Phase Transitions on Networks:

a) Persistence as order parameter b) Approach to mean field

Collaborators

Covered in this talk

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In related works

Dr. A. R Sonawane

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Dynamics without Hamiltonian or Lyapunov function

Often spatiotemporal dynamics in which different parts in space behave differently.
No long-range order in space.
But lasting memory.

Partial Differential Equations:

Dissipative solitons.

Coupled Oscillators:

Chimera States

Coupled Maps.

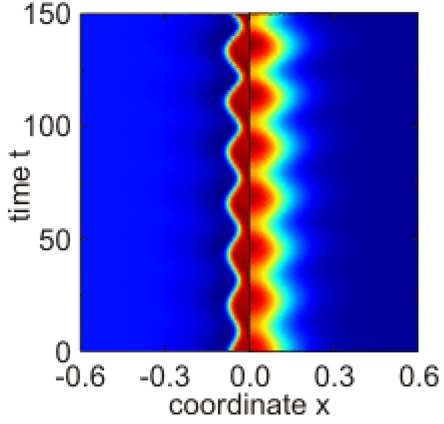
Localized Chaos.

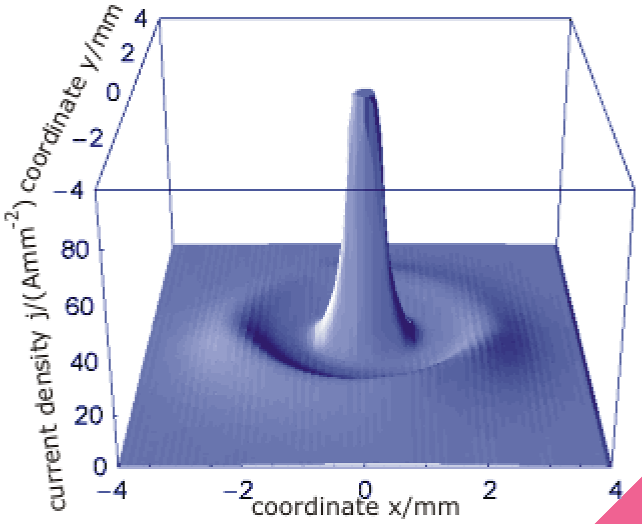
Cellular Automata:

Gliders etc.

Dissipative Solitons

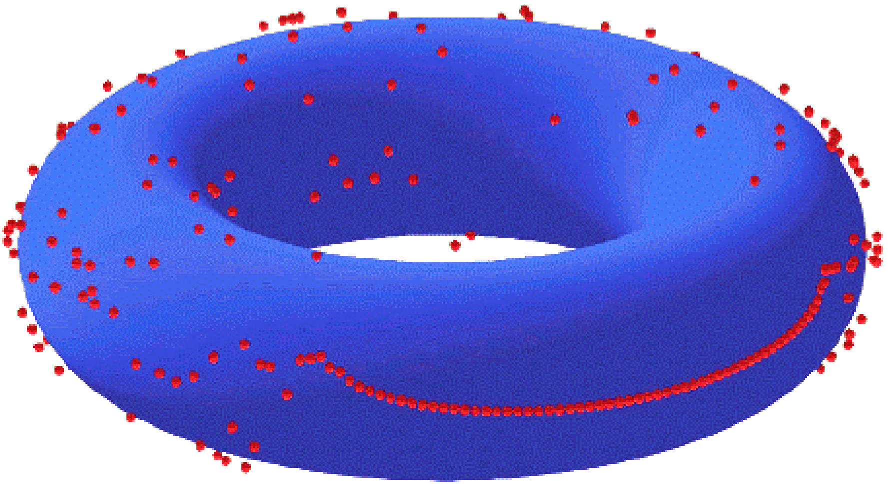
Found in reaction-diffusion equations, Ginzburg-Landau, Swift-Hohenberg, in fact all standard pattern forming equations. Growing number of experiments.

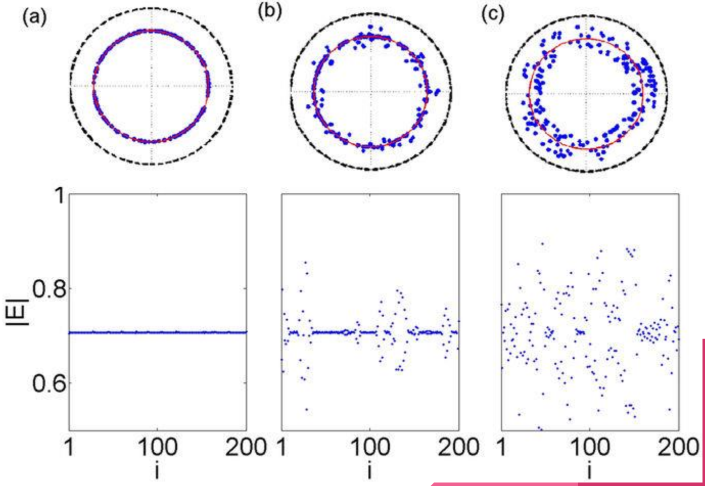




Chimera states

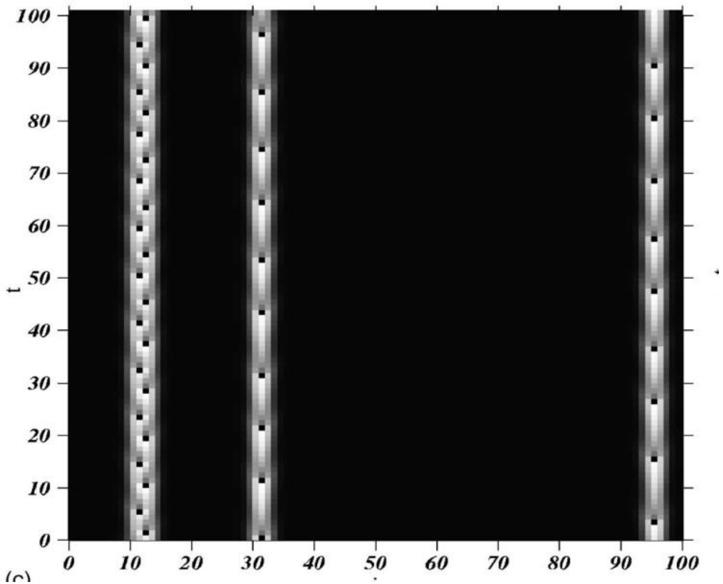
Homogeneous states coexisting with inhomogeneous state. Phase oscillators and coupled lasers.

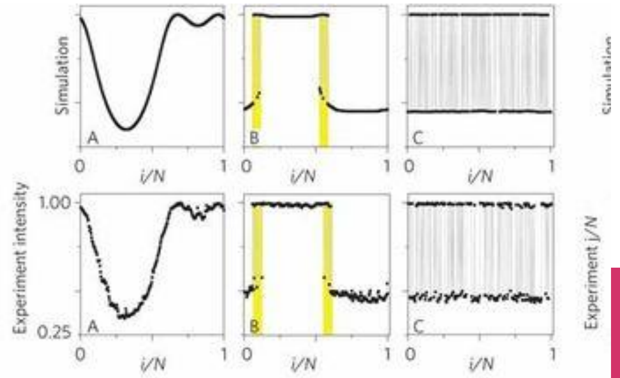




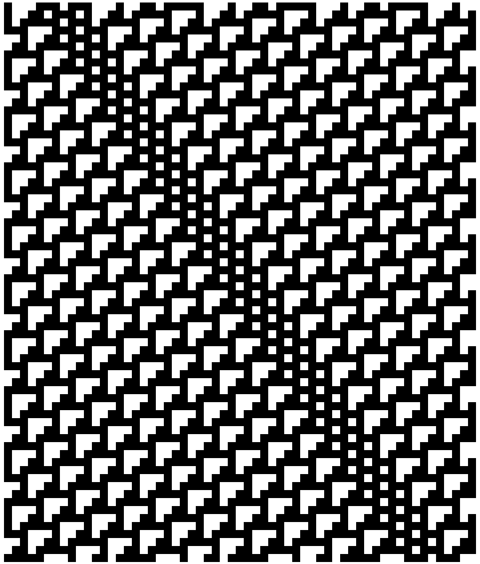
Coupled map lattice: Localized chaos.

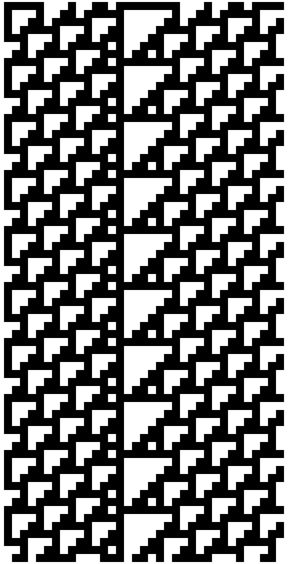
Both numerical simulations (Gupte etc) as well as experimental realizations (Roy etc.)





Cellular automata: Rule 110





You may change parameter to approach ‘chimera’

Thus we are talking of phase with state which is partially arrested.

Multiple co-existing behaviors.

We try to treat this as a phase transition.

For fully absorbed states, order parameter is well defined.

Persistence as order parameter

How long you remembered initial state.

Order parameter for fully absorbing states is easy to define.

Synchronization --- variance.
Number of active particles etc.

For partially absorbed states , there is no long range order in space, but they remember things for long.

Some reference to time essential.

We say that the sites which never forgot their initial condition, did not deviate from this state even once have persisted.

Nonzero persistence is signature of fully-partially arrested state.
Like EA order parameter, but stronger.

Coupled map lattices.

PDE- Continuous space-time

Coupled oscillators: Discrete space

Coupled maps –Discrete time

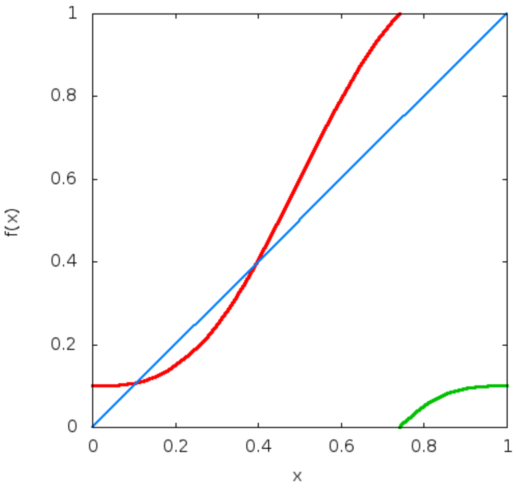
(CA-Discrete variable value as well.)

A generic definition in 1-d nearest neighbor coupling is

$$x(i,t+1)=h_0\, f(x(i,t)) + h_1\, g(x(i+1,t-1)) + h_{-1}\, h(x(i-1,t-1))$$

Usually $h_0 + h_1 + h_{-1} = 1$

Coupled circle maps



We take unstable fixed point as reference and divide phase space in two parts.

Coupled circle maps on small world.

We study transition from chimera/ partially arrested state to spatiotemporal chaos for coupled circle maps. Here, some maps are stuck and some are not.

Whenever you have multistability in the system, say some initial condition goes to attractor A and some to attractor B.

If they are uncoupled, they go to their respective attractors. This behavior should persist for small coupling.

Absence of any variational function means ground state is not unique.

How to define persistence.

We take fixed point of the map as reference point and divide in + and - zones

If the slope at fixed point is positive, + are expected to stay + and – are expected to stay -.

If slope is negative, + will be –. It'll return to + on two iterations. Thus, + are expected to stay + at all even times and – are expected to stay -.

Deviant lot which departs from what is expected, has not persisted.

Persistent sites are ones which did not depart from expected behavior even once.

The phase diagram

SFP- Synchronized Fixed point

FP/PO- Coexistence of Fixed point

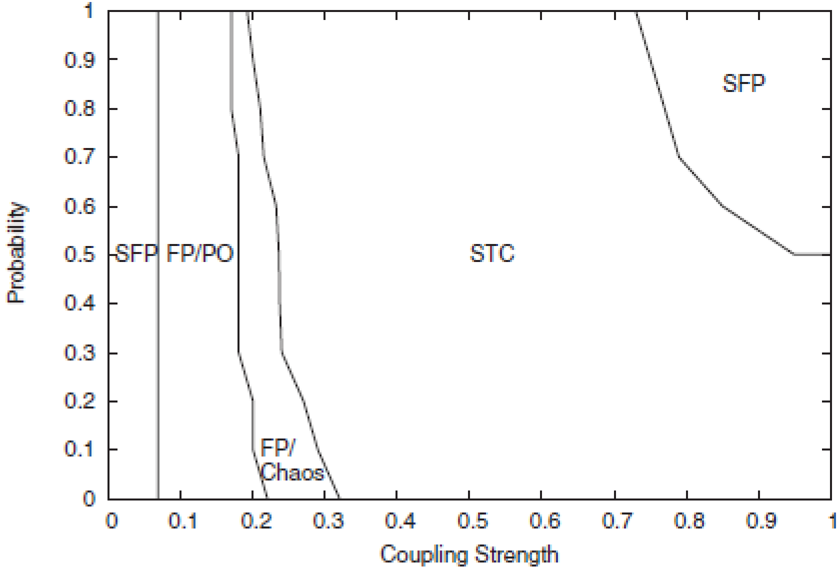
and periodic orbit

FP/Chaos- Coexistence of Fixed point

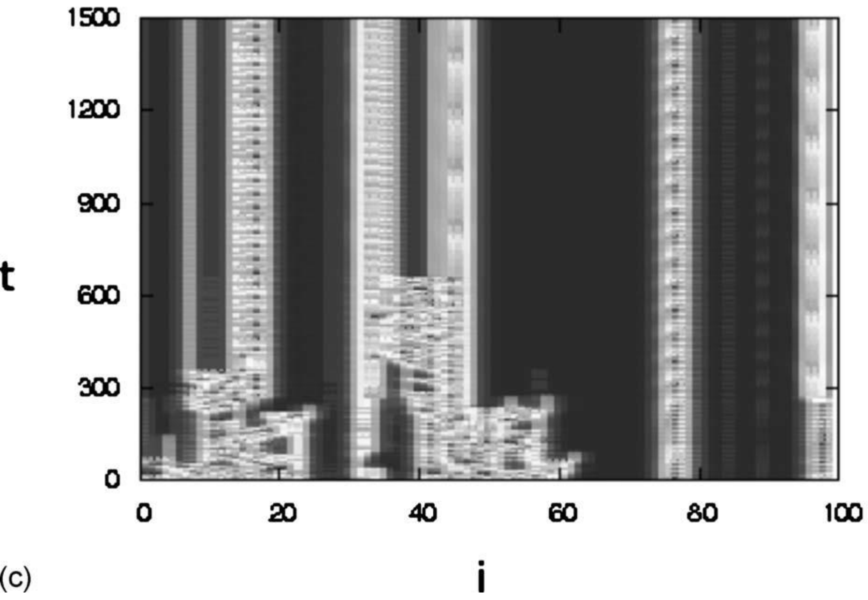
and chaos

STC- Spatiotemporal Chaos

SFP- Synchronized Fixed Point



We focus on FP/Chaos – spatiotemporal chaos



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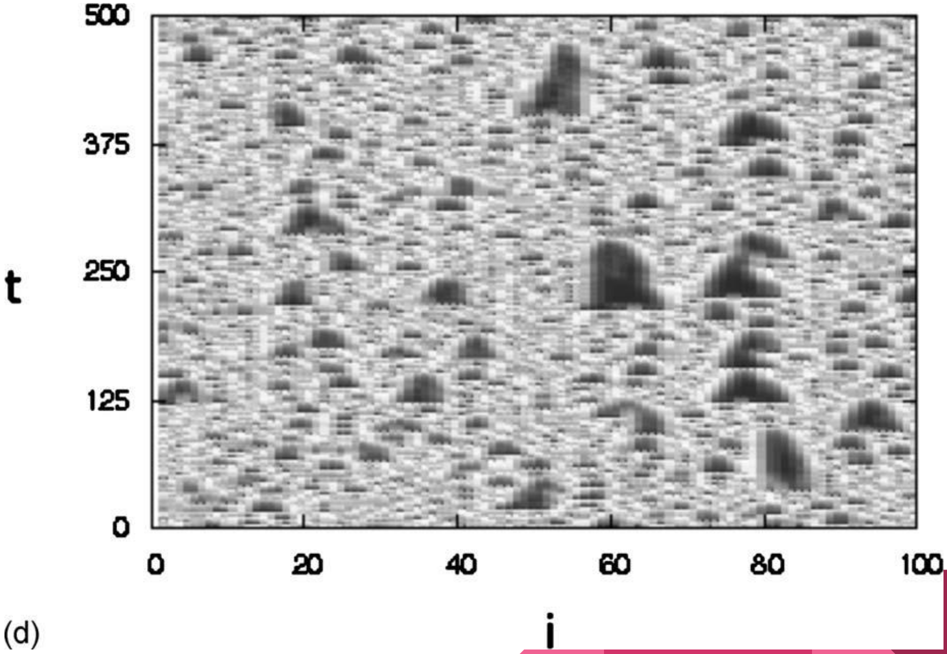
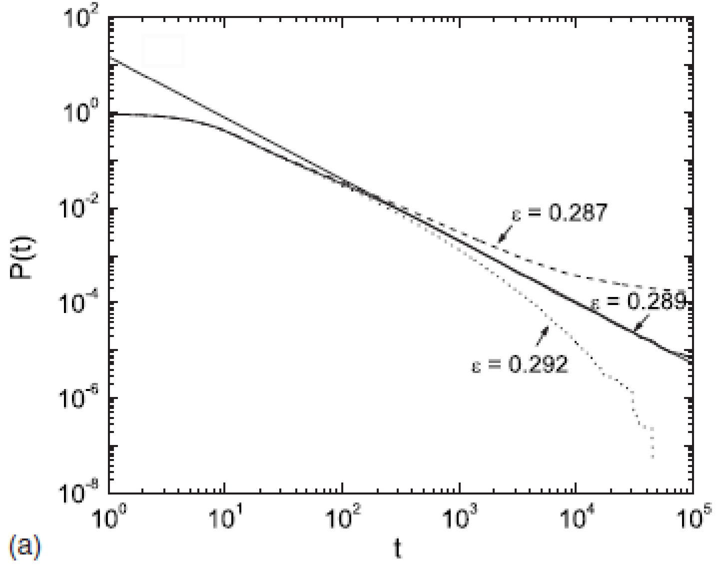
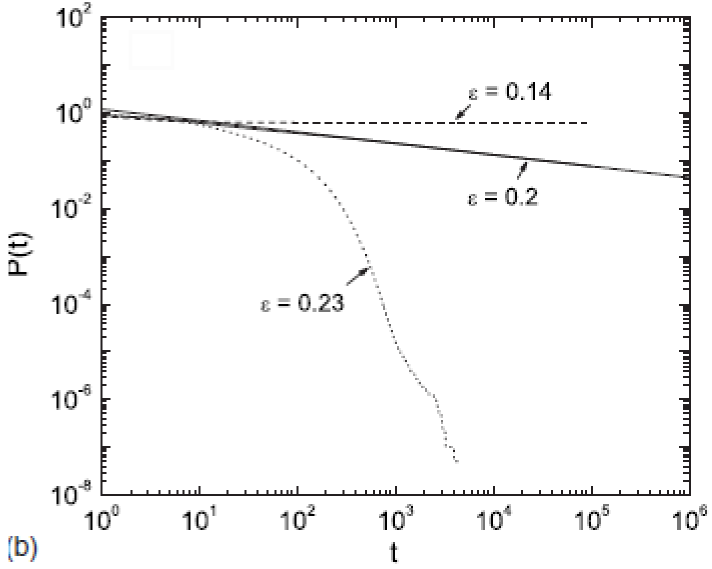


(d)

Persistence as order parameter

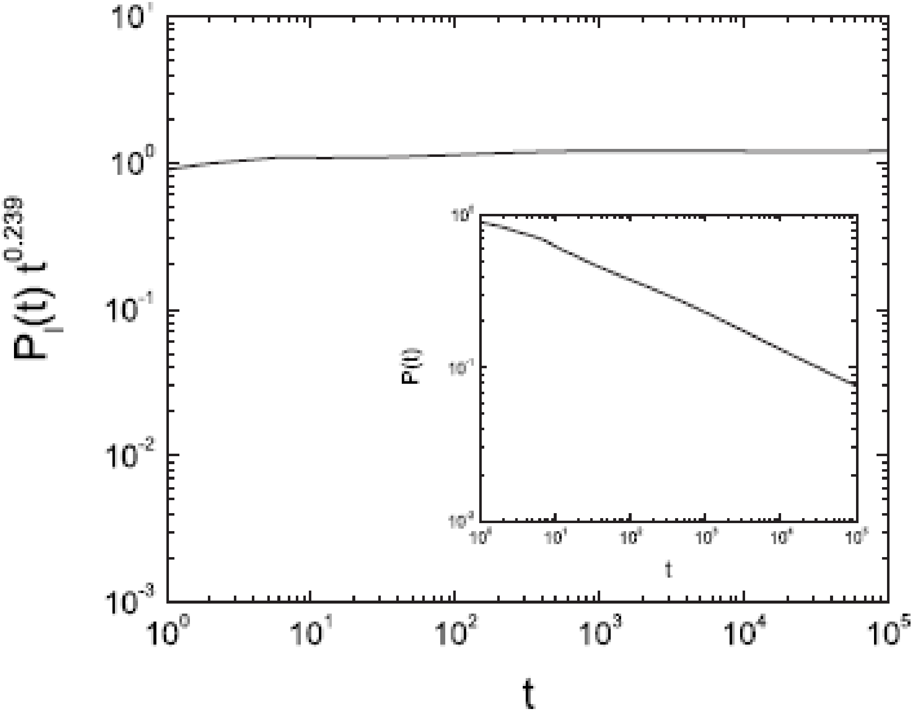


p=0.1 exponent 1.291

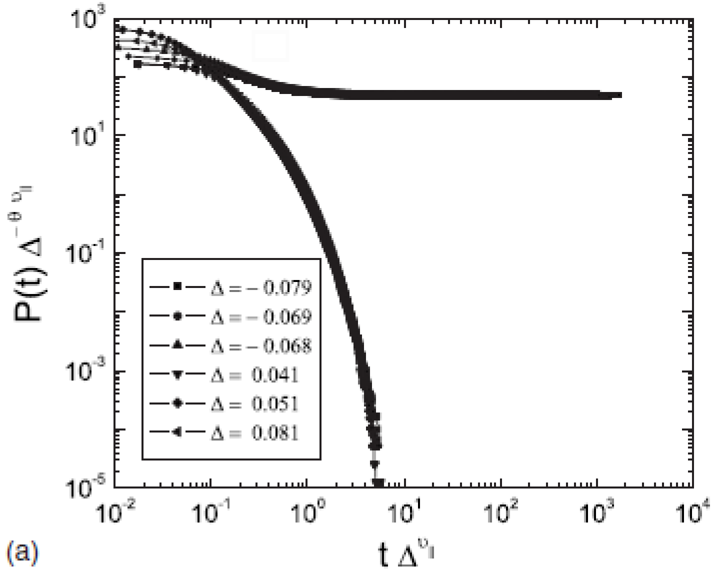


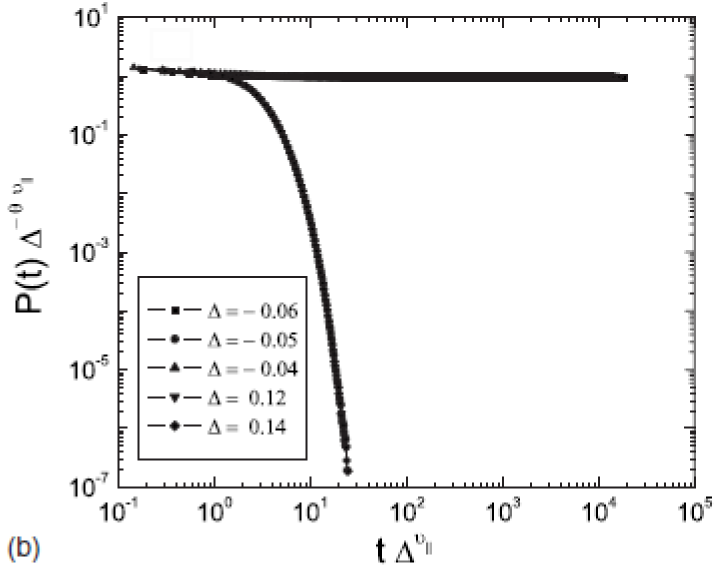
p=0.8 exponent 0.239

Further evidence for power law decay



Off-critical scaling collapse

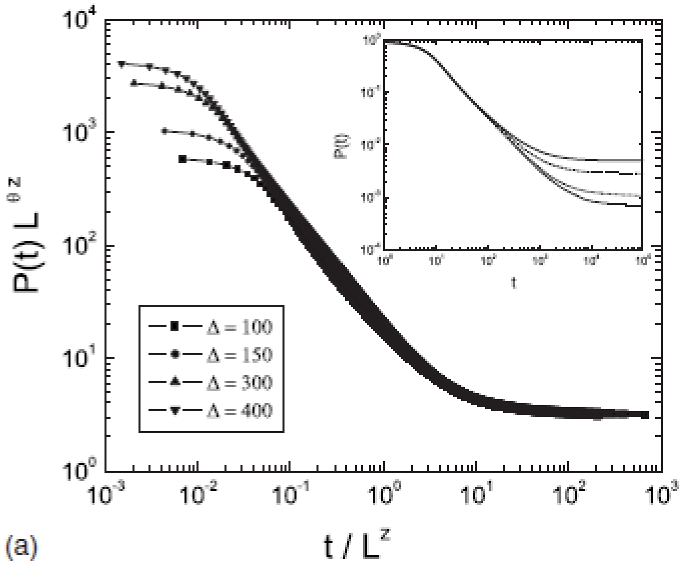


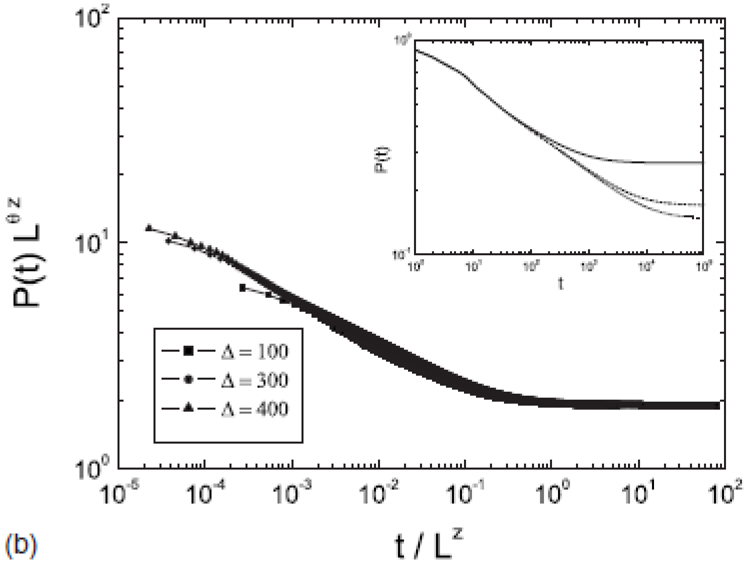


p=0.1 (exponent 1.6)

p=0.8 (exponent 0.47)

Finite size scaling





p=0.1 z= 1.085

p=0.8 z=1.786

Zoo of exponents without any systematic trend

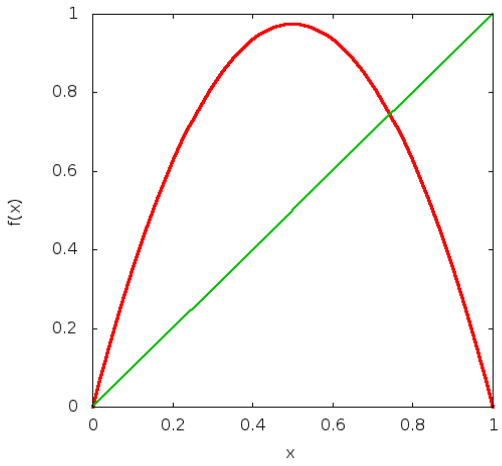
Prob. (p)	Critical point (ϵ_c)	Persistence exponent (θ_l)	$\nu_{\|}$	z
0.0	0.31799 ± 0.00002	1.565 ± 0.003	1.293 ± 0.003	1.145 ± 0.003
0.1	0.289 ± 0.0002	1.291 ± 0.004	1.60 ± 0.02	1.085 ± 0.003
0.2	0.271 ± 0.0002	1.247 ± 0.003	1.82 ± 0.02	1.146 ± 0.002
0.3	0.2348 ± 0.0001	0.344 ± 0.002	1.84 ± 0.02	1.650 ± 0.003
0.4	0.2330 ± 0.0003	0.670 ± 0.002	1.13 ± 0.01	2.134 ± 0.002
0.5	0.2360 ± 0.0003	1.775 ± 0.002	0.65 ± 0.02	4.70 ± 0.03
0.6	0.2358 ± 0.0002	1.510 ± 0.002	0.93 ± 0.02	3.50 ± 0.02
0.7	0.2140 ± 0.0003	0.461 ± 0.002	0.20 ± 0.02	1.586 ± 0.002
0.8	0.200 ± 0.003	0.239 ± 0.002	0.470 ± 0.003	1.786 ± 0.002
0.9	0.192 ± 0.002	0.149 ± 0.001	0.70 ± 0.02	2.13 ± 0.02
1.0	0.190 ± 0.001	0.115 ± 0.001	0.45 ± 0.02	2.33 ± 0.03

Coupled logistic maps on small world networks

We partition the space by unstable

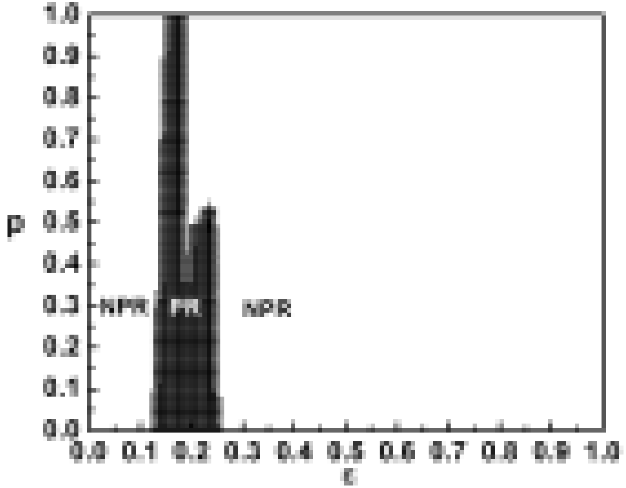
fixed point.

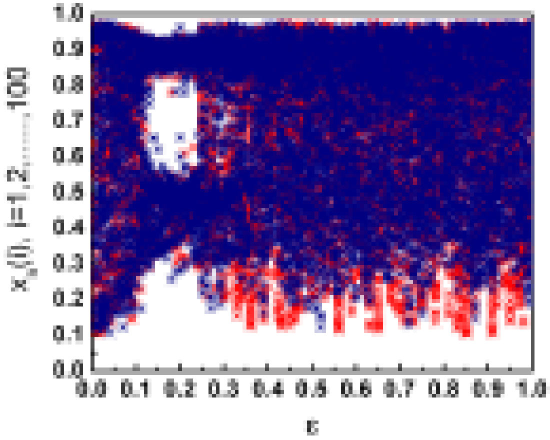
But now the persistence is modulo-2.



We couple to 2 neighbors on either side, four sites overall.

Phase Diagram and Bifurcation Diagram

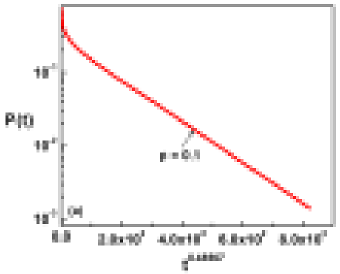


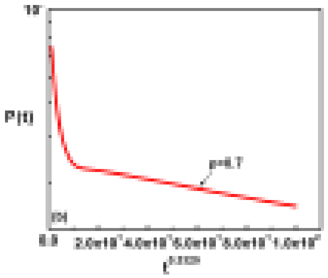


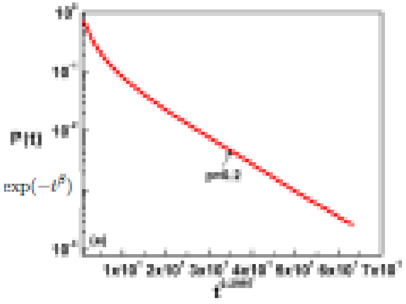
Two band chaos to fully developed chaos. Loss of coarse grained memory.

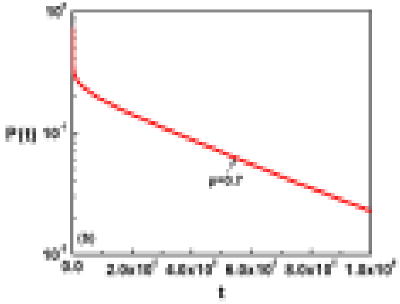
But we do not have power law decay.
Stretched Exponential ~ exp($-t^{\beta}$)

Border I p=0.1
and p=0.7





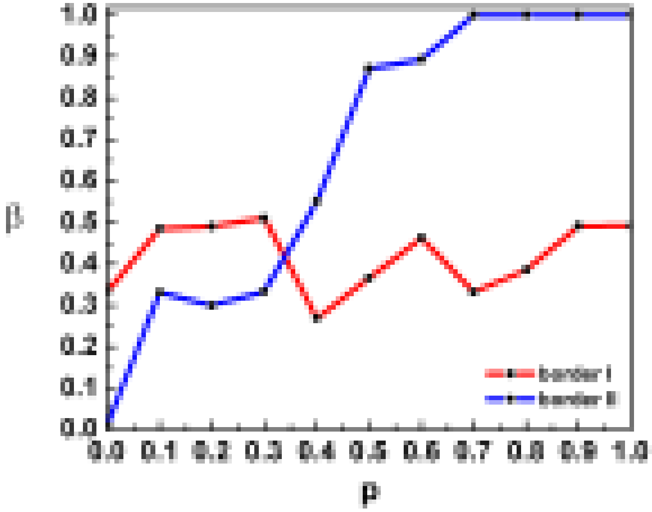




Border II p=0.1 and
p=0.7

The behavior is different at two borders

Exponent β is bounded by ½ on border 1. It becomes fully exponential on border II.



Exchange Frustration

With 1 neighbor on either side, we have antiferromagnetic long-range order.

(Persistence exponent has been found in it and is found to be 3/8.)

With 2 neighbors with antiferromagnetic interaction, frustration sets in.

Small world couplings do not even have bipartite nature. But locally it is antiferromagnetic.

Glasses are dynamically frustrated systems. They are known to have a stretched exponential relaxation. We have such system.

Maybe that is reason.

Persistence studies are like survival analysis in statistics.

How long a bulb, a car, a switch, a human being will survive.

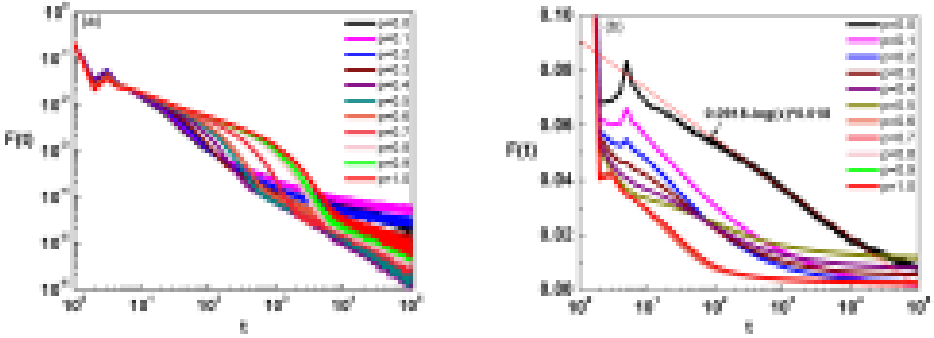
We need good fitting functions. (So that we can find fair warranty, insurance etc.)

One of most popular fitting function is Weibull distribution. Stretched exponential is its complementary cumulative distribution.

When mortality rate is nonstationary, decreases in time and there are multiple causes of failure, it is expected.

Multiple overlapping time-scales.

Let us find mortality rate, i.e. rate of switching.



Rate of switching decays power-law on border I.

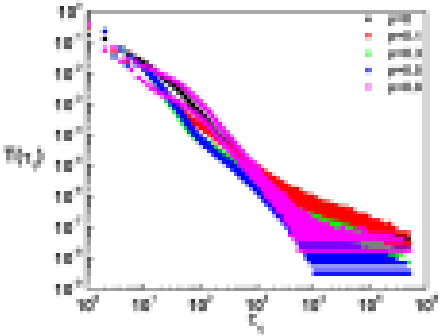
It shows logarithmic decay followed by saturation on border II.

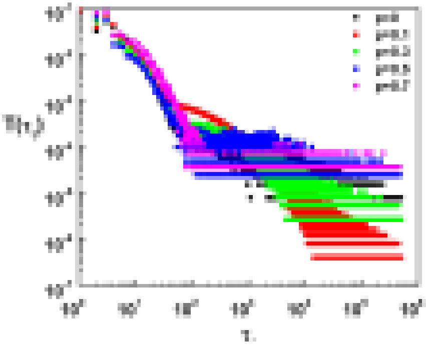
Constant rate leads to exponential.

Trap times:

How long the site remains stuck before flipping. More like return time. Usually return time statistics is different from first passage time statistics.

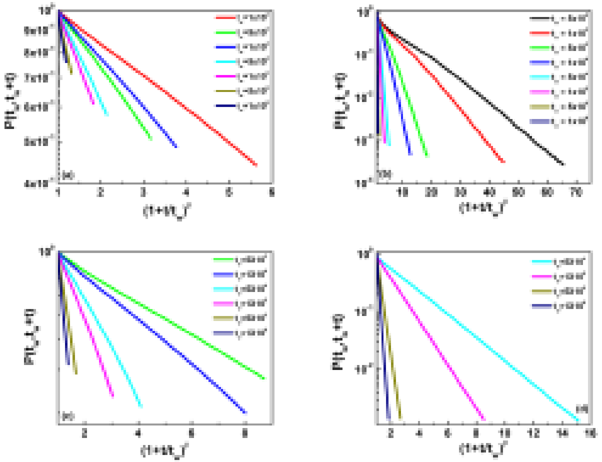
Power-law on border II. Very long lived traps on border I.





Glassy Dynamics

If we wait for long time and then compute persistence, it is still stretched exponential.

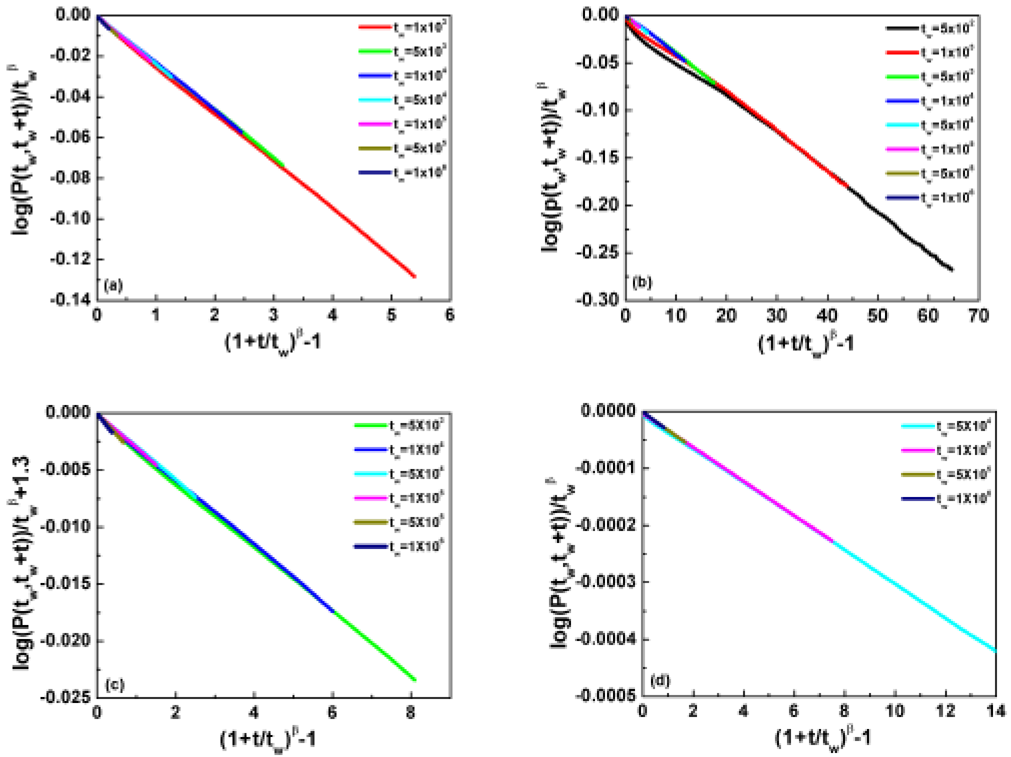


$$P(t_w, t_w + t) = \frac{\exp(-r(t + t_w)^\beta)}{\exp(-r t_w^\beta)}$$

$$P(t_w, t_w + t) = -\exp(r t_w^\beta)\exp((1 + t/t_w)^\beta - 1)$$

$$\frac{\log(P(t_w, t_w + t))}{t_w^\beta} = -r((1 + t/t_w)^\beta - 1)$$

Waiting-time Scaling: Like glasses f (t/t_w)

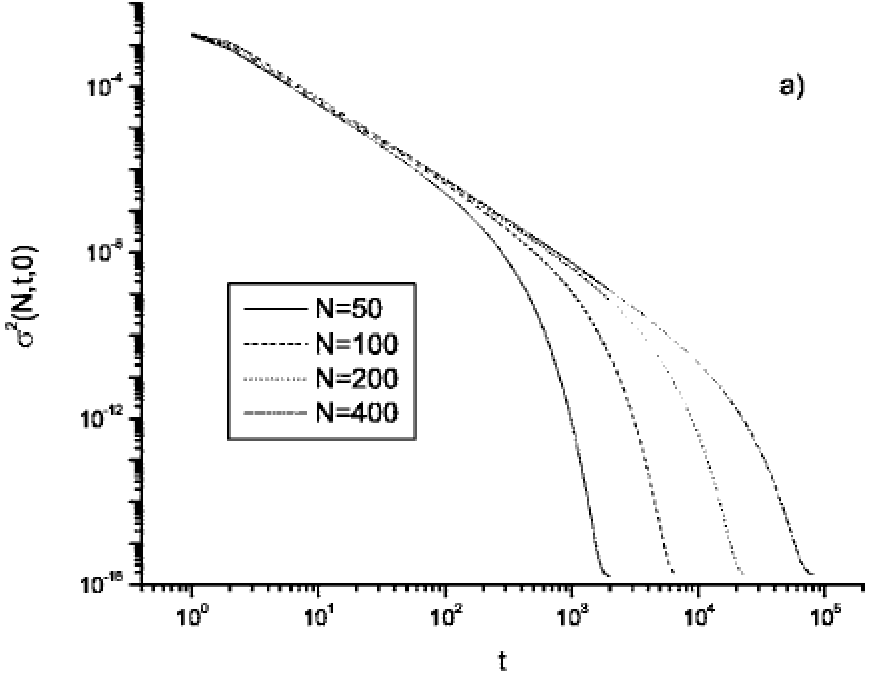


Local Global Coupling

It shows chaotic synchronization.

Order parameter decays like mean field, exponents can be defined etc.

Interesting thing is persistence can be defined in this system without apparent periodicity.



Majority persistence

Say there are 700 sites above average and 300 below average.

We do not know what they should do. We take a clue from social sciences and proclaim that whatever majority does is right. Site can flip as many times as it likes, it is persistent if it does what majority does.

Say 300 of 700 went below average and 400 went above average, 400 have persisted.

If 200 of 300 went below average and 100 went above average, 200 persisted.

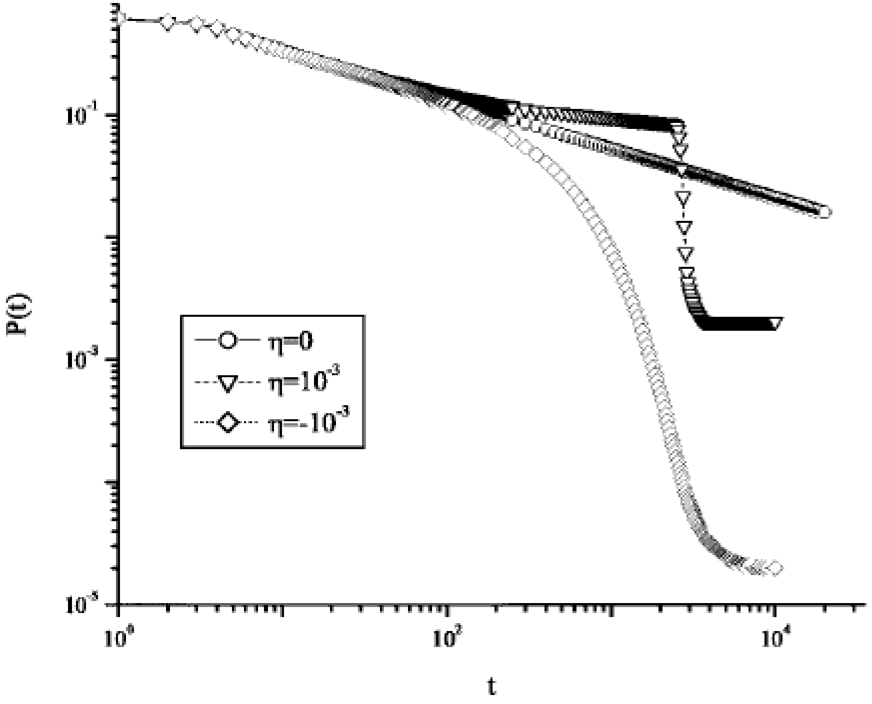
Now we form further groups among persisted sites.

It shows power law at critical point!

This happens for logistic as well as tent map.

The exponents are different though.

This definition has not been extended to chaotic synchronization in other complex networks where synchronization occurs.



Phase Transitions on Small-world

Are they in mean-field universality class.

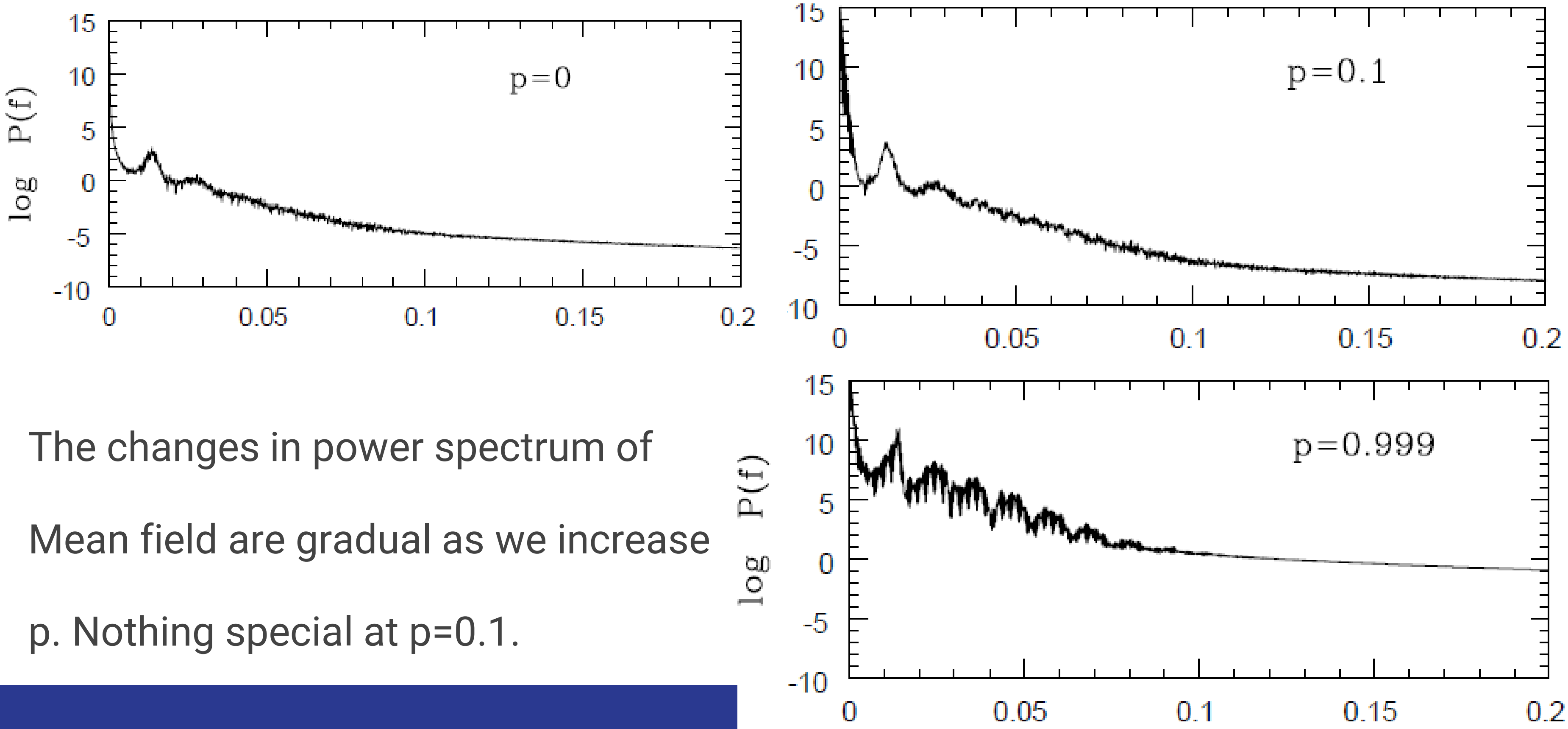
In equilibrium systems, there is no doubt.

They are always in mean field universality class. For any nonzero p. The characteristic distance scales logarithmically. So infinite dimensional system. (Small p – bigger lattice is necessary.)

In nonequilibrium systems, there are several possibilities.

a) In general, mean field behavior will not set in for infinitesimal or even small p.
b) There is a possibility that mean field behavior will set in as network becomes fully nonlocal.
c) Time-scales matter.
d) There is a curious case in which we have equilibrium behavior for infinitesimal p.

Coupled Hindmarsh-Rose Neurons



The changes in power spectrum of Mean field are gradual as we increase p. Nothing special at p=0.1.

SIR Model

0 susceptible

1-4 Infectious

5-13 Refractory

Put this dynamics on a lattice. Now add short-cuts.

The finding was that there is a transition to oscillatory state for nonzero p.
This critical point does not go to zero for infinite size.
[Phys. Rev. Lett. 86, 2909 (2001)]

We change dynamics

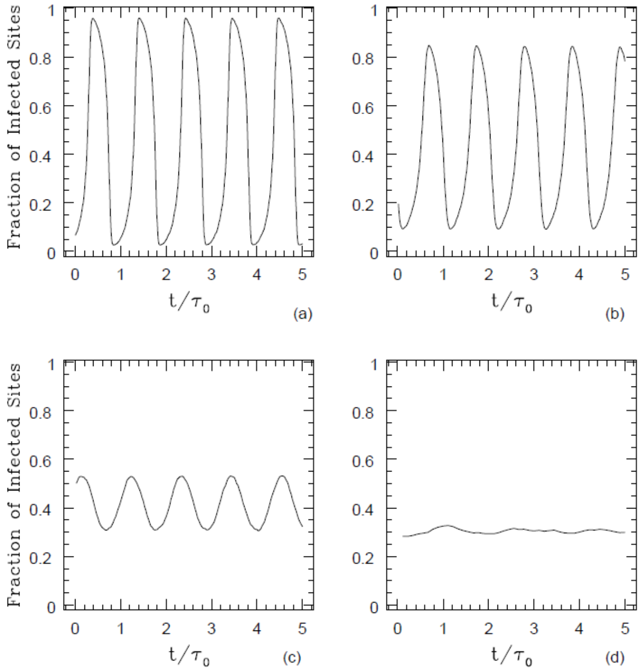
In stead of 4 and 13, let us take

16 and 52, or

32 and 104, or

64 and 208

For p=0.1, the oscillations become apparent for slow, adiabatic driving.

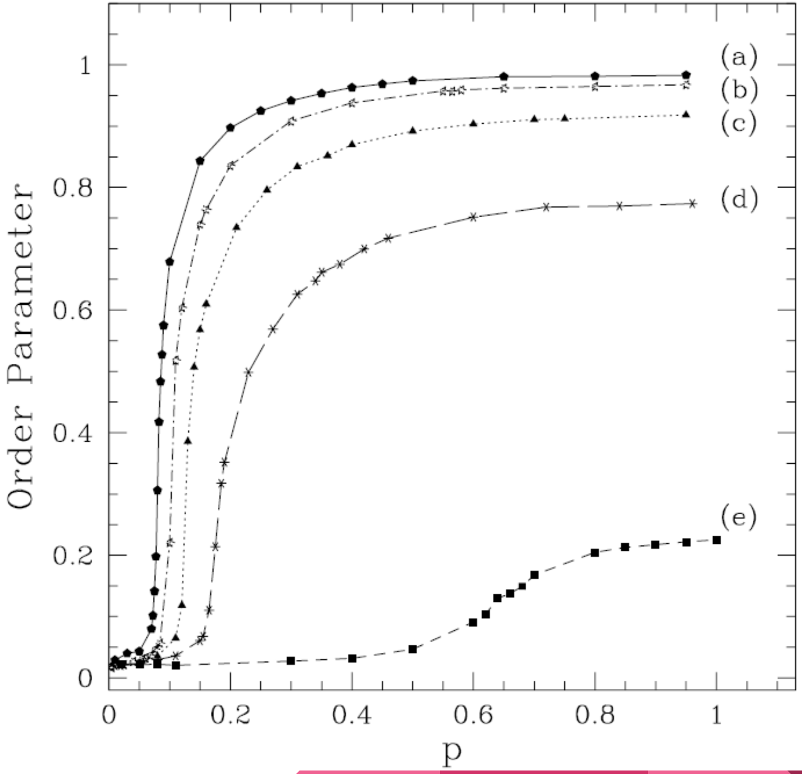


Order parameter

The transition point approaches zero

as we drive slowly.

So topology is not everything. Different dynamical systems will have different transitions on same topology.



Domany-Kinzel Automata on SW networks

If neighbors are 1 and 0, site becomes 1 with prob. p1

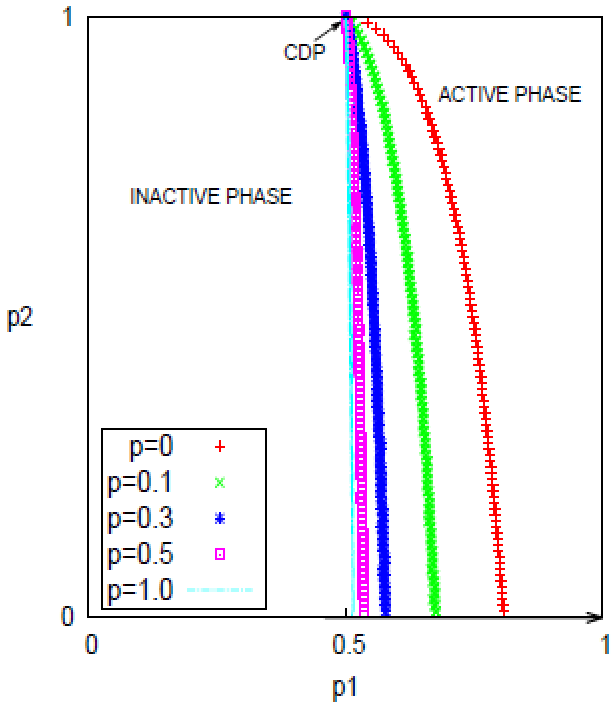
If both neighbors are 1, site become 1 with prob. p2

p2=1 is special class called compact DP.

(It has 2 absorbing states)

As p increases, critical line tends to p1=1/2

For p2=1, critical value is always p1=1/2



For compact directed percolation: Mean field

For any value of p, the order parameter decays as 1/t.

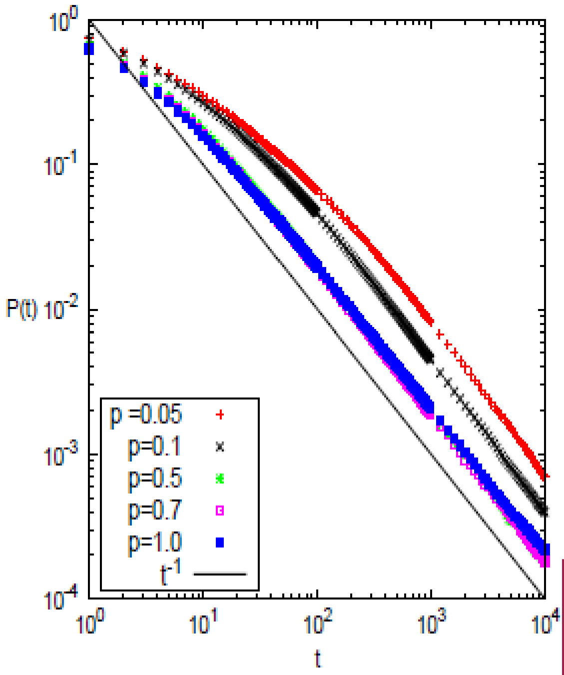
The underlying topology is not even bidirectional.

Mean field sets in, as soon as some fraction of

nonlocal connections are created.

In 1-d, equivalent to Glauber Ising model.

But on small world, no clear mapping.

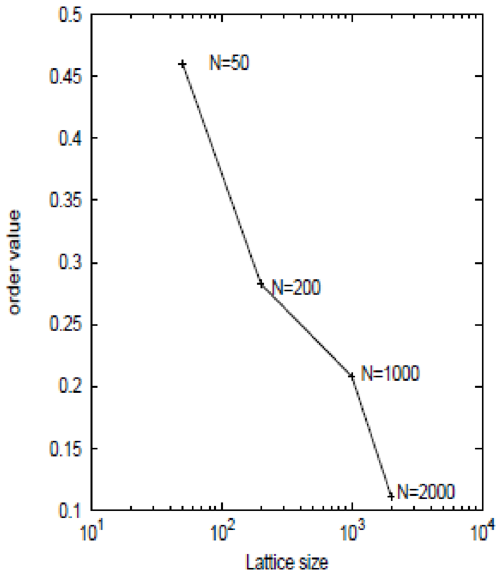


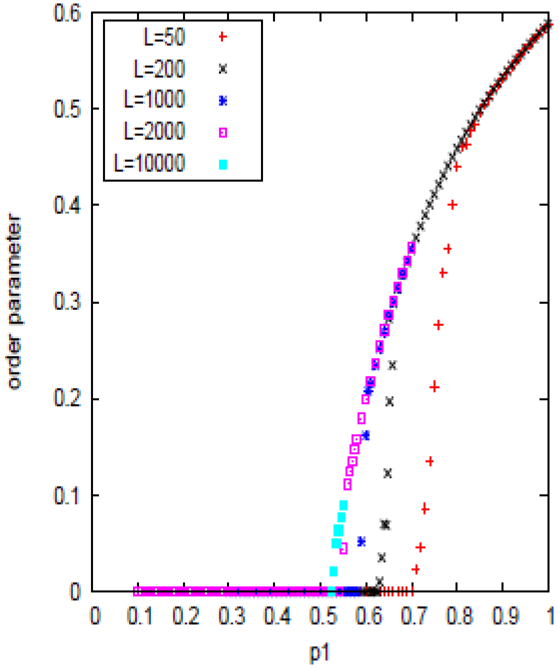
For DP, is it first order?

Order parameter shows jump at critical point.

But jump goes to zero for infinite lattice.

Second order.

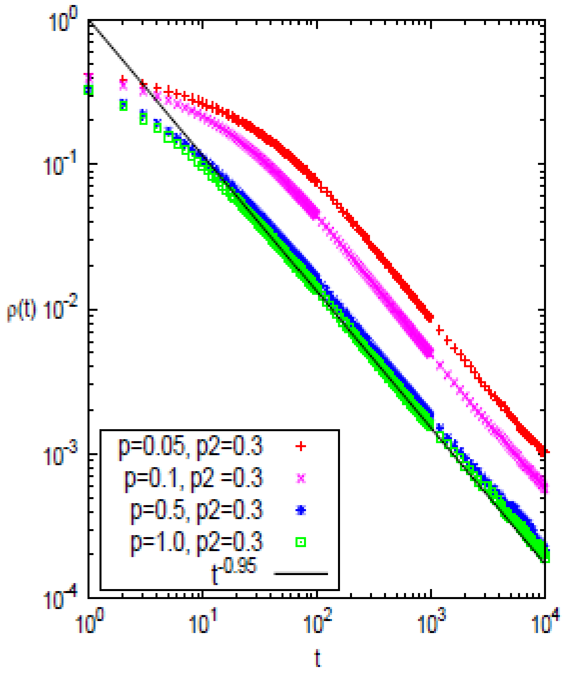




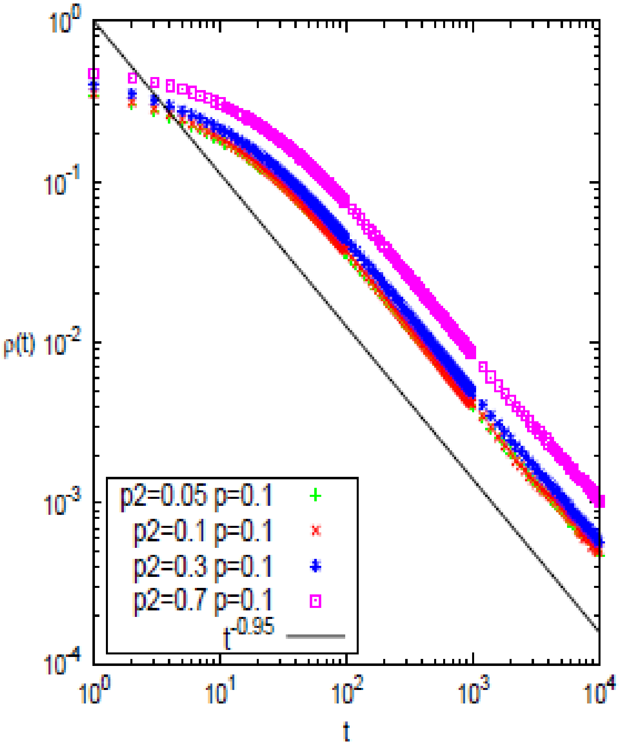
The critical exponent is not 1.

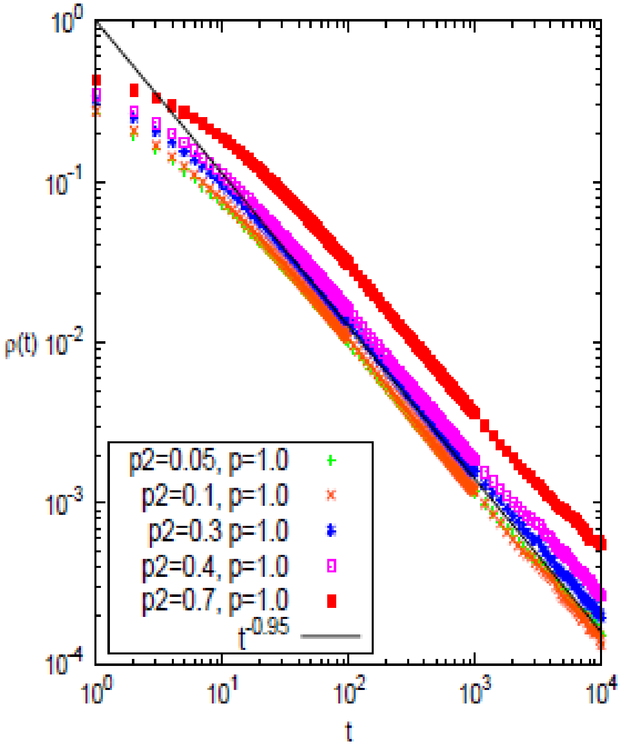
From p=0.05 to 1, we have same exponent.

It is 0.95



Same behavior for different values of p2





So SW had different universality class (!?)

Of course number of connections matter. More connections, faster will be approach to mean field. No references saying so. But we have seen that.

Complex interplay of topology and dynamics.

These could be entirely different phase transitions.

For absorbing phase transitions/ chimera.

Universality/ At times

Universality-class using short time dynamics (persistence).

Transition to absorbing or partially absorbing states in nonequilibrium systems can be studied using persistence.

Persistence has to be carefully defined, modulo-2, majority, whatever…

Conditions for transitions on small-world to be mean field need further studies.

Publications

Covered in this talk

AV Mahajan, PMG, JSTAT, 023212 (2018), A.V. Mahajan and PMG, PRE 81, 056211 (2010), AV Mahajan, MA Saif, PMG, EPJ (ST), 222 895 (2013), PMG and S. Sinha, *PRE* 72. 052903 (2005) PMG and C-K Hu, PRE 73, 036212 (2006), PMG and S. Sinha, IJBC, 16, 2767 (2006) ,

Related (CML)

B.L. Dutta and P.M. Gade, Chaos, 23, 033138 (2013), PMG and G.G. Sahasrabudhe PRE 87, 052905 (2013), PMG, D.V. Senthilkumar, S. Barve and S. Sinha PR E 75, 066208 (2007).

Related (Others)

M. A. Saif and PMG JSTAT P03016 (2010), M B Matte and PMG JSTAT 116203 (2016), A. R. Sonawane and PMG, Chaos, 21, 013122 (2011), N. Shambharkar and PMG (preprint)

Thank You!