Rationality and Complexity in Games

Rajiv Sethi

Dynamics of Complex Systems July 2019

Overview

Lecture I: Fundamentals and Paradoxes

- Players, Strategies, and Payoffs
- Nash Equilibrium
- Extensive Form Games
- Paradoxes

Lecture II: Complexity and Rationality

- Repeated Games
- Finite Automata
- Procedural Rationality
- Disequilibrium Dynamics

Experiment

- Ten players
- Each person chooses a (rational) number in the interval $[0, 100]$
- We compute half the average
- Player closest to half the average gets \$1000 (shared equally if tied)
- All others get \$0

Games

A game is defined by

- A set of players $\{1, ..., n\}$
- For each player i, a set of strategies S_i
- A payoff function $u : S_1 \times ... \times S_n \to \mathbf{R^n}$

Pure strategies involve a complete plan of contingent actions

Chess: $\approx 10^{47}$ board positions; more pure strategies than atoms in universe

Mixed strategies involve randomizations over the set of pure strategies

Nash Equilibium

A strategy profile $(s_1^*, ..., s_n^*) \in S_1 \times ... \times S_n$ is a Nash equilibrium if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for all i and all $s_i \in S_i$.

No player can improve her payoff with a unilateral change of strategy

A strategy profile is a strict Nash equilibrium if the inequality holds strictly

Chess has an (unknown) equilibrium in pure strategies; not all games do

Equilibrium in the Half-the-Average game?

Public Goods

	H	M	L
H	$6, 6$	$3, 7$	$0, 8$
M	$7, 3$	$4, 4$	$1, 5$
L	$8, 0$	$5, 1$	$2, 2$

Public Goods

	H	M	L
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All-Pay Auction

Consider an object with value $v = \frac{5}{2}$ and two bidders

Simultaneous bids, nonnegative integers, ties broken at random

Strategies and payoffs:

	0	1	2
0	$\frac{5}{4}, \frac{5}{4}$	$0, \frac{3}{2}$	$0, \frac{1}{2}$
1	$\frac{3}{2}, 0$	$\frac{1}{4}, \frac{1}{4}$	$-1, \frac{1}{2}$
2	$\frac{1}{2}, 0$	$\frac{1}{2}, -1$	$-\frac{3}{4}, -\frac{3}{4}$

Does a pure strategy equilibrium exist?

Nash's Existence Theorem

Does every game have an equilibrium?

If S_i is a compact and convex subset of a Euclidian space, and u is a continuous function, then an equilibrium exists

Proof: Application of Kakutani's Fixed Point Theorem

Corollary: Every finite game has a mixed strategy equilibrium

Coordination and Hawk-Dove games have pure and mixed equilibria

Mixed Strategy Equilibria

The coordination game

	Left	Right
Up	5, 5	0, 3
Down	3, 0	4, 4

has a symmetric mixed strategy equilibrium with probabilities $(2/3, 1/3)$

The Hawk-Dove game

	Hawk	Dove
Hawk	0, 0	8, 2
Dove	2, 8	5, 5

has a symmetric mixed strategy equilibrium with probabilities $(3/5, 2/5)$

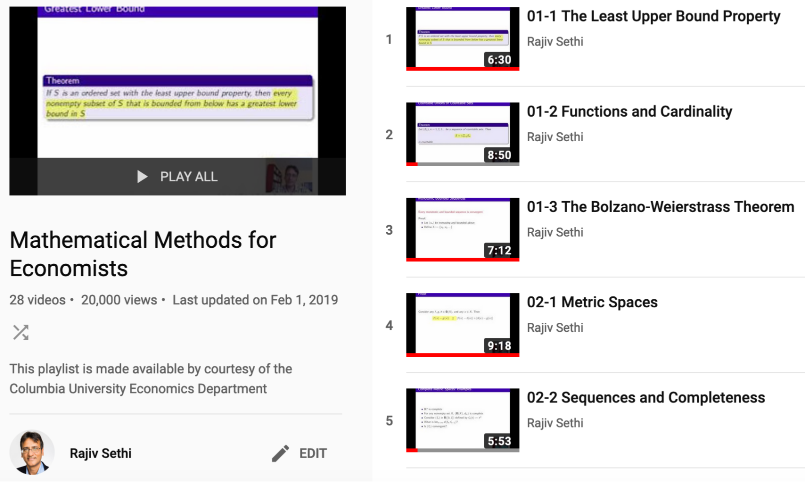
All-Pay Auction

Strategies and payoffs:

	0	1	2
0	$\frac{5}{4}, \frac{5}{4}$	$0, \frac{3}{2}$	$0, \frac{1}{2}$
1	$\frac{3}{2}, 0$	$\frac{1}{4}, \frac{1}{4}$	$-1, \frac{1}{2}$
2	$\frac{1}{2}, 0$	$\frac{1}{2}, -1$	$-\frac{3}{4}, -\frac{3}{4}$

Symmetric mixed strategy equilibrium with distribution $(1/5, 3/5, 1/5)$

Equilibrium payoffs are $\frac{1}{4}$, collusive payoffs $\frac{5}{4}$


Greatest Lower Bound
Theorem
If S is an ordered set with the least upper bound property, then every nonempty subset of S that is bounded from below has a greatest lower bound in S
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Rajiv Sethi
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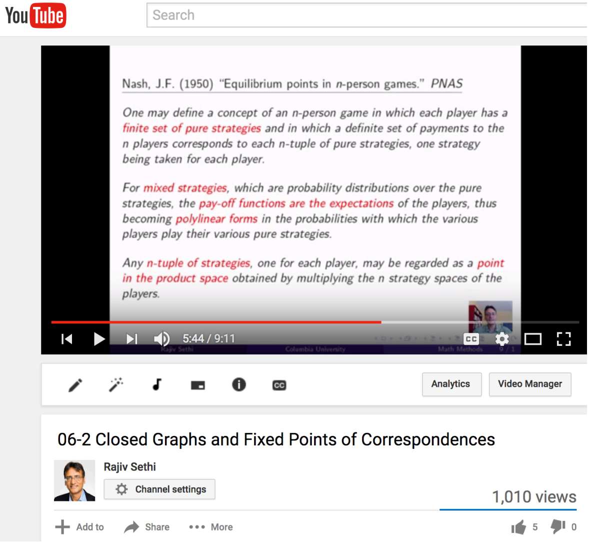
1
6:30
01-1 The Least Upper Bound Property
Rajiv Sethi
2
8:50
01-2 Functions and Cardinality
Rajiv Sethi
3
7:12
01-3 The Bolzano-Weierstrass Theorem
Rajiv Sethi
4
9:18
02-1 Metric Spaces
Rajiv Sethi
5
5:53
02-2 Sequences and Completeness
Rajiv Sethi


Search
Nash, J.F. (1950) "Equilibrium points in n-person games." PNAS
One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player.
For mixed strategies, which are probability distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.
Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players.
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06-2 Closed Graphs and Fixed Points of Correspondences
Rajiv Sethi
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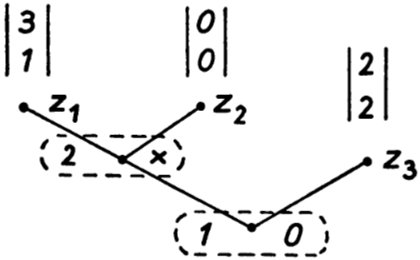
SPIELTHEORETISCHE BEHANDLUNG EINES OLIGOPOLMODELLS MIT NACHFRAGETRÄGHEIT

TEIL I: BESTIMMUNG DES DYNAMISCHEN PREISGLEICHGEWICHTS

von

REINHARD SELTEN

Frankfurt/M.



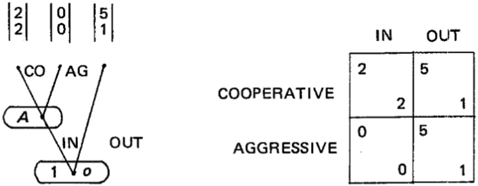
	Left	Right
Left	3, 1	0, 0
Right	2, 2	2, 2

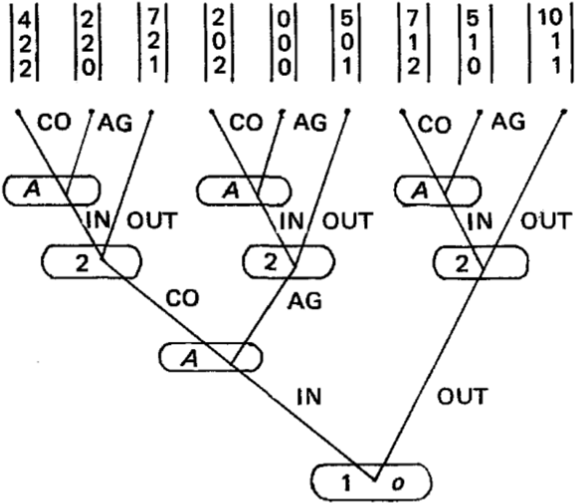
REINHARD SELTEN

THE CHAIN STORE PARADOX

TABLE I
Player A's partial payoffs and player k's payoff.

player k's decision	player A's decision in period k	player k's payoff	player A's partial payoff for period k
IN	COOPERATIVE	2	2
IN	AGGRESSIVE	0	0
OUT	–	1	5




4 2 2
2 2 0
7 2 1
2 0 2
0 0 0
5 0 1
7 1 2
5 1 0
10 1 1
CO
AG
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A
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IN
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2
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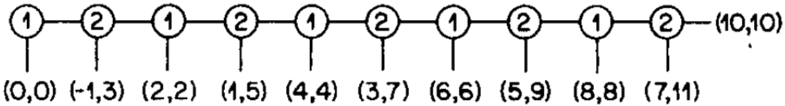
1
o

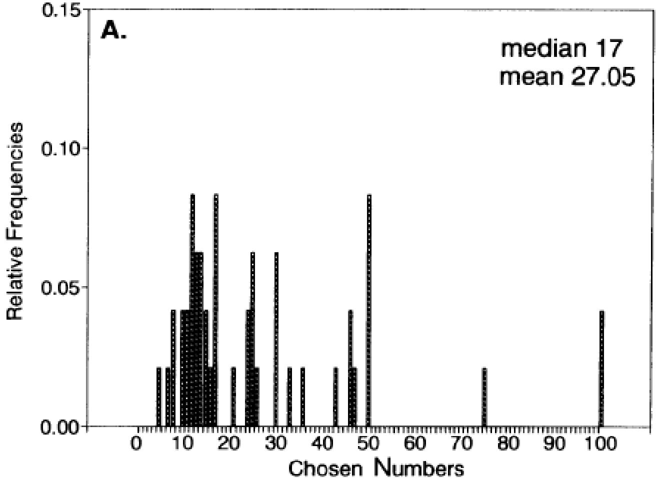
Games of Perfect Information, Predatory Pricing and the Chain-Store Paradox

ROBERT W. ROSENTHAL



Unraveling in Guessing Games: An Experimental Study

By Rosemarie Nagel*



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Level-k Reasoning

Is there a Nash equilibrium in the half-the-average game?

Level k models

- Level-0 players choose uniformly at random
- Level-k players choose based on belief that others are level $k-1$
- Distribution of types implies distribution of strategies

Fits behavior better than Nash in half-the-average and related games

Level-k Equilibrium in the Public Goods Game

	H	M	L
H	6, 6	3, 7	0, 8
M	7, 3	4, 4	1, 5
L	8, 0	5, 1	2, 2

Level-0 players choose uniformly at random

What about level-k for $k \geq 1$?

What does the model predict?

Sampling Equilibrium

Consider symmetric two-player game with m pure strategies (actions)

Let $p = (p_1, ..., p_m)$ denote an arbitrary mixed strategy

Suppose each action is sampled once against p

Let $w_i(p)$ denote probability that action i results in highest payoff

Then p^* is a sampling equilibrium if

$$w_i(p^*) = p_i^*$$

for all i

Interpretation: steady state of a population with inflows and outflows

Sampling and Nash in All-Pay Auctions

	0	1	2
0	$\frac{5}{4}, \frac{5}{4}$	$0, \frac{3}{2}$	$0, \frac{1}{2}$
1	$\frac{3}{2}, 0$	$\frac{1}{4}, \frac{1}{4}$	$-1, \frac{1}{2}$
2	$\frac{1}{2}, 0$	$\frac{1}{2}, -1$	$-\frac{3}{4}, -\frac{3}{4}$

Is the Nash equilibrium $p = (1/5, 3/5, 1/5)$ also a sampling equilibrium?

$$w_0(p) = \left(\frac{1}{5} \times \frac{4}{5}\right) + \left(\frac{4}{5} \times \frac{1}{5} \times \frac{1}{5}\right) = \frac{24}{125} \neq \frac{1}{5}$$

Sampling Equilibrium in the Public Goods Game

	H	M	L
H	$6,6$	$3,7$	$0,8$
M	$7,3$	$4,4$	$1,5$
L	$8,0$	$5,1$	$2,2$

The strict Nash equilibrium at $p^* = (0,0,1)$ is a sampling equilibrium

But there is a second sampling equilibrium at $p^* = (0.20, 0.28, 0.52)$

Which one should we expect to see?

Stability

Sampling dynamics:

$$\dot{p}_i > 0 \text{ if and only if } w_i(p) > p_i$$

Example:

$$\dot{p}_i = w_i(p) - p_i$$

A stable sampling equilibrium is stable rest point of the sampling dynamics

Which of the sampling equilibria in the public goods game is stable?

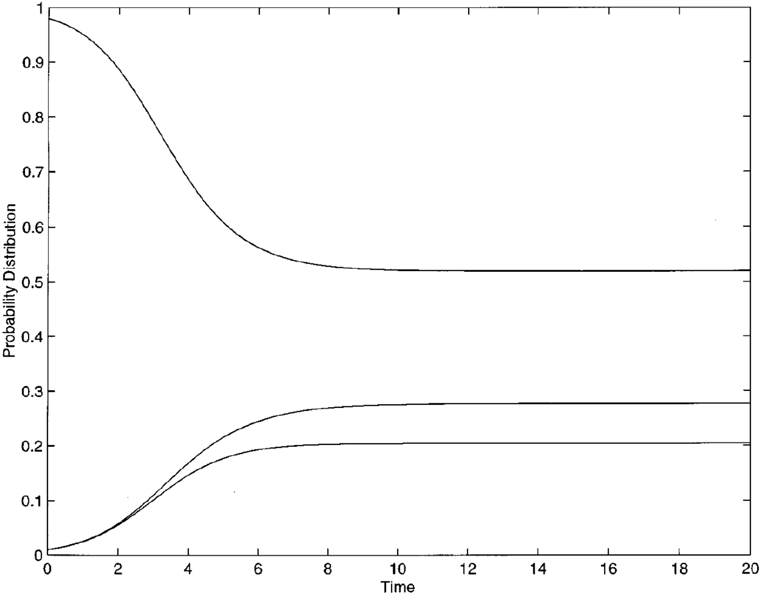


FIG. 1. Convergence to the interior $S(1)$ equilibrium.

Stability of Equilibria in Games with Procedurally Rational Players[1]

Rajiv Sethi

DEFINITION. An action profile $(a_q, a_q, \ldots, a_q)$ in a symmetric n-player game is *inferior* if, for every $i \neq q$, there exists $j(i) \neq q$ such that

$$u(a_{j(i)}, a_i, a_q, \ldots, a_q) > u(a_q, a_q, \ldots, a_q).$$

It is *twice inferior* if, for every action $i \neq q$, there exist $j(i) \neq q$ and $k(i) \neq q$ such that $j \neq k$ and

$$u(a_{j(i)}, a_i, a_q, \ldots a_q) \geq u(a_{k(i)}, a_i, a_q, \ldots, a_q) > u(a_q, a_q, \ldots, a_q).$$

THEOREM 1. *If $(\alpha^*, \ldots, \alpha^*)$ is an inferior strict Nash equilibrium of a symmetric game with three or more players, then α^* is unstable under the sampling dynamics* (1).

THEOREM 2. *If (α^*, α^*) is a twice inferior strict Nash equilibrium of a symmetric two-player game, then α^* is unstable under the sampling dynamics* (1)

Stable Sampling Equilibrium in Common Pool Resource Games

Juan Camilo Cárdenas [1], César Mantilla [2] and Rajiv Sethi [3,*]

CPR game: n players, actions $A = \{a_1, ..., a_m\}$ (resource extraction levels)

Aggregate extraction is

$$X = \sum_{j=1}^{n} x_j$$

where x_i is extraction of player i

Payoff to player i depends on own action and aggregate action

$$\pi_i = g(x_i, X)$$

where g is increasing in first argument and decreasing in second

Divergence between individual incentives and collective interests

Nash equilibrium has inefficiently high extraction

American Political Science Review Vol. 86, No. 2 June 1992

COVENANTS WITH AND WITHOUT A SWORD: SELF-GOVERNANCE IS POSSIBLE

ELINOR OSTROM, JAMES WALKER, and ROY GARDNER
Indiana University, Bloomington

Norms from outside and from inside: an experimental analysis on the governance of local ecosystems

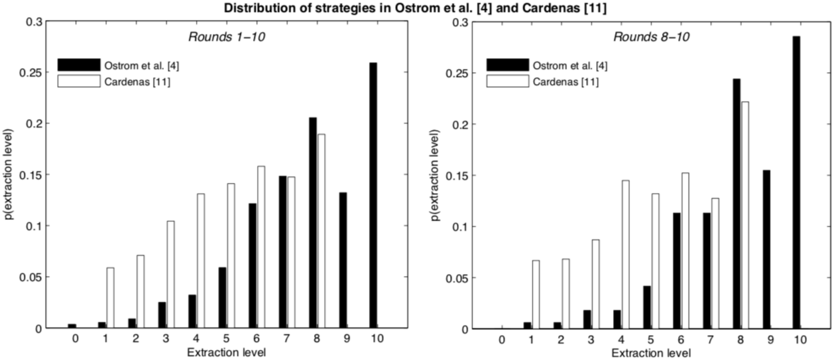
Juan-Camilo Cardenas*

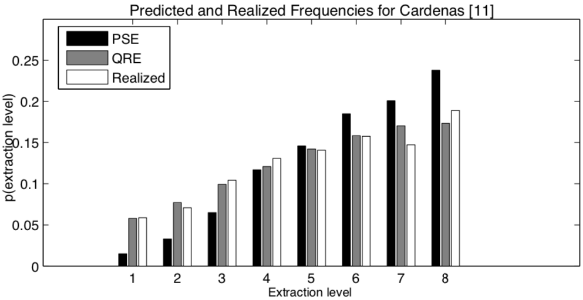
Experiments

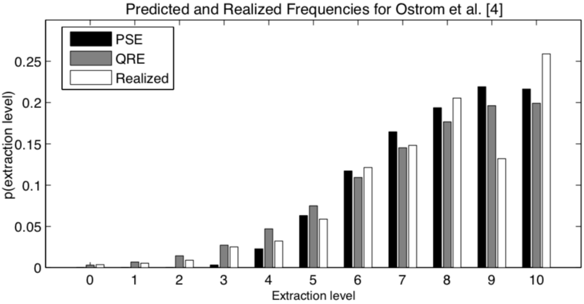
n players, token endowment e, divided between two markets, payoffs:

$$\pi_i = w(e - x_i) + \frac{x_i}{X} f(X)$$

	Cárdenas [11] **Students**	**Ostrom *et al.* [4]**
a. Experimental setting		
Number of subjects	230	56
Subjects per group	5	8
Number of rounds	10	10
Action set	$\{1,\ldots,8\}$	$\{0,\ldots,10\}$
Nash equilibrium	8	8
Surplus maximizing per-capita extraction	1	4.5


Distribution of strategies in Ostrom et al. [4] and Cardenas [11]
Rounds 1–10
Rounds 8–10
Ostrom et al. [4]
Cardenas [11]
p(extraction level)
Extraction level
0
0.05
0.1
0.15
0.2
0.25
0.3
0 1 2 3 4 5 6 7 8 9 10


Predicted and Realized Frequencies for Cardenas [11]
PSE
QRE
Realized
p(extraction level)
0.25
0.2
0.15
0.1
0.05
0
1 2 3 4 5 6 7 8
Extraction level


Predicted and Realized Frequencies for Ostrom et al. [4]
PSE
QRE
Realized
p(extraction level)
0.25
0.2
0.15
0.1
0.05
0
0 1 2 3 4 5 6 7 8 9 10
Extraction level

Efficiency and stability of sampling equilibrium in public goods games

César Mantilla[1] | Rajiv Sethi[2] | Juan Camilo Cárdenas[1]

Public Goods Game

Symmetric game, n players, action set $A = \{0, 1, ..., e\}$

Interpretation: contributions to public good, given endowment e

Payoff to a player who contributes a when aggregate contribution is S:

$$(e - a) + f(S),$$

where f is (weakly) concave and satisfies $f(0) = 0$.

Linear case: each unit contributed yields some μ to each player:

$$(e - a) + \mu S$$

If $\mu n > 1$ and $\mu < 1$, this represents a social dilemma

- $\mu n > 1$ implies socially optimal for all to contribute fully
- $\mu < 1$ implies individually rational to contribute zero

Example

Linear case, $n = 3$, $A = \{0, 1\}$, payoffs:

		$S - a_i$		
		0	1	2
a_i	0	1	$1 + \mu$	$1 + 2\mu$
	1	μ	2μ	3μ

where $3\mu > 1$

If $\mu < 1$ NE has no contribution, if $\mu > 1$ then NE full contribution

All strict Nash equilibria are sampling equilibria

But what about stable sampling equilibia?

	Nash	Stable Sampling
$\mu \in (1/3, 1/2)$	(1,0)	(1,0)
$\mu \in (1/2, 1)$	(1,0)	(0.72,0.28)
$\mu > 1$	(0,1)	(0.28,0.72)

So the strict Nash equilibrium is unstable even when it is efficient

Prediction: behavioral heterogeneity even in simple environments

Example

Linear case, $n = 2$, $A = \{0, 1, 2\}$, payoffs:

		$S - a_i$		
		0	1	2
	0	2	$2 + \mu$	$2 + 2\mu$
a_i	1	$1+\mu$	$1+2\mu$	$1+3\mu$
	2	2μ	3μ	4μ

where $2\mu > 1$

Public goods game if $\mu \in (1/2, 1)$; efficient dominant strategy if $\mu > 1$.

If $\mu < 1$ NE has no contribution, if $\mu > 1$ then NE full contribution

Now stable sampling differs only when the Nash equilibrium is inefficient

	Nash	Stable Sampling
$\mu \in (1/2, 2/3)$	(1,0,0)	(1,0,0)
$\mu \in (2/3, 1)$	(1,0,0)	(0.52,0.28,0.20)
$\mu > 1$	(0,0,1)	(0,0,1)

Conclusions

Nash equilibrium works well in some cases, poorly in others

Same goes for alternative models (level k, sampling)

No solution concept has universal applicability

Approach to games must be context dependent

Further Reading

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