78**, 144301 (2008).

A completely different Scaling

$$F_k = 1 - 4(e^{-\pi\tau k^3/2} - e^{-\pi\tau k^3}) \sin^2(4\delta k t)$$

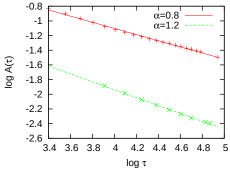
- An exponential decay:

$$D_{non-ad}(t) \sim \exp\{-2^{14/3} N\delta^2 t^2/(3\pi\tau)\}.$$

- Scaling of $\ln D_{non-ad}(\sim \tau^{-1})$ is completely different!!

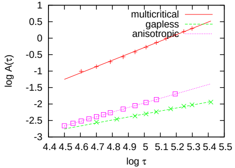
Numerical Justification

Non-linear Quenching



Slope $\simeq -\alpha/(\alpha+1)$

Different Qubit-environment interactions



Fairly good agreement with analytical predictions.

What do we learn?

• Points to the existence of a universality in the scaling of the decoherence factor

• not necessarily identical to the scaling of the defect density.

• May be identical for quenching through isolated critical points.

• Clear deviation for quenching through critical lines.

Points to ponder

• Integrable system reducible to two-level problems....

What happens beyond that?

Is there is a Universality?

In next few slides, we discuss how the standard Kibble-Zurek scaling gets modified (in some cases drastically) in different situations. The anisotropic case is given as a home-work while the multi critical point case was discussed at some length. Then I have mentioned briefly sudden quenching and its connection to the fidelity susceptibility.

Topics could not be discussed are placed after the summary. These include a discussion of fidelity susceptibility and also some other special situations where Kibble-Zurek gets modified

Quenching through a 2-D semi-Dirac point

The semi-Dirac Hamiltonian

$$H = \begin{pmatrix} \mu(t) & iv_F|k_y| + \frac{k_x^2}{2m} \\ -iv_F|k_y| + \frac{k_x^2}{2m} & -\mu(t) \end{pmatrix},$$

The mass term μ is quenched linearly as $\mu(t) = t/\tau$ through Anisotropic QCP at $\mu = 0$

The defect density

$$n \sim \int dk_x \int dk_y \exp\left[-\pi\tau\left(v_F^2|k_y|^2 + \frac{k_x^4}{(2m)^2}\right)\right] \sim 1/\tau^{3/4}$$

Dutta, Singh and Divakaran, EPL (2009).

Quenching through a (d, m) AQCP

Let us use the phase space argument:

$\Delta \sim k_i^{z_1}$ for m (parallel) momentum components

$\Delta \sim k_i^{z_2}$ for $d - m$ (perpendicular) components; $z_2 > z_1$.

The gap scales as $\Delta \sim \tau^{-r\nu_1 z_1/(1+r\nu_1 z_1)}$

For parallel components: $k_{||} \sim \tau^{-r\nu_1/(1+r\nu_1 z_1)}$

For perpendicular components: $k_\perp \sim \tau^{-(r\nu_1 z_1/z_2)/(1+r\nu_1 z_1)}$

the phase space available for quasiparticle excitations,

$$\Omega = k_{||}^m k_\perp^{d-m}$$

$$n \sim \tau^{-[m+(d-m)z_1/z_2]\nu_1 r/(1+r\nu_1 z_1)}$$

Hickichi, Suzuki and Sengupta, Phys. Rev. B **82**, 174305 (2010)

Quenching through the Multi-critical point

Quench both $\gamma = J_x - J_y = t/\tau$ and $h - 1 = t/\tau$ with MCP at $\gamma = h - 1 = 0$; two parameter quenching

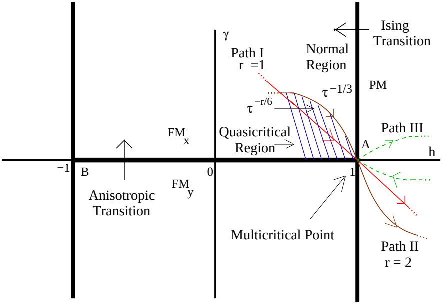
$\nu = 1/2$ and $z = 2$;expected resultDefect density: $n \sim \tau^{-1/4}$

What one finds instead Defect density: $n \sim \tau^{-1/6}$

Why this anomaly?

- U. Divakaran, V. Mukherjee, A. Dutta and D. Sen, J. Stat. Mech. P02007 (2009).
- S. Deng, G. Ortiz and L. Viola, EPL **84**, 67008 (2008); Phys. Rev. B **80** 241109 (R) (2009).

Approaching the Multicritical Point along different Paths



- We approach the MCP along a path $h - 1 \sim \gamma^r$. The parameter r determines the path.

Results for General path: Defect density: $n \sim \tau^{-r/6}$ for $r \leq 2$

$n \sim \tau^{-1/3}$ for $r > 2$.

Quasicritical Points

Two level Systems: Choosing $\lambda_1 = h - 1$ and $\lambda_2 = \gamma$; for the linear path $\lambda_1 = \lambda_2$

$$H_k(t) = (\lambda_1 + k^2)\sigma^z + \lambda_2 k \sigma^x$$

- • The energy gap is minimum when $(\lambda + k^2) = 0$
- • These are so called Quasicritical Points.
- • The minimum Gap Scales as k^3
- • Effective dynamical exponents: $z_{qc} = 3$ and hence $\nu_{qc} = \frac{1}{3}$.

  <u>References</u>:

  - V. Mukherjee and A. Dutta, EPL, **92** 37104 (2010).

linearization method in the vicinity of the minimum gap point

consider a generic path $\lambda_1 = \lambda_2^r sgn(\lambda_2)$ where $\lambda_2 = t/\tau$.

The two-level Hamiltonian
$H_k(t) = \left(|\frac{t}{\tau}|^r sgn(t) + k^2\right)\sigma^z + k\frac{t}{\tau}\sigma^x(1)$

Note that we can not use the LZ formula here: a) Off-diagonal term time dependent; b) Non-linear variation of parameter

In the large τ limit, the maximum contribution to the defect density comes from the vicinity of the minimum gap point at $t = t_0$ which is given by

$\left(|\frac{t_0}{\tau}|^r sgn(t) + k^2\right) = 0$; the MCP is at $t = 0$.

linearizing around t_0 (valid for $1 \leq r$), $we have$

$$H_k(t) = \left(\frac{t - t_0}{\tau_{\rm eff}}\right)\sigma^z + k^{2/r+1}\sigma^x \qquad (2)$$

Linearization method continued .

Note that the renormalized rate of quenching: $\tau_{\rm eff} = \tau k^{2(r-1)/r}$; for $r = 1, \tau_{\rm eff} = \tau$

Now apply LZ formula to Eq. (2); $P_k = \exp(-\pi k^{4/r+2}\tau_{\rm eff})$ and integrating over k, one finds $n \sim \tau^{-r/6}$

What happens for $r \geq 2$.

Minimum gap is at the MCP $t = 0$; no additional minimum gap point

From (1) and (2); one finds :$\tau_{\rm eff}/(\tau/k) \sim k^{(2-r)/r} \to \infty$ for $r > 2$; This means that τ/k (which is the rate of quenching $\gamma = \lambda_2$) is relevant; this is essentially quenching along the Ising critical line at $h = 1$ dictated by

$$H_k = k^2\sigma_z + k\frac{t}{\tau}\sigma_x$$

yielding $n \sim \tau^{-1/3}$.

Summary for a generic path

- • For a given r, the system crosses the quasicritical point with effective $z_{qc} = \frac{2}{r} + 1$
- • For $r > 2$, the system does not cross any quasicritical point
- • $r = 2$ is the limiting path; $z_{qc} \to z_{mc} = 2$

A similar situation does not arise close to a QCP: the minimum gap is at the QCP and there is NO scaling of the γ term which is just a constant.

Scaling Relation: In fact, the quasicritical exponents that appear in the scaling.

Lessons Learnt: Kibble-Zurek is not that Universal.

Quasi-critical points and path-dependent exponents.

References

V. Mukherjee, and A. Dutta, EPL, **92** 37104 (2010)

Sudden Quench of small amplitude starting from the critical point

- Small amplitude starting from the critical point $\lambda = 0$
- Protocol is not important
- Magntitude of Quench λ_f

Scaling

- $k \sim \lambda_f^{\nu}$ Modes $k < \lambda_f^{\nu}$ excited with $p_k = 1$
- $n_{ex} \sim \lambda_f^{\nu d}$

From Fidelity Susceptibility

- $\langle \psi_0(\lambda + \delta) | \psi_0(\lambda) \rangle|^2 = 1 - \delta^2 \chi_F$
- $\delta = \lambda_f$, $\chi_F = \lambda_f^{\nu d - 2}$ for $\lambda_f \gg L^{-1/\nu}$ implies $n_{ex} \sim \lambda_f^{\nu d}$
- $\chi_F = \lambda_f^2 L^{1/\nu - d}$ for $\lambda_f \ll L^{-1/\nu}$ implies $n_{ex} \sim \lambda_f^2 L^{2/\nu - d}$

C. de Grandi, V. Gritsev and A. Polkovnikov, Phys. Rev. B **81**, 012303 (2010).

Summary

- Quenching across a quantum critical point: Generation of defects
- Slow Quenching: $n \sim \tau^{-\nu d/(\nu z+1)}$
- has been shown exactly for transverse Ising/ XY spin chains
- Reducible to two-level systems: Landau-Zener formula
- Non-integrable models: in a longitudinal field
- Adiabatic perturbation theory

Modifications

a) Non-linear quenching $\lambda(t) \sim |t/\tau|^r \mathrm{sgn(t)}$: $\nu \to r\nu$

b) Multicritical point:

Anisotropic Quantum Critical points, extended quantum critical region, spin chain coupled to a heat bath, systems with quenched disorder.

Quenching and Quantum information

Quenching through the QCP: Generates defects which in turn result in non-zero quantum correlations

Concurrence and Discord satisfies Same scaling as defect?

Qubit coupled to bath: Scaling of the decoherence factor?

Sudden Quenching: Fidelity susceptibility

Thanks to my collaborators

Dr. Uma Divakaran, Dr. Victor Mukherjee, Shraddha Sharma, Tanay Nag, Ayoti Patra, Aavishkar Patel, Sthitadhi Roy, Manisha Thakurathi

Prof. Diptiman Sen, Prof. Anantoli Polkovnikov, Prof. R. R. P. Singh

A. Dutta, U. Divakaran, D. Sen, B. K. Chakrabarti, T. F. Rosenbaum and G. Aeppli, arxiv: 1012:0653 v2.

The following topics could not be discussed;

a) Modification of the Kibble Zurek Scaling following a quench through a gapless phase illustrated using the example of the two dimensional Kitaev model on a honeycomb lattice

The scaling of the fidelity susceptibility and sudden quenching starting from some quantum critical points and also the notion of a generalized fidelity susceptibility

Modifications of the Kibble-Zurek Scaling: Gapless Surface

A d-dimensional quantum system quenched across a $(d-m)$-dimensional critical hyperspace on which the energy gap vanishes.

the phase space available for excitations gets modified to $\Omega \sim k^m$

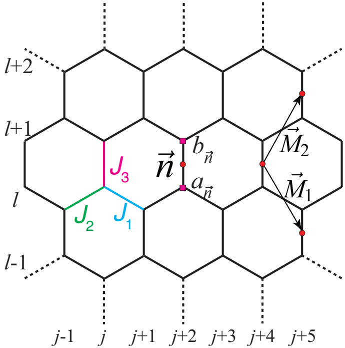
The defect density

$$n \sim \int_0^\infty d^m k \ p_k \left(k\tau^{\nu/\nu z+1}\right) \sim 1/\tau^{m\nu/(\nu z+1)}$$

Kitaev Honeycomb model: $d = 2, m = 1, \nu = z = 1$, the defect density $n \sim 1/\sqrt{\tau}$.

Sengupta, Sen and Mondal, Phys. Rev. Lett. **100** (2008) 077204

Kitaev Model on a honeycomb lattice



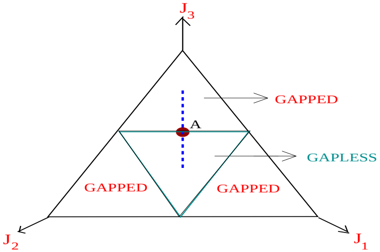
$$H_{2d} = \sum_{j+l=\mathrm{even}} \left(J_1 \sigma^x_{j,l} \sigma^x_{j+1,l} + J_2 \sigma^y_{j-1,l} \sigma^y_{j,l} + J_3 \sigma^z_{j,l} \sigma^z_{j,l+1} \right),$$

$$H' = \sum_{\vec{k}} \left(a^{\dagger}_{\vec{k}} \quad b^{\dagger}_{\vec{k}} \right) H_{\vec{k}} \left(\begin{array}{c} a_{\vec{k}} \\ b_{\vec{k}} \end{array} \right),$$

$$\begin{aligned} H_{\vec{k}} &= \alpha_{\vec{k}}\, \sigma^1 + \beta_{\vec{k}}\, \sigma^2, \\ \alpha_{\vec{k}} &= 2[J_1 \sin(\vec{k}\cdot\vec{M}_1) - J_2 \sin(\vec{k}\cdot\vec{M}_2)], \\ \beta_{\vec{k}} &= 2[J_3 + J_1 \cos(\vec{k}\cdot\vec{M}_1) + J_2 \cos(\vec{k}\cdot\vec{M}_2)]. \end{aligned}$$

$$E^{\pm}_{\vec{k}} = \pm \sqrt{\alpha^2_{\vec{k}} + \beta^2_{\vec{k}}}.$$

Phase Diagram: Gapless Phase for $|J_1 - J_2| \leq J_3 \leq (J_1 + J_3)$



Anisotropic Quantum Critical Point

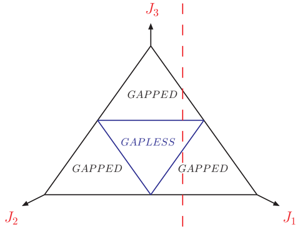
$$\begin{aligned}\alpha_{\vec{k}} &= \sqrt{3}(J_2 - J_1)dk_x + 3(J_1 + J_2)dk_y, \\ \beta_{\vec{k}} &= J_1\left(\frac{\sqrt{3}}{2}dk_x - \frac{3}{2}dk_y\right)^2 + J_2\left(\frac{\sqrt{3}}{2}dk_x + \frac{3}{2}dk_y\right)^2,\end{aligned}$$

$\alpha_{\vec{k}}$ varies linearly in one particular direction in the plane of (dk_x, dk_y), while $\beta_{\vec{k}}$ varies quadratically in any direction.

- • A: Anisotropic QCP $d = 2$, $m = 1$. $\nu_{||} = 1/2$ and $\nu_{\perp} = 1$.

Hickichi, Suzuki and Sengupta, Phys. Rev. B **82**, 174305 (2010)

The two-dimensional Kitaev model on a hexagonal lattice



$J_3 = t/\tau$ is linearly quenched from $-\infty < t < \infty$

Using the Landau-Zener transition formula given by

$$p_{\vec{k}} = \exp(-2\pi\tau[J_1 \sin(\vec{k}\cdot\vec{M}_1) \ - \ J_2 \sin(\vec{k}\cdot\vec{M}_2)]^2)$$

In the limit $\tau \to \infty$ when only modes close to the gapless modes contribute, the defect density is found to scale as $1/\sqrt{\tau}$.

The ground state Quantum Fidelity

We consider the Hamiltonian

$$H(\lambda) = H_0 + \lambda H_I; \quad H(\lambda)|\psi_0(\lambda)\rangle = E_0|\psi_0(\lambda)\rangle$$

where $|\psi_0(\lambda)\rangle$ is the ground state wave function.

- λ is the driving term. The QCP is at $\lambda = 0$.
- The quantum fidelity: modulus of the overlap between two ground state corresponding to parameters λ and $\lambda + \delta$

$$F(\lambda, \delta) = |\langle\psi_0(\lambda)|\psi_0(\lambda + \delta)\rangle|$$

- In the limit $N \to \infty$, the fidelity vanishes in the thermodynamic limit. Anderson's Orthogonality Catastrophe
- What happens for finite N? Indicator of Quantum Criticality: Shows a dip close to it

Small L and $\delta \to 0$ limit: Fidelity Susceptibility

Finite size system How can it indicate criticality?

Length-scales of the problem: $\xi (= \lambda^{-\nu})$, L and $\delta^{-\nu}$

Consider the limit when $\delta \to 0$ and small L

$$F(\lambda, \delta) = 1 - \frac{1}{2} \delta^2 L^d \chi_F(\lambda)$$

Fidelity susceptibility $\chi_F = -\frac{2}{L^d} \lim_{\delta \to 0} (\ln F / \delta^2) = -\frac{1}{L^d} \partial^2 F / \partial \delta^2$

$$\delta^2 L^d \chi_F(\lambda) << 1$$

Points to Note:

- The parameter δ is factored out. χ_F depends on λ only.
- The quantum fidelity is close to unity; Can not describe Anderson's Orthogonality Catastrophe

P. Zanardi and N. Paunkovic, Phys. Rev. E **74** 031123; Gu, S. J., Int. J. Mod. Phys. B **24**, 4371; Gritsev, V., and A. Polkovnikov, 0910.3692.

Scaling of Fidelity Susceptibility close to a QCP

$\delta \to 0$ and small L

$$F(\lambda, \delta) = 1 - \frac{1}{2}\delta^2 L^d \chi_F(\lambda)$$

Fidelity susceptibility $\chi_F = -\frac{2}{L^d} \lim_{\delta \to 0}(\ln F/\delta^2) = -\frac{1}{L^d}\partial^2 F/\partial\delta^2$

$$\delta^2 L^d \chi_F(\lambda) << 1$$

- $\chi_F \sim \lambda^{\nu d - 2}$ for $\xi \ll L$.
- $\chi_F \sim L^{2/\nu - d}$ for $\xi \gg L$:

Quantum Critical Scaling: $\nu d < 2$

Caveat No dip the fidelity for $\nu d > 2$.

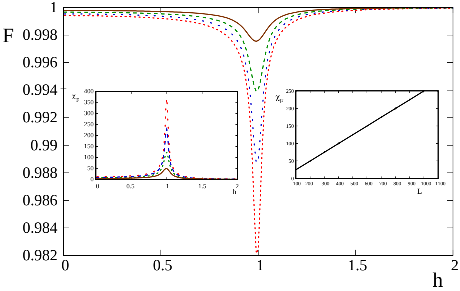
This immediately gives the scaling of the defect density following a quench of small amplitude starting from the QCP discussed before.

Verification for a transverse Ising Chain

We consider the transverse Ising Chain Hamiltonian

$$H = -\sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z$$

QCP at $\lambda = h - 1 = 0$, exponents $\nu = z = 1$.



For large system size; Deviates from unity away from QCP

Derivation: scaling of Fidelity Susceptibility close to a QCP

Using the perturbation expansion:

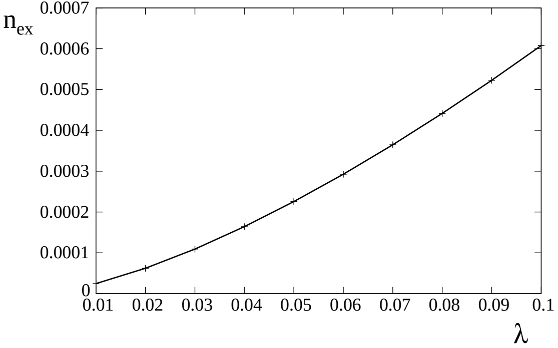
$$\chi_F(\lambda) = \frac{1}{L^d} \sum_{m \neq 0} \frac{|\langle \psi_m(\lambda)| H_I |\psi_0(\lambda)\rangle|^2}{[E_m(\lambda) - E_0(\lambda)]^2}.$$

χ_F shows a scaling behavior with exponent given in terms of some of the critical exponents.
Compare χ_F with the ground state specific heat density

$$\chi_E = -\frac{1}{L^d} \frac{\partial^2 E_0}{\partial \lambda^2} = -\frac{2}{L^d} \sum_{m \neq 0} \frac{|\langle \psi_m(\lambda)| H_I |\psi_0(\lambda)\rangle|^2}{E_m(\lambda) - E_0(\lambda)}$$

We note the difference in the denominator. Stronger divergence is expected.
Close to a QCP: $\chi_E \sim |\lambda|^{-\alpha}$ and the hyperscaling relation $2 - \alpha = \nu(d + z)$.

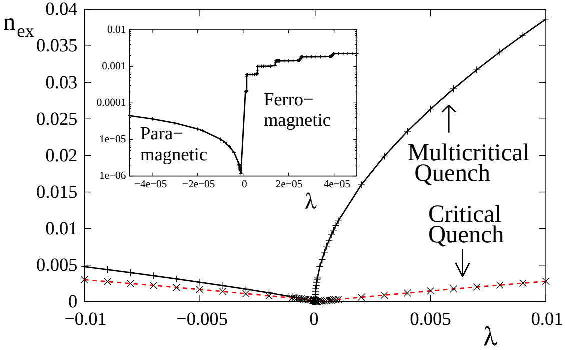
Small Quench starting from the AQCP using fidelity susceptibility



- Scaling with λ: $n_e x \sim \lambda_f^2 \chi_F \sim \lambda_f^{\nu_{||} m + (d-m)\nu_\perp}$
  for $|\lambda| \gg L_{||}^{-1/\nu_{||}}, L_\perp^{-1/\nu_\perp}$
- In the present case: $n_{ex} \sim \lambda_f^{3/2}$

V. Mukherjee, and A. Dutta, Phys. Rev. B **83** 214302 (2011).

Small Quench starting from the MCP



- • Small system size Kinks in the defect in the ferro side; minimum gap points
- • Para side: $n_{ex} \sim \lambda_f^{3/2}$ and Ferroside $n_{ex} \sim \lambda_f^{1/2}$.

References V. Mukherjee, A. Polkovnikov and A. Dutta, Phys. Rev. B **60** 344 (2011).

M. Rams and B. Damski, arxiv:1104.4102 (2011).

Concept of Generalized fidelity Susceptibility

Consider a generic Quench: $\lambda(t) = \delta \frac{t^r}{r} \Theta(t)$

- $r = 0$: rapid quench of amplitude δ
- $r = 1$: Slow linear Quench

Probablity of excitation: $P_{ex} = \delta^2 L^d \chi_{2r+2}$ where

$$\chi_m = \frac{1}{L^d} \sum_{n \neq 0} \frac{|\langle|\psi_n|H_I|\psi_0\rangle|^2}{(E_n - E_0)^m}$$

- $\chi_{2r+2} \sim \delta^{(\nu d - 2 - 2\nu z r)/(1+\nu z r)}$
- $n_{ex} \sim \delta^{d\nu/(z\nu r+1)}$; $r = 0$, sudden quench $n_{ex} \sim \lambda_f^{d\nu}$
- $r = 1$, the KZ Scaling: $n_{ex} \sim \tau^{-\nu d/(\nu z+1)}$ with $\delta = 1/\tau$.
- Can be generalized to an AQCP

Grandi, Gritsev and Polkovnikov, Phys Rev B **81** 012303 (2010);
Mukherjee and Dutta, Phys. Rev. B (2011)