Synchronization, multi-cluster oscillation death and chimera states in a network of oscillators: Role of initial conditions

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Outline

- Introduction
- Network of Stuart-Landau oscillators
- Numerical results and certain quantitative measures
- Mean field model
- Summary & conclusions

Introduction

- Oscillatory systems are ubiquitous in nature – reveal the processes underlying the development of several collective phenomena in complex systems of oscillatory units
- Examples include a variety of asymptotic states such as synchronization, oscillation death (OD), amplitude death (AD), chimera, traveling waves and so on
- OD corresponds to stabilization of newly created inhomogeneous steady states (IHSS) arising due to symmetry breaking of a homogeneous steady state (HSS)
- Amplitude death (AD) via Turing type pitchfork, saddle-node or transcritical bifurcations
- Chimera states are spatio-temporal patterns with coexisting coherent and incoherent groups
- A combined effect of chimera and OD states may lead to another emergent pattern known as the CD state [A. Zakharova, M. Kapeller, and E.Schöll, Phys. Rev. Lett **112**, 154101 (2014)]

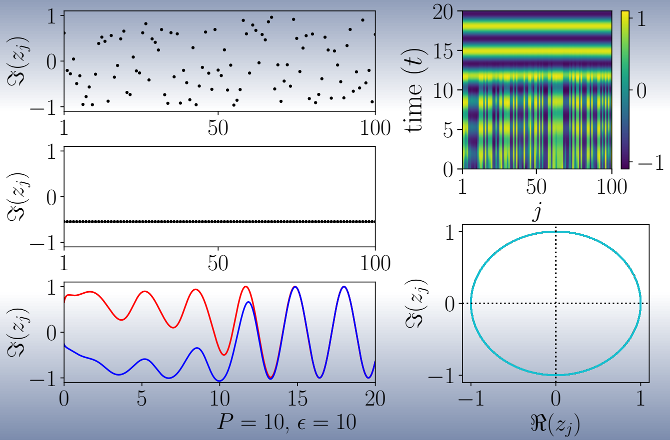
Network of Stuart-Landau oscillators

$$\dot{z}_j = (\lambda + i\omega - |z_j|^2)z_j + \frac{\epsilon}{2P}\sum_{k=j-P}^{j+P}\left[\Re(z_k) - \Re(z_j)\right], \; j = 1, 2, ..., N \qquad (1)$$

- $z \in \mathbb{C}$, $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{R}$, ϵ and P/N are the coupling parameters (P - number of nearest neighbours in each direction of the ring)
- the indices j and k are modulo N, $\Re(\ldots)$ denotes the real part, and $\epsilon > 0$ is the coupling strength
- breaking of rotational symmetry is necessary for the existence of nontrivial steady states.
- all the oscillators are identical and their parameters are fixed: eg. $\lambda = 1$, $\omega = 2$ and $N = 100$
- **Initial conditions:** Two initial anti-phase clusters of equal size, $n_1 = n_2 = n = N/2$: the oscillators $z_1, z_2, \ldots, z_n$ are set to values $(x_j, y_j) = (-1, +1)$ (upper branch) while the oscillators $z_{n+1}, z_{n+2}, \ldots, z_{2n}$ are set to values with $(x_j, y_j) = (+1, -1)$

Network of Stuart-Landau oscillators

- Parameters
  - $P = 10$
  - $\epsilon = 10$
- Initial conditions
  - random (-1,1)
- Time series
  - $\Im(z_{20})$
  - $\Im(z_{60})$
- Complete synchronization



Network of Stuart-Landau oscillators

- Parameters

  $P = 15$

  $\epsilon = 22$

- Initial conditions

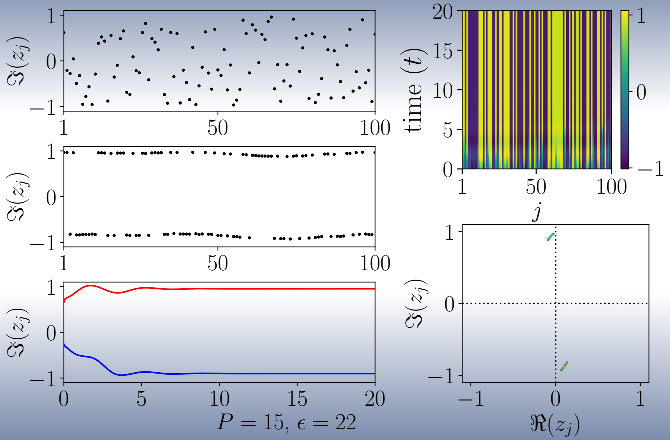
  random (-1,1)

- Time series

  $\Im(z_{20})$

  $\Im(z_{60})$

- Stable IHSS (chimera death)



Network of Stuart-Landau oscillators

- Parameters

  $P = 15$

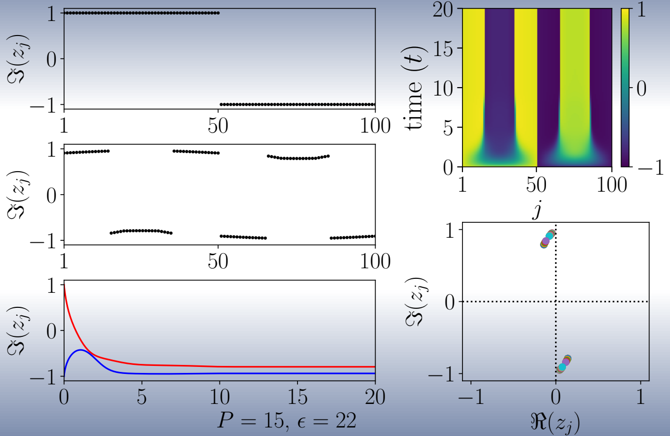
  $\epsilon = 22$

- Initial conditions
  - $-1 + i \; (j \leq 50)$
  - $+1 - i \; (j > 50)$
- Time series

  $\Im(z_{20})$

  $\Im(z_{60})$

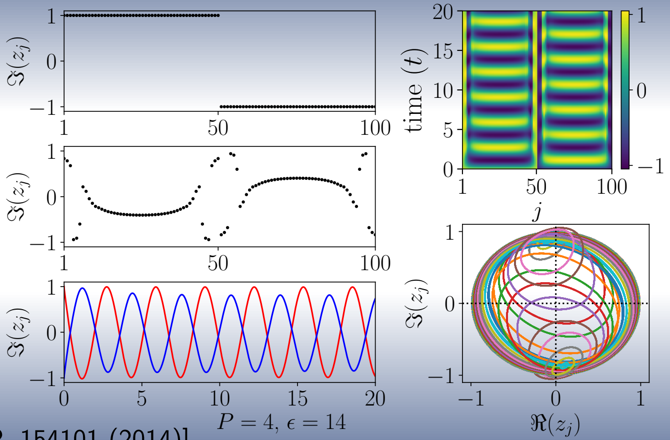
- 3-OD state



A. Zakharova, M. Kapeller, and E.Schöll, Phys. Rev. Lett 112, 154101 (2014)

Network of Stuart-Landau oscillators

- Parameters
  - $P = 4$
  - $\epsilon = 14$
- Initial conditions
  - $-1 + i \; (j \leq 50)$
  - $+1 - i \; (j > 50)$
- Time series
  - $\Im(z_{20})$
  - $\Im(z_{60})$
- Amplitude chimera [PRL **112**, 154101 (2014)]



Characterization of asymptotic states

- transform the original state variables $\{x_i, i = 1, 2, \ldots, N\}$ to new variables $w_i = x_i - x_{i+1}$
- when the i^{th} and $i+1^{th}$ oscillators are oscillating coherently, the value of w_i is minimum
- w_i takes a value between $\pm|x_{i,max} - x_{i,min}|$, where $x_{i,max}$ and $x_{i,min}$ are the upper and the lower bounds, respectively, of the allowed values of x_i
- **coherent state:** all the w_i's take a minimum value for all times, while in the case of an incoherent state, these w_i's get distributed between $\pm|x_{i,max} - x_{i,min}|$
- **chimera state:** some of the w_i's may take the same value while the others may be distributed over the above range

R. Gopal et al., Phys. Rev. E 89, 052914 (2014).

K. Premalatha et al., Phys. Rev. E 91, 052915 (2015).

Characterization of asymptotic states

- the standard deviation for the asymptotic state is calculated as

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(w_i - \langle w_i\rangle)^2}, \quad \text{where} \quad \langle w_i\rangle = \frac{1}{N}\sum_{i=1}^{N} w_i. \tag{2}$$

- $\sigma = 0$ for coherent states and $\sigma \neq 0$ for both incoherent and chimera states
- the chimera and incoherent states are distinguished by quantifying the distribution $\{w_i\}$
- this is done by dividing the oscillators into M number of even bins of equal length $n = N/M$ and a local standard deviation

$$\sigma(m) = \sqrt{\frac{1}{n}\sum_{j=n(m-1)+1}^{mn}[w_j - \langle w_j\rangle]^2}, \quad m = 1, 2, \ldots, M \tag{3}$$

Characterization of asymptotic states

- the strength of incoherence (SI) as,

$$SI = 1 - \frac{1}{M}\sum_{m=1}^{M} s_m, \quad s_m = \Theta(\delta - \sigma(m)), \tag{4}$$

  $\Theta(\delta - \sigma(m))$ is a Heaviside step function, and δ is predefined threshold
- when $\sigma(m)$ is less than δ, the values $s_m = 1$, otherwise $s_m = 0$
- the strength of incoherence takes values $SI = 1$ or $SI = 0$ or $0 < SI < 1$ for incoherent, coherent and chimera (or cluster) states, respectively

Characterization of asymptotic states

- a discontinuity measure (η) is calculated, based on the distribution of s_m, as

$$\eta = \frac{1}{2}\sum_{m=1}^{M}|s_m - s_{m+1}|, \quad (s_{M+1} = s_1) \tag{5}$$

- for chimera state, η takes value "1" and a positive integer value greater than "1" for multichimera states
- the measures SI and η are sufficient to characterize chimera, coherent or incoherent states (in the oscillatory regime)
- however, an additional measure from the time series of the original state variables x_i is required to characterize oscillation death (OD) states

Characterization of asymptotic states

- D is defined as

$$D = \frac{1}{N}\sum_{i=1}^{N}\Theta\left(\sum_{l=1}^{T}|x_{t_l,i} - x_{t_{l-1},i}| - \delta_1\right)$$

  $x_{t_l,i}$, $x_{t_{l-1},i}$ are state variables of i-th oscillator at time t_l and it's previous time t_{l-1} with T sufficiently large
- measure the sum of the difference in the value of states variable at some consecutive steps
- If the system approaches steady state (no matter homogeneous or inhomogeneous) there will be no change in the state values and D will be "0" for a pre-defined positive threshold δ_1
- If there is in any sort of oscillation, then D has real value unity (i.e., "1") for proper choice of δ_1

Characterization of asymptotic states

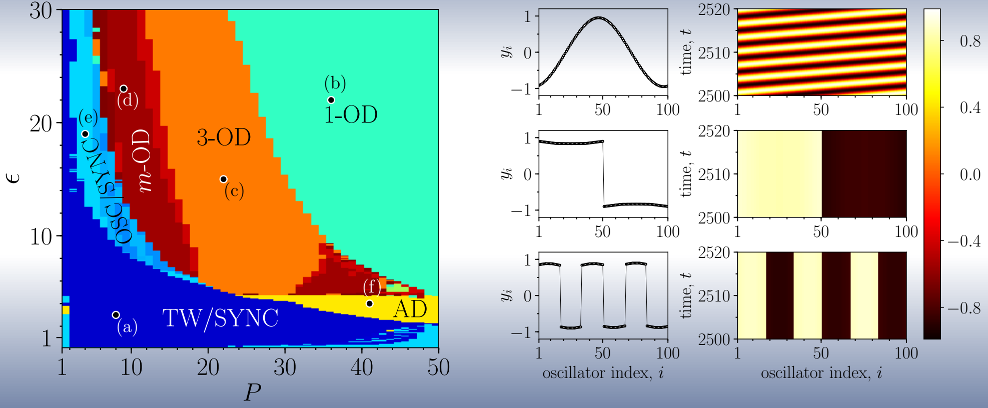
Table: Characterization of dynamical states

D	SI	S	DM (η)	Dynamical state
0	0	0	0	AD
0	$0 <$ SI < 1	0	1	1-OD
0	$0 <$ SI < 1	0	$\eta \geq 3$	η-OD
0	$0 <$ SI < 1	$0 <$ S < 1	2	1-CD
0	$0 <$ SI < 1	$0 <$ S < 1	$\eta \geq 6$	$(\frac{\eta}{2})$-CD
1	0	0	0	SYNC/TW
1	$0 <$ SI < 1	0	2	2-CLT
1	$0 <$ SI < 1	0	$\eta \geq 3$	η-CLT
1	$0 <$ SI < 1	$0 <$ S < 1	2	1-chimera
1	$0 <$ SI < 1	$0 <$ S < 1	$\eta \geq 3$	$(\eta - 1)$-chimera

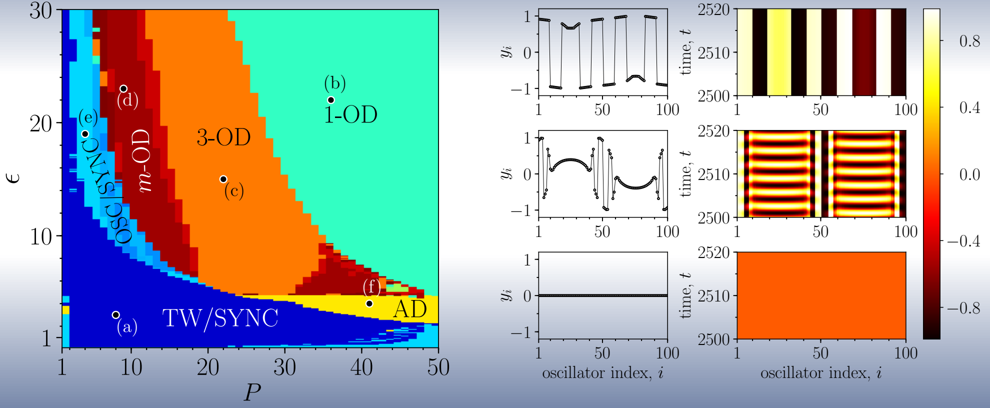
Characterization of asymptotic states

- If D is zero, then the final state is AD (or OD) whenever $SI = 0$ (or $0 < SI < 1$)
- First step is to check whether the value of D is zero or unity
- S – the modified strength of incoherence obtained by removing isolated points of discontinuities (if exists) in the values of the array w_i
- $S = 0$ implies that there exists few clusters of the asymptotic states, the number of such clusters can be determined from the value of DM
- If DM $= 1$, then the network is in 1-OD state. DM $= \eta \geq 3$ corresponds to the state of η-OD
- whenever $0 < SI < 1$ together with $0 < S < 1$ and DM $= \eta = 2$, the coupled system is in 1-CD state (for which there must be some non-isolated points of discontinuities in the array of w_i that can't be removed)

Network of Stuart-Landau oscillators



Network of Stuart-Landau oscillators

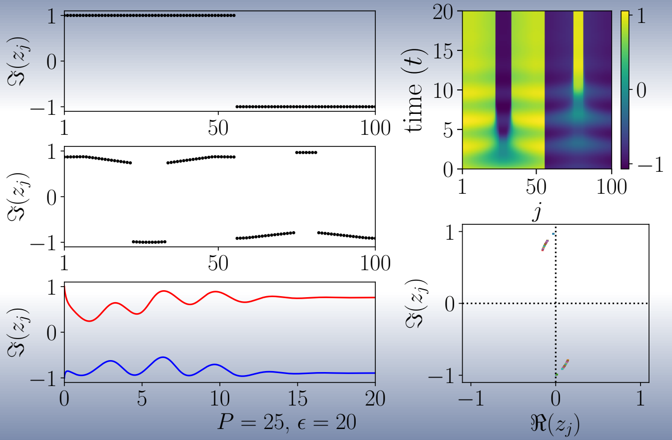


Network of Stuart-Landau oscillators

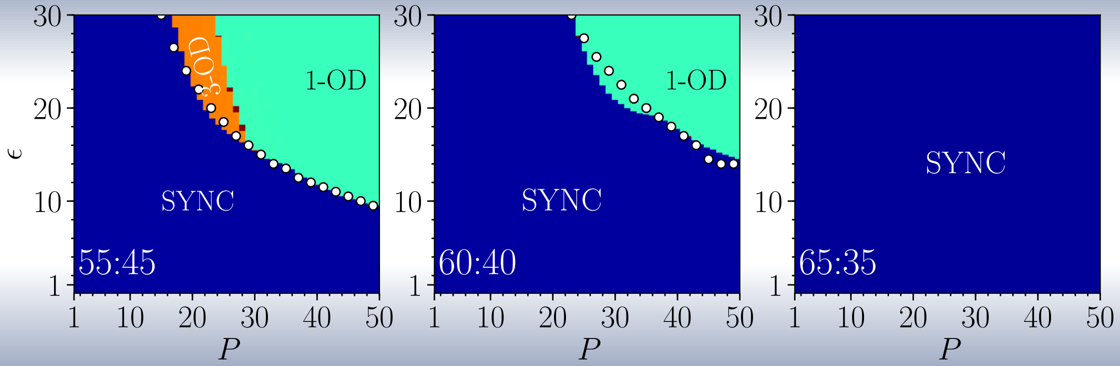
- For a range of weak coupling, the system emerges into a state of traveling wave (TW) or synchronization (SYNC) for almost all the coupling range
- This dynamical behavior persists for stronger interaction too, but only in the small coupling range
- In a narrower strip above the TW/SYNC region, chimera-like desynchronized oscillatory states (OSC) and SYNC coexist
- In the parameter plane above the OSC/SYNC region, the system settles to the multicluster-OD ($m(\geq 5)$-OD) states
- The oscillators populate newly created IHSS states, leading to the formation of several spatially distributed alternate (antiphase) clusters
- An increasing coupling range reduces the number of clusters resulting in a dominant 1-OD region although there exist small patches of m-OD states and an AD regime

Network of Stuart-Landau oscillators

- influence of asymmetry in initial clusters' size on the asymptotic states of the network, $n_1 \neq n_2$
- for instance, consider $n_1 = 55$, $n_2 = 45$ where $\Delta n = |n_1 - n_2| = 10$)
- an outright change in the phase diagram is seen; the regions of OD (1-, 3-), particularly, got shrunken
- a large region of CS or SYNC state emerges that engulfs the TW and m-OD and other regions of the complex dynamics

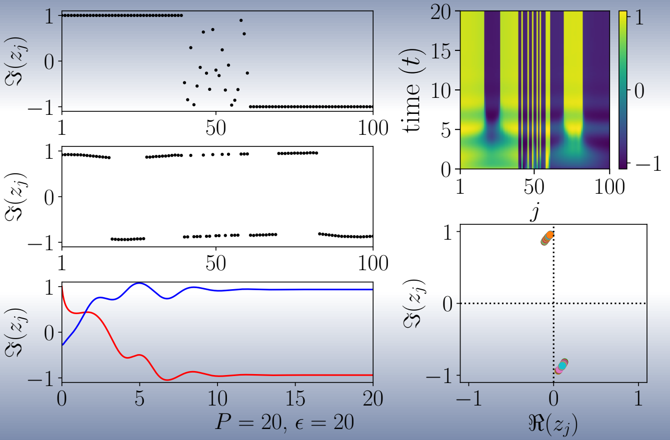


Network of Stuart-Landau oscillators

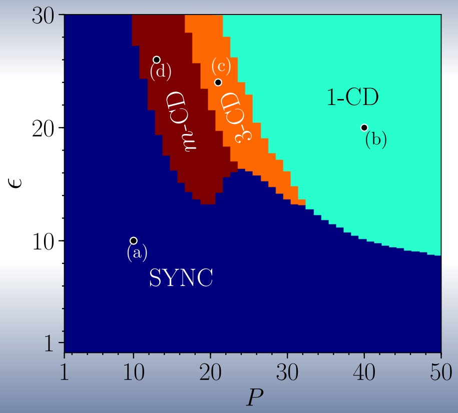


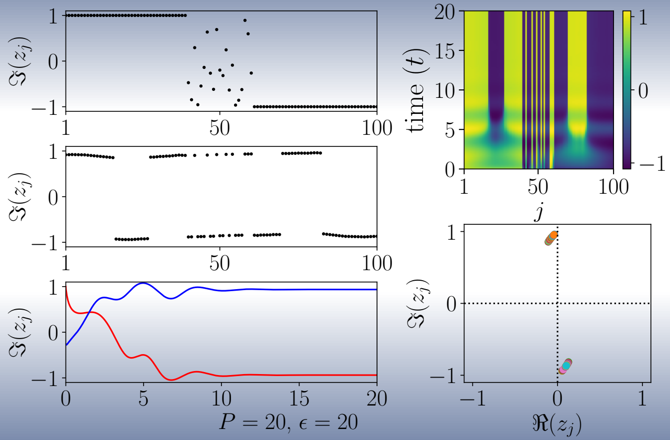
Network of Stuart-Landau oscillators

- initial clusters while assimilating an oscillator crew with random initial states – start with initial antiphase clusters of size $\tilde{n} = N/2 - 10$
- the oscillators $z_1, z_2, \ldots, z_{\tilde{n}}$ of the first cluster are set to $(x_j, y_j) = (-1, +1)$
- the oscillators $z_{\frac{3\tilde{n}}{2}+1}, z_{\frac{3\tilde{n}}{2}+2}, \ldots, z_N$ are set to $(x_j, y_j) = (+1, -1)$
- the remaining intermediate oscillators $z_{\tilde{n}+1}, \ldots, z_{\frac{3\tilde{n}}{2}}$ are assigned uniformly distributed random initial states $\in (-1, +1)$



Network of Stuart-Landau oscillators





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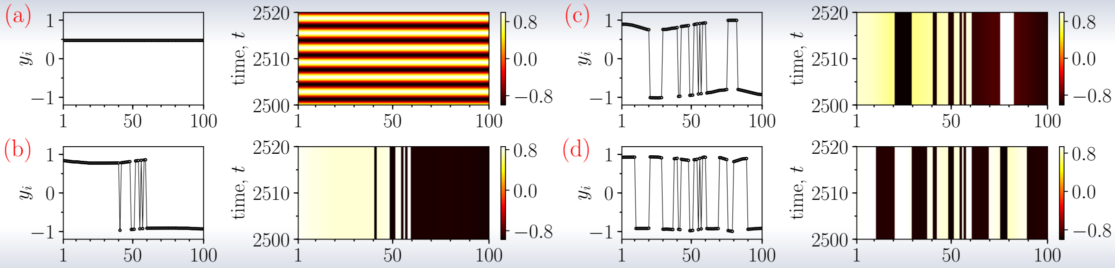

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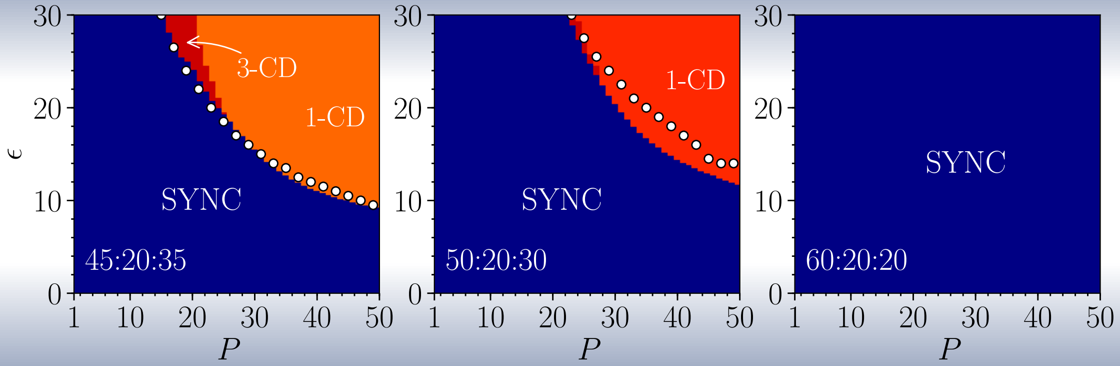
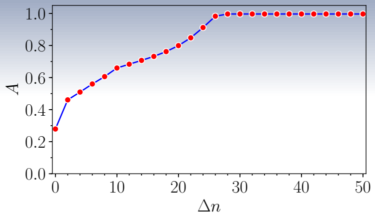
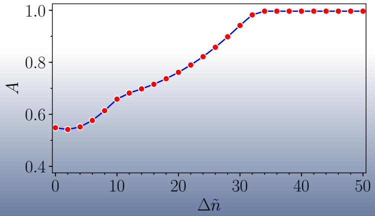


Normalized synchronization area

- sufficient number of points T from the $\epsilon - P$ parameter plane and identify the number of points Q that lead to complete synchrony of the system for each and every value of $\Delta n \in [0,\ 50]$
- A is defined to be the fraction Q/T
- $A \sim 1$ corresponds to the situation of complete synchrony for almost all the possible values of ϵ and P
- $A \simeq 0$ indicates the absence of synchrony for any ϵ and P
- $T = 300 \times 50 = 15000$





Mean field model

- Mean filed approximation to reduce the network of oscillators
- Assumption: the oscillators in both the initial clusters possess the same phase and amplitude whatever be their sizes (or the value of Δn)
- the intra-cluster interaction terms vanish and leave us only the inter-cluster interactions
- the oscillators on the upper branch with indices j $(j = 1, 2, \ldots, n_1)$ and the lower branch with indices k $(k = 1, 2, \ldots, n_2)$ interact with P neighbours on both sides of the ring
- as long as $P < 2n_1$ and $P < 2n_2$, a maximum of $P - j + 1$ oscillators' interaction are relevant for the j-th oscillator on the n_1-cluster as the first $j - 1$ interactions are from the same cluster
- depending on the value of P, this number of relevant interactions can be zero or even be n_1

Mean field model

- the number of relevant interactions for j-th oscillator is

$$\max\{\min\{n_1, P-j+1\}, 0\}, \quad j = 1, 2, ..., n_1. \tag{6}$$

- with similar arguments, the number of relevant oscillators interacting with the k-th oscillator on the n_2-cluster will be

$$\max\{\min\{n_2, P-k+1\}, 0\}, \quad k = 1, 2, ..., n_2. \tag{7}$$

- the mean relevant interaction of the whole n_1- cluster is obtained by summing up the above values for each oscillator and dividing by the product Pn_1 of the number of neighbours P on each side and the number of oscillators n_1 in the cluster

$$I(P, n_1) = \frac{1}{Pn_1}\sum_{j=1}^{n_1}\max\{\min\{n_1, P-j+1\}, 0\}. \tag{8}$$

Mean field model

- in a similar fashion, the mean relevant interaction of the n_2- cluster is given as

$$I(P,n_2) = \frac{1}{Pn_2}\sum_{k=1}^{n_2} \max\{\min\{n_2, P-k+1\}, 0\}. \tag{9}$$

- helps to reduce the network of N oscillators to a system of two coupled oscillators with mean relevant interaction strengths $I(P,n_1)\epsilon$ and $I(P,n_2)\epsilon$ for n_1- and n_2- clusters:

$$\dot{z}_1 = (\lambda + i\omega - |z_1|^2)z_1 + \epsilon I(P,n_1)\left[\Re(z_2) - \Re(z_1)\right], \tag{10a}$$

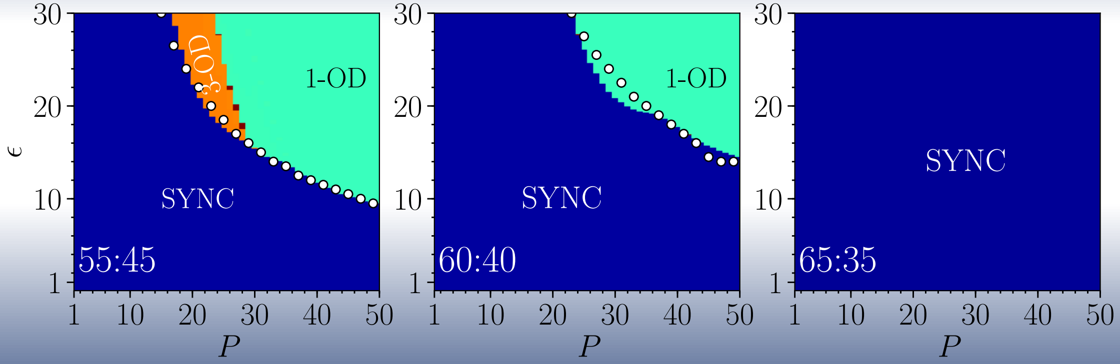
$$\dot{z}_2 = (\lambda + i\omega - |z_2|^2)z_2 + \epsilon I(P,n_2)\left[\Re(z_1) - \Re(z_2)\right]. \tag{10b}$$

- synchronization error $E = \left\langle \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \right\rangle_t$

  where $z_1 \equiv (x_1, y_1)$, $z_2 \equiv (x_2, y_2)$ and $\langle \cdots \rangle_t$ is the time average

Mean field model

- using the mean field model the boundary of CS in the parameter plane is plotted in white circles it falls almost exactly on the separating line of SYNC



Summary & Conclusions

- Studied the behaviour of a network of Stuart-Landau oscillators with symmetry breaking coupling by playing with the choice of initial conditions
- These network of oscillators exhibits a variety of asymptotic states: Synchronization, traveling waves, multicluster oscillation death, amplitude death, and chimera death
- Certain quantitative measures – to identify different asymptotic states
- Increasing asymmetry in the initial cluster size favours complete synchronization state for a broad range of coupling parameters
- The network model can also be reduced using the mean-field approximation that reproduces the dynamical features of the original network

S. Majhi et al, Asymmetry in initial cluster size favors symmetry in a network of oscillators, Chaos (2018) Accepted.