Optimized Evolution of Networks

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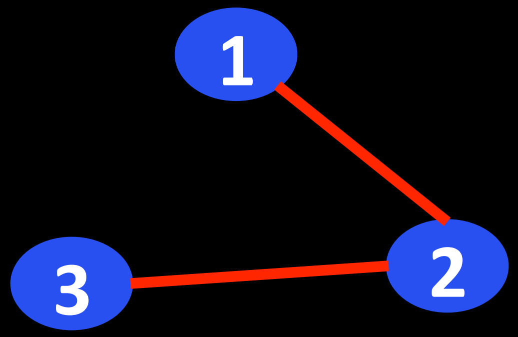
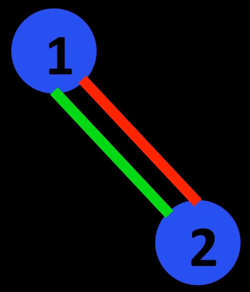
Network’s Spectra + Optimization + Multiplex Networks

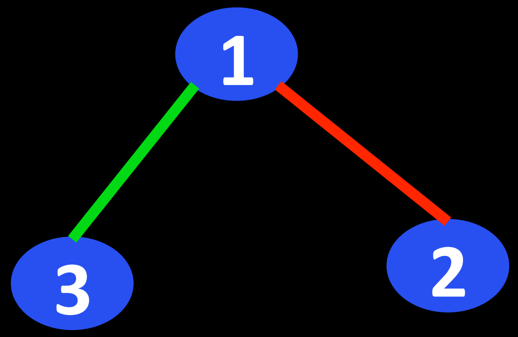
Regulating properties of one network (layer) by appropriate “multiplexing”

Single layer Networks:
one type of interactions



There may exist more than one type of interaction among the same units

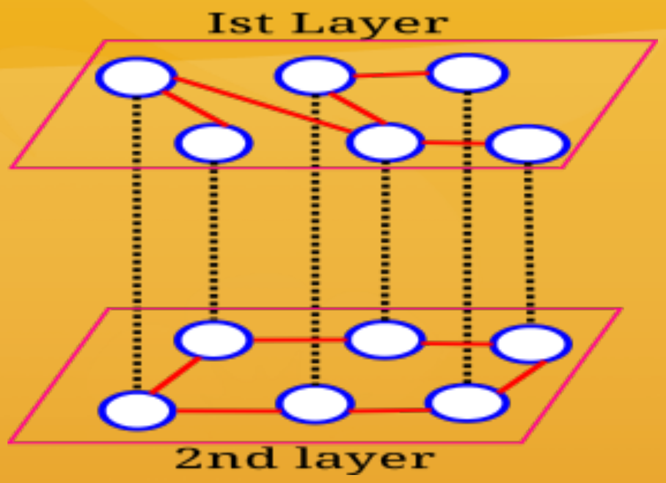




Multiplex Networks

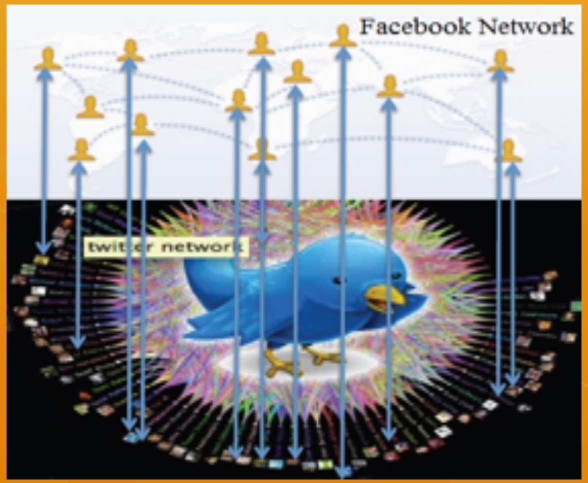
Multiplex or Multilayer networks

A multiplex network is a set of N nodes interacting in m layers, each reflecting a distinct type of interaction among the same units



- ✓ In human brain, different regions can be seen connected by functional and structural neural networks [Bullmore, and Sporns, Nat. Rev. Neurosci (2010)]

- ✓ Transport network: Different layers can be Air, train and bus transportation networks [Boccaletti et al. Phys. Rep. (2014)]



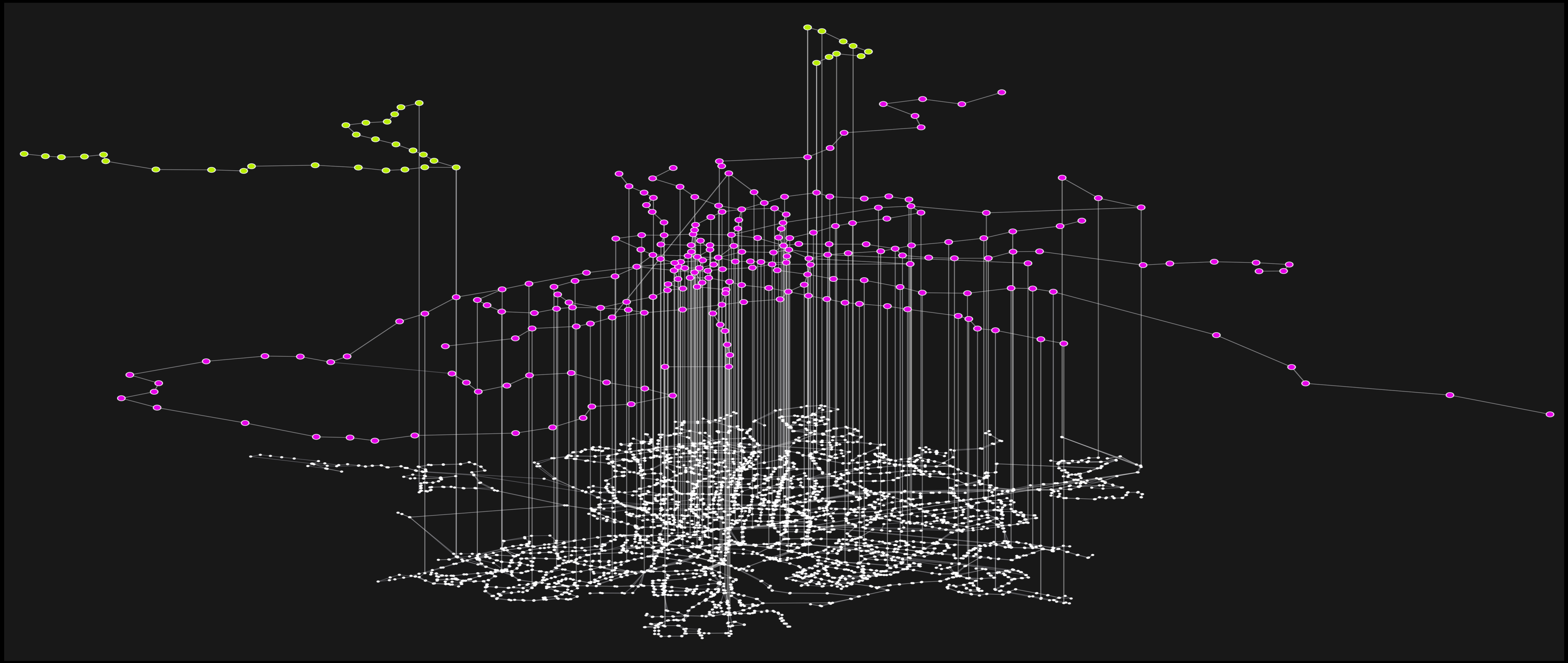
- ✓ In social networks people may be connected because of belonging to the same family, being friend or work [Camellia Sarkar, Alok Yadav and SJ, EPL (2016a)]

Transport system as Network: Single layer representation

Gray lines represent passenger flows along direct connections between 4069 airports worldwide. Geographic regions are distinguished by color

The Hidden Geometry of Complex, Network-Driven Contagion Phenomena, Brockmann and Helbing. Science 2013

Multilayer transportation system

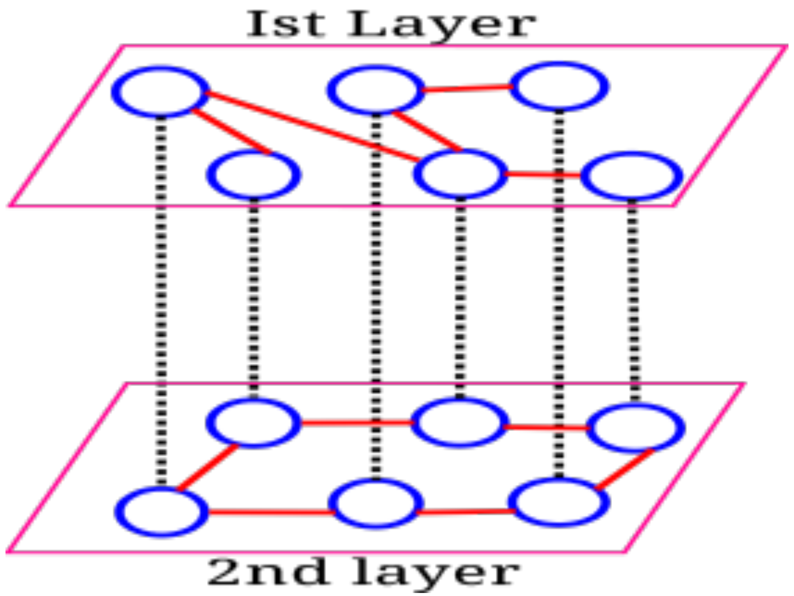


Madrid multilayer transportation system; tram (yellow nodes), metro (purple nodes) and buses (white nodes)

Aleta, Meloni, and Moreno. A Multilayer perspective for the analysis of urbantransportation systems, 2017

➢Multiplex framework provides understanding to various dynamical features of underlying real-world systems which are beyond the limit of *single network incorporating only one type of coupling behavior*

Ignoring impact of multiplexity (Layer 2)
may result in wrong prediction for the behavior of a system (Layer 1)



A strike of the bus service may result in overloading the rail and air traffic routes

Multiplexing and Synchronization. Dwivedi, Baptista and SJ, PRE 2017

We can control properties of the entire system (multilayer network) by tuning structural properties of only one *accessible* layer

We focus on:

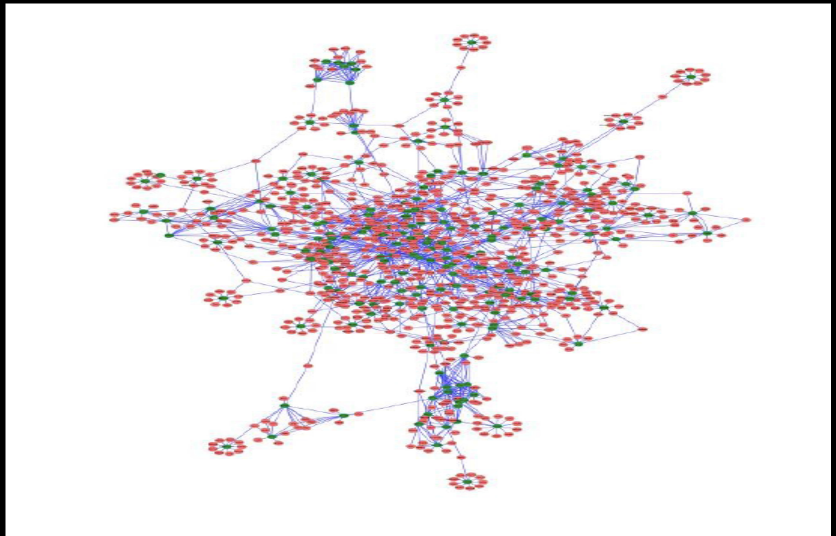
Eigenvector localization of complex networks

Localization properties of eigenvectors have diverse applications ranging from detection of influential nodes to disease-spreading phenomena in underlying networks

Principle Eigenvector (PEV)

- PEV is useful for getting insight into the propagation or localization of perturbation in the underlying systems (Golstev et. al. PRL 2012)
- Provides insight to disease spreading, for instance using SIS model (Satorras and Castellano, Sci. Rep. 2016)
- PEV localization provides insight into the propagation of perturbation in mutualistic ecological networks (Suweis et al., Nat. Commun. 2015)
- In the analysis the existence of rare-regions in brain networks study (Moretti and Munoz, Nat. Commun. 2013)
- To enhance the efficiency of Google matrix (Ermann, Frahm and Shepelyansky, Rev. Mod. Phys. 2015)
- Eigenvector localization has been used to detect communities in multilayer and temporal networks (Taylor, Caceres and Mucha, arxiv:1609.04376)

Adjacency matrix of single layer networks



$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 1 & 1 & 0 & \dots \\ \dots & 1 & 0 & 1 & 1 & \dots \\ \dots & 1 & 0 & 1 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

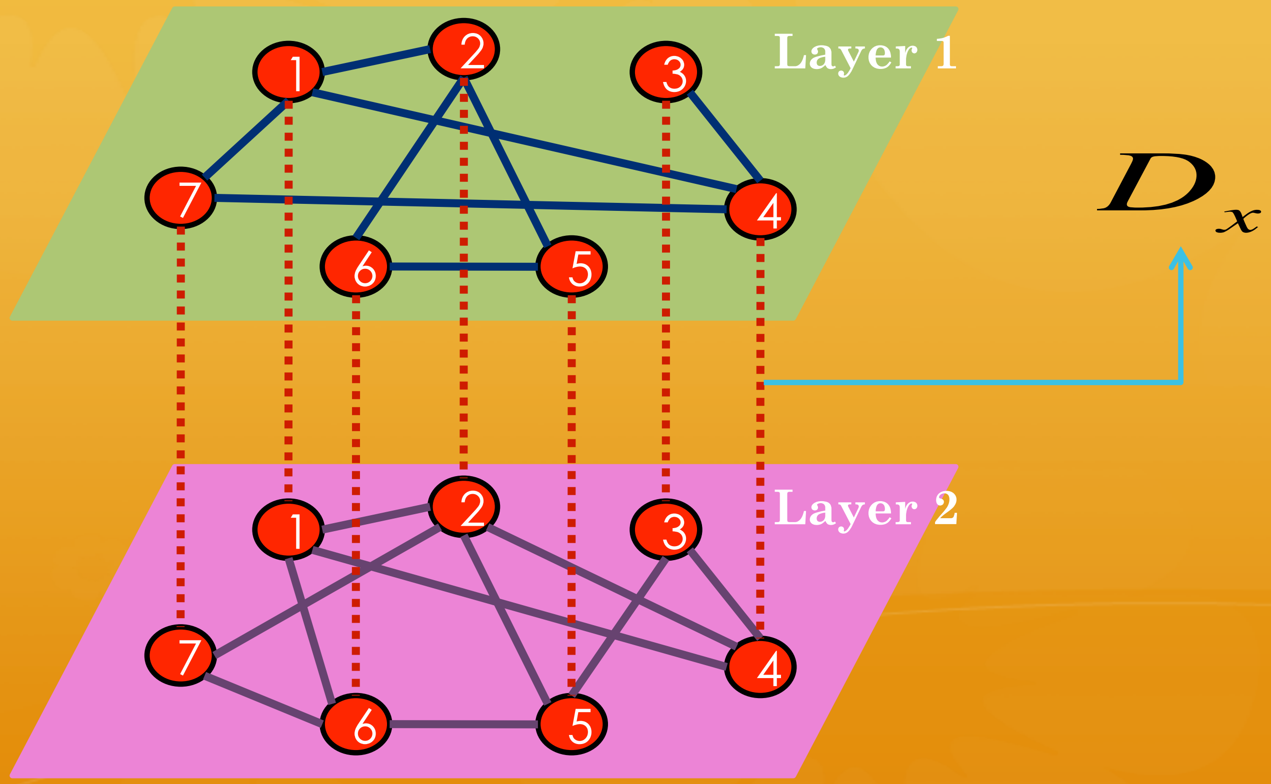
Eigenvalues of the underlying Adjacency matrix are called the spectra of network

$$\{\lambda_i\},\ \ i = 1, \dots, N$$

Corresponding eigenvectors are:

$$X_1, X_2, \cdots, X_N$$

Adjacency matrix of Multi-layer networks



$$\begin{pmatrix} A_1 & D_x I \\ D_x I & E_y A_2 \end{pmatrix}$$

D_x : strength to which activities of layer 1 affects layer 2

E_y : relative coupling strength of the layers

The localization of an eigenvector:

- Few components of a vector take very high values
- Rest of the components take very small values

Measure of Eigenvector Localization:

$$Y_{X_k} = \sum_{i=1}^{N} x_i^4, \tag{1}$$

where x_i is the ith component of the normalized eigenvector, X_k with $k \in \{1,2,\ldots,N\}$, in the Euclidean norm. A delocalized eigenvector with component $[1/\sqrt{N}, 1/\sqrt{N}, \ldots, 1/\sqrt{N}]$ has the IPR value $1/N$, whereas the most localized eigenvector with components $[1,0,\ldots,0]$ leads to an IPR value equal to 1. For a connected network, the IPR value lies between these two extreme values.

For given size (number of nodes *N*) and building cost (number of connections in the network *NC*), what network structure will correspond to the most localized PEV ??

Theoretical Framework:

- For give N (size of the system) and NC (building cost), if we can enumerate all the possible configurations, the network corresponding to the highest IPR value will be our desired one

- Enumerating all the network configurations for a given N and NC is computationally exhaustive

- We formulate this problem through an optimization technique

Our optimization aims at maximizing IPR

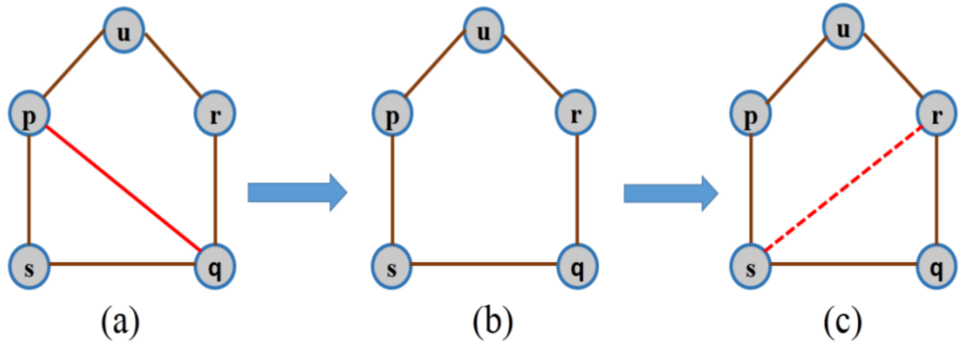
We construct a network which has a highly localized principal eigenvector using an optimization technique for the network evolution

Step1: Calculate the IPR value of principal eigenvector of matrix A

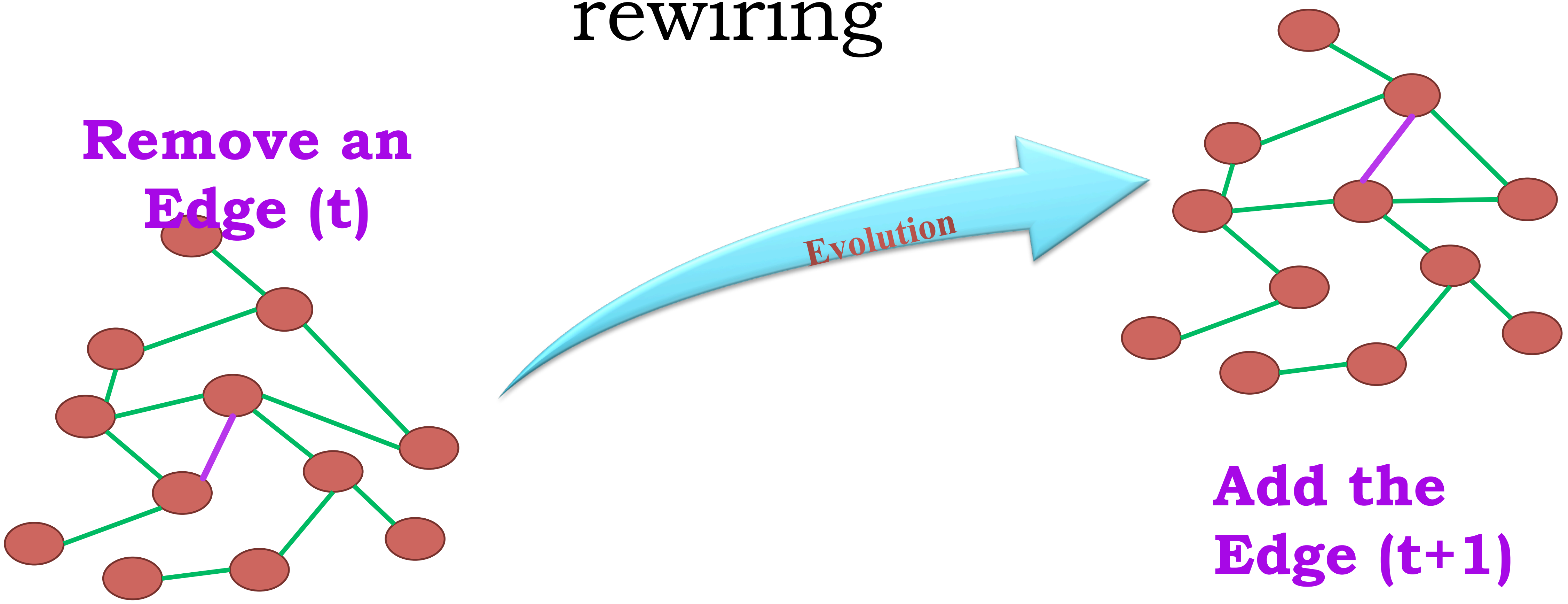
Step2: Rewire an edge uniformly and independently at random in G and denote the new network as G’ and the adjacency matrix as A‘

Step3: Calculate IPR of principal eigenvector of A‘

Step4: If IPR(X1') > IPR(X1), replace A with A‘

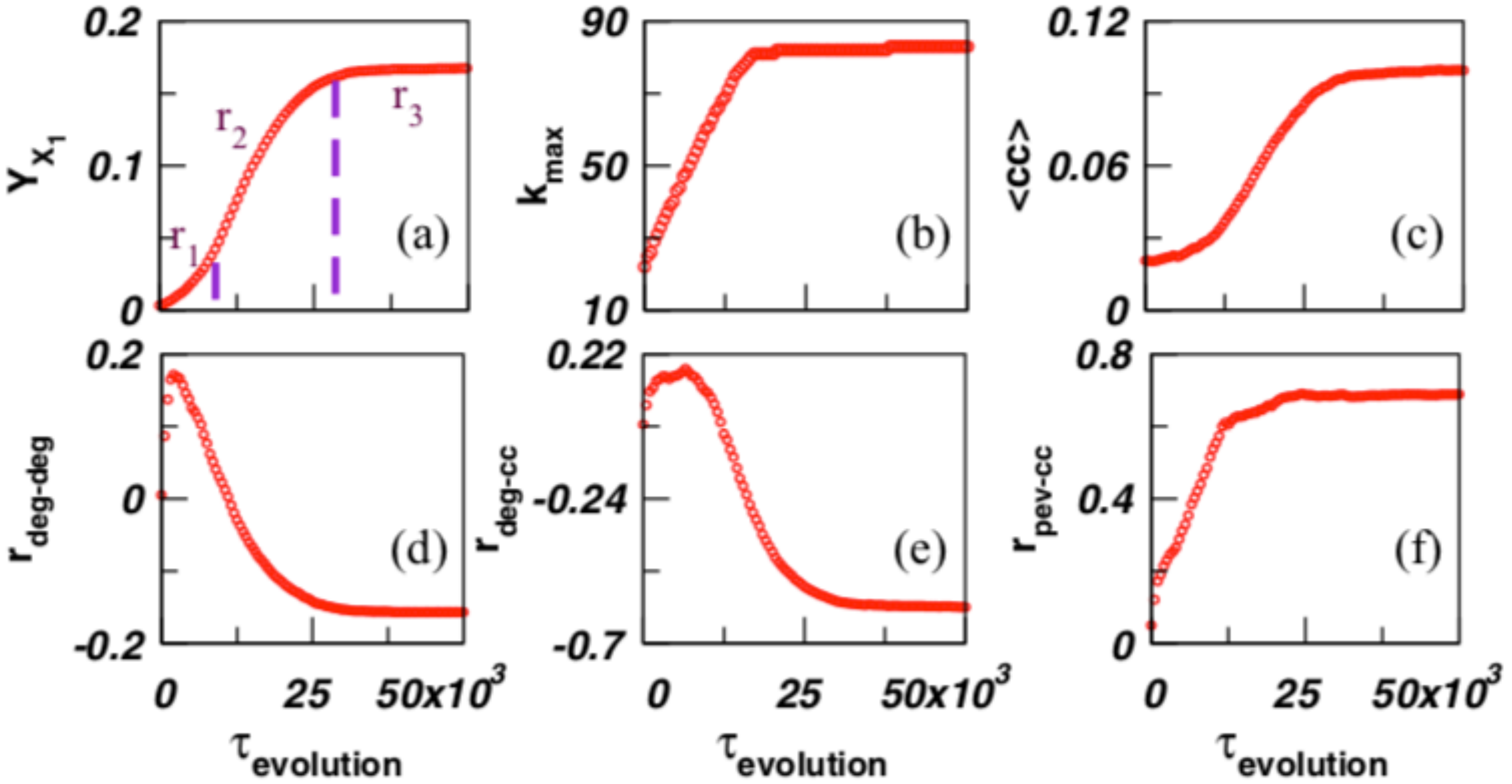


Evolution of a layer (network) by Edge rewiring



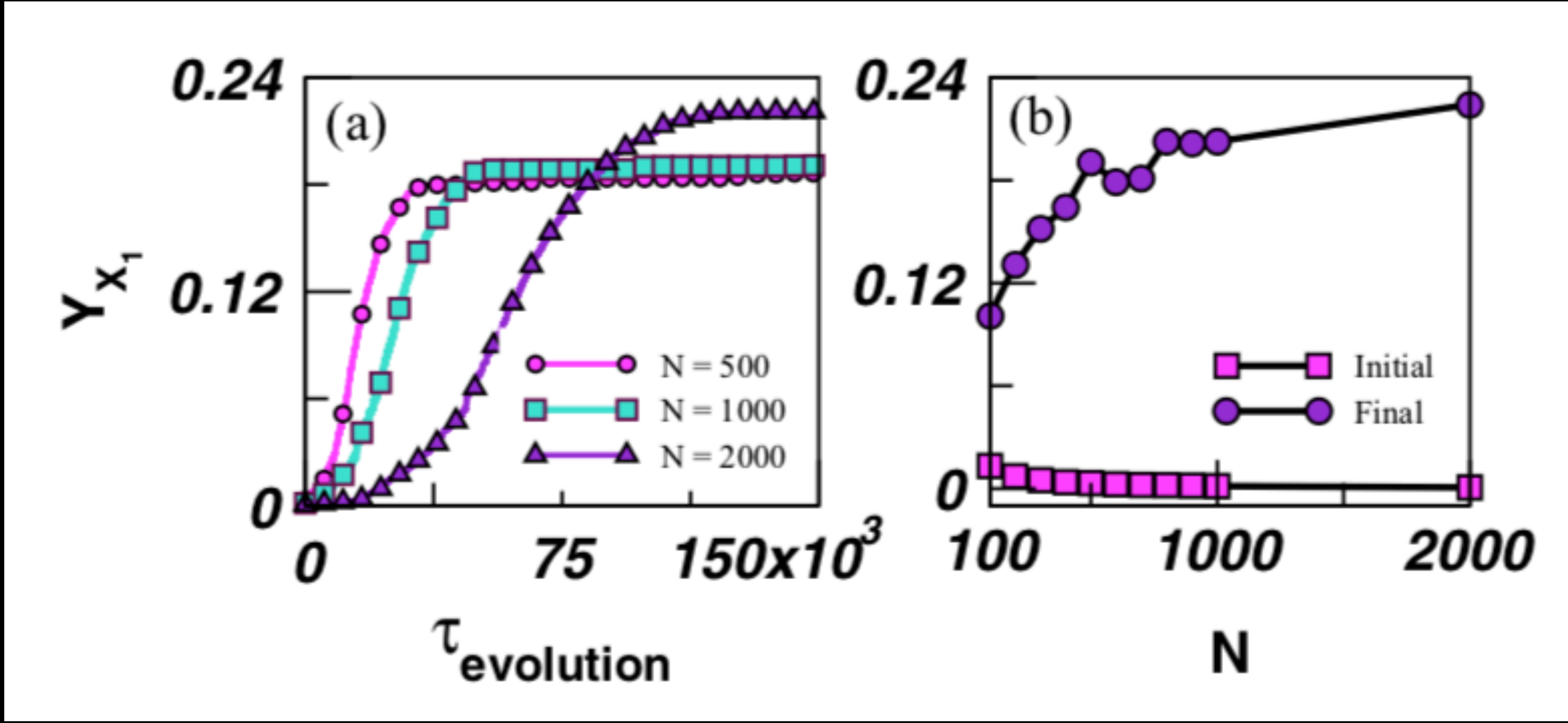
$$IPR(t+1) > IPR(t)$$

Optimized evolution of the network



network possess various special network architecture features: high degree, cluster coefficient, negative degree-degree correlations

For Different Network Size



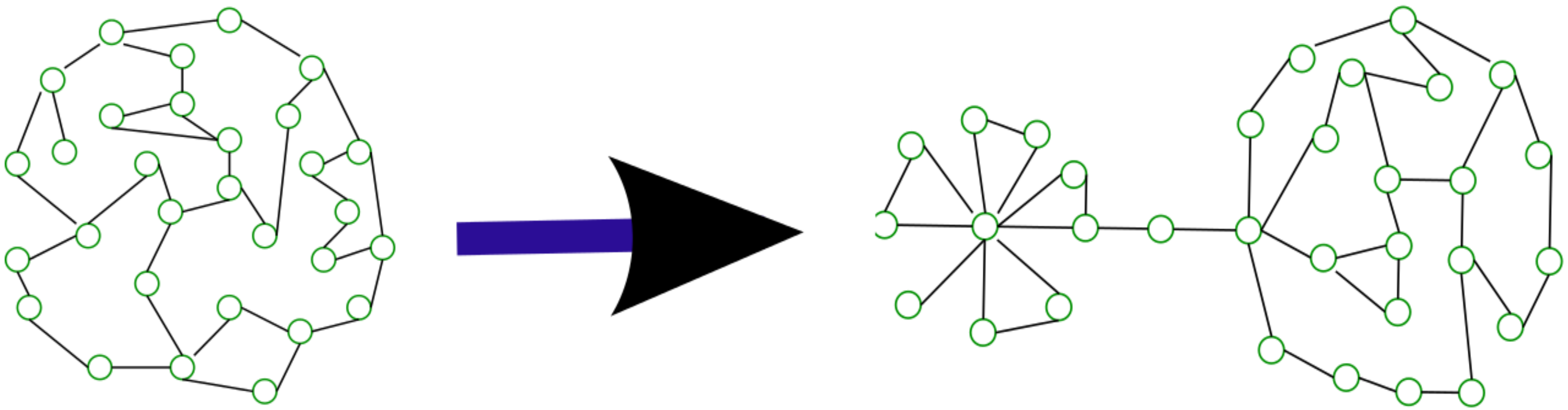
Comparison with Corresponding Model Networks

A model network with the same degree sequence, same average clustering coefficient or having the same degree-degree correlation as in the optimized networks

Index	ER_{opt}^{conf}	ER_{opt}^{cc}	$ER_{opt}^{r_{deg-deg}}$
N	500	500	500
$\langle k \rangle$	10	10	10
N_c	2499 ± 55	2499	2499
k_{max}	90 ± 7	100	100
λ_{max}	12.67 ± 0.46	11.36	11.60
$r_{deg-deg}$	-0.02 ± 0.005	-0.17	-0.16
IPR	0.05 ± 0.006	0.02	0.03
$\langle CC \rangle$	0.02 ± 0.003	0.14	0.01

- *IPR value is much lesser* than that of the network created through the evolution
- *The optimized network contains some other properties* achieved during the network evolution

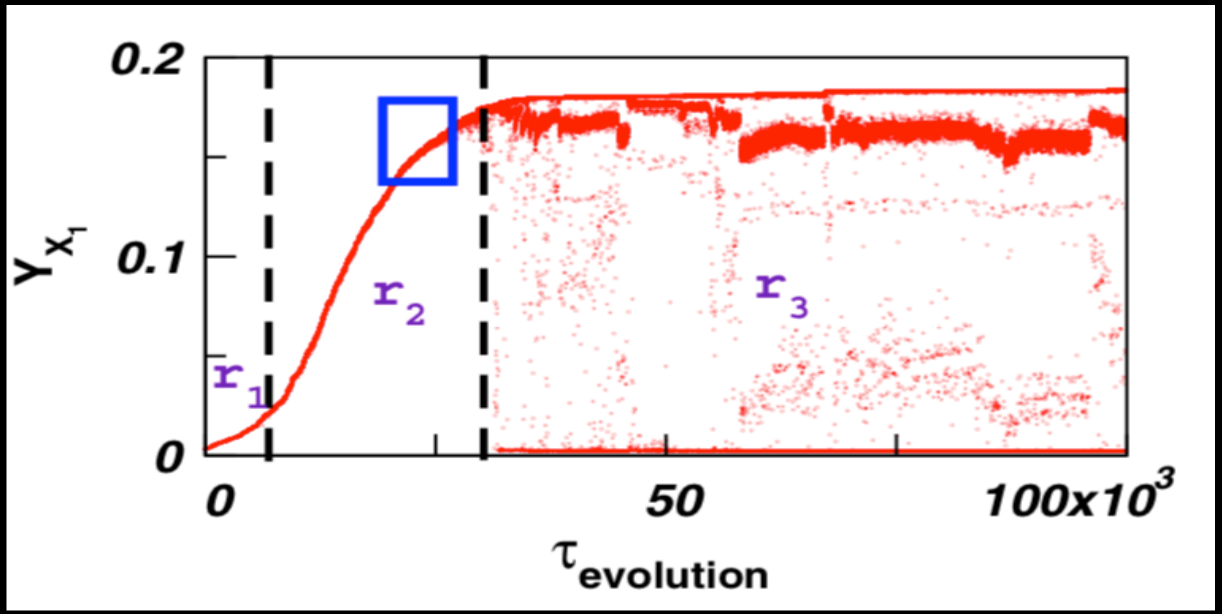
Evolution of Random Networks using IPR as fitness function:



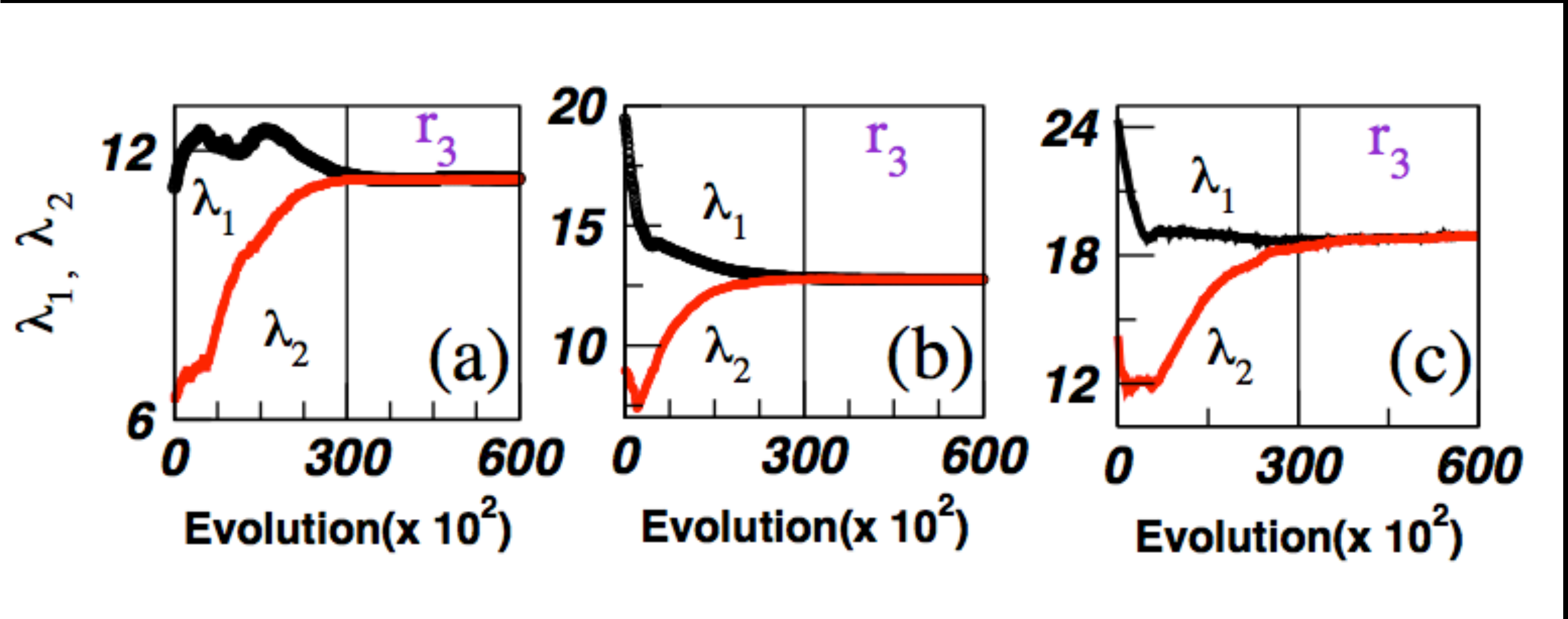
Initial network (ER network) Optimized network

The optimized structure is robust against change in the initial network structure

Eigenvector localization and relation with eigenvalues:



Degenerate eigenvalues:



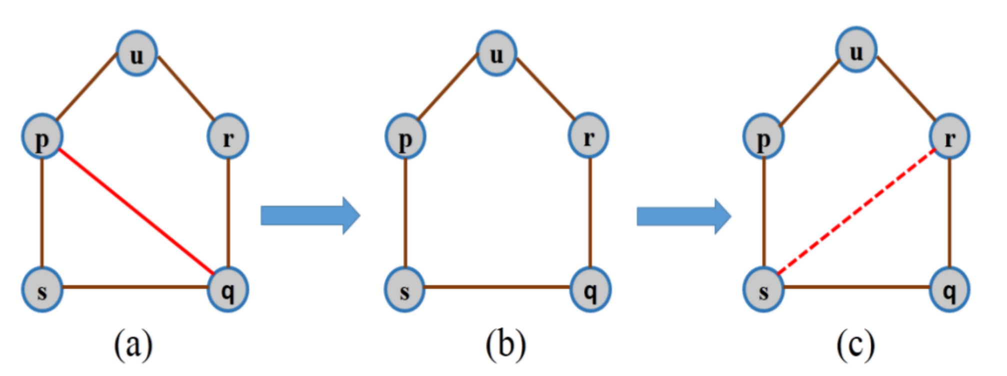
In saturation regime the largest two eigenvalues become equal. Initial networks are
(a) Random (b) Scale free
(c) C. Elegans, as initial networks

Analytical Derivation

A^t is the initial network at step t, A^{t+1} is the adjacency matrix after a single edge rewiring

$$\Delta a_{ij} = \begin{cases} -1 & \text{for edge removal} \\ 1 & \text{for edge addition} \end{cases}$$

$$A^t \xrightarrow[\Delta a_{qp}]{\Delta a_{pq}} A^{t'} \xrightarrow[\Delta a_{sr}]{\Delta a_{rs}} A^{t+1}$$



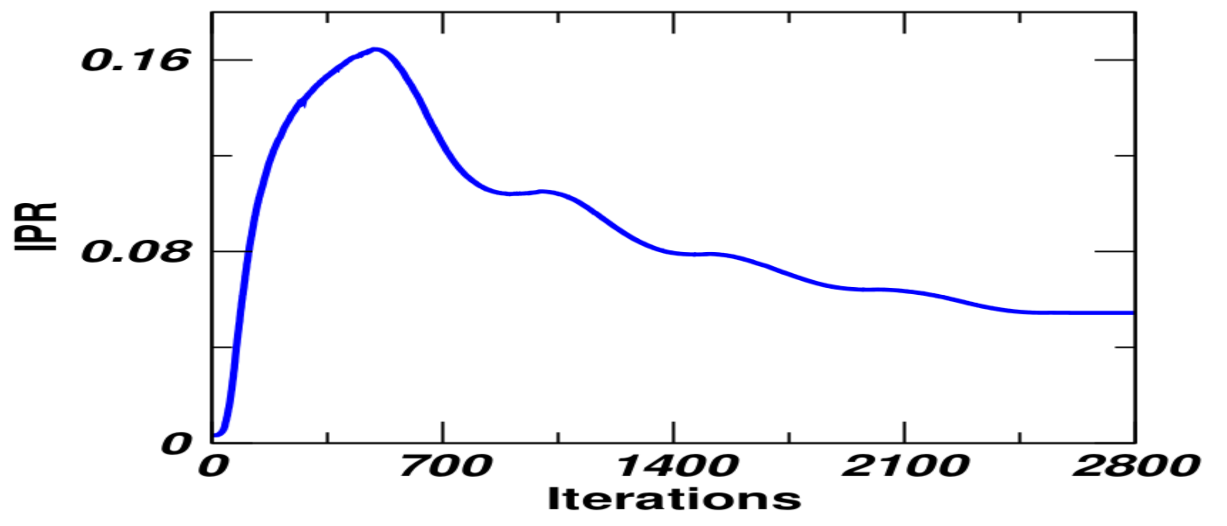
Changes in eigenvector entries:

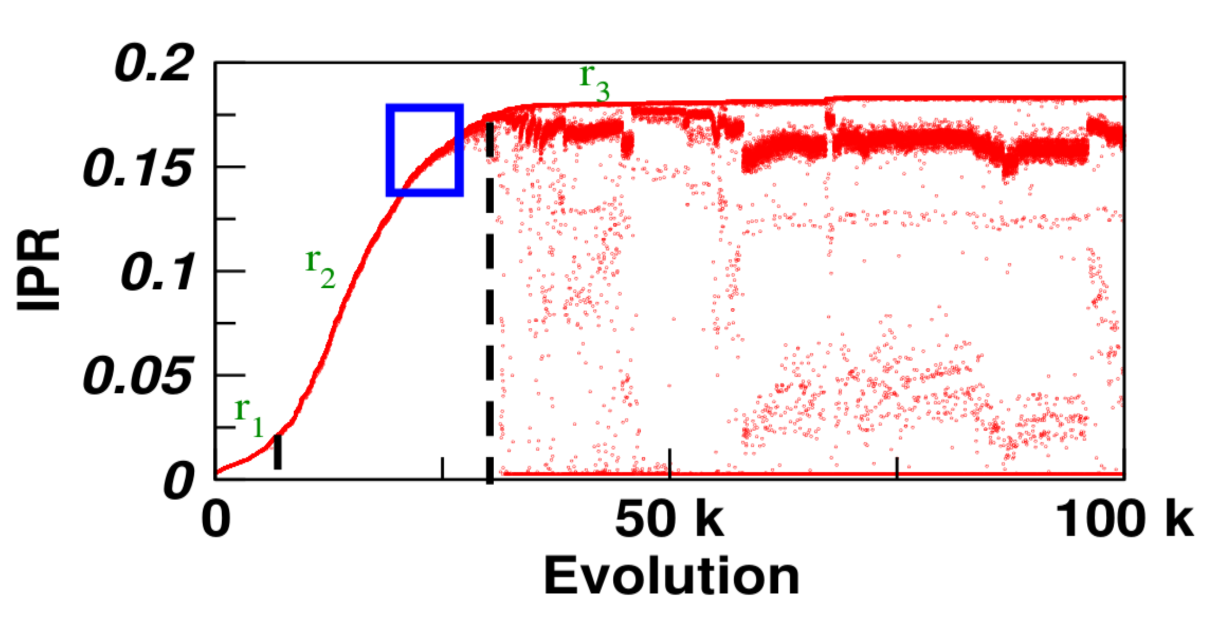
Hence IPR value will increase if the changes of the IPR value $\Delta IPR^{t+1} > 0$. Thus,

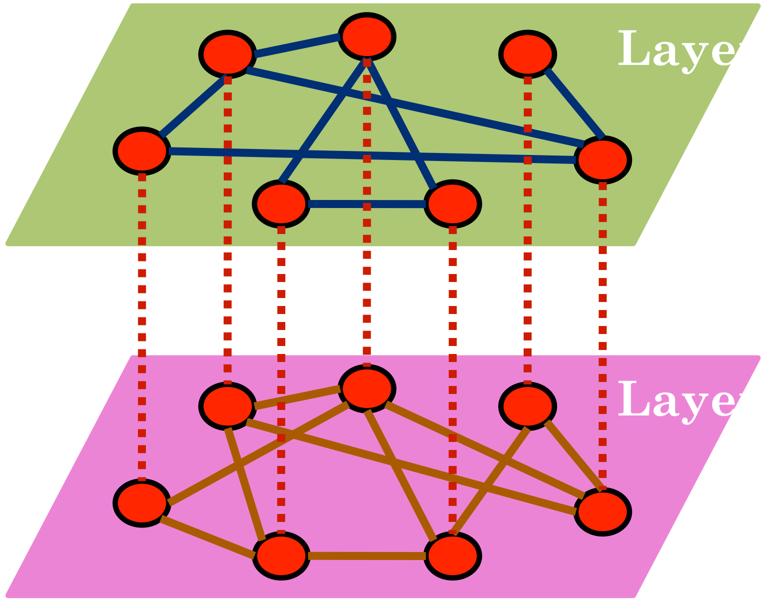
$$\left[\frac{\lambda^t x_r^t x_s^t[(x_r^t)^2 + (x_s^t)^2]}{(\lambda^t)^2 - 1} + \frac{[(x_r^t)^4 + (x_s^t)^4]}{(\lambda^t)^2 - 1}\right] > \frac{x_p^t x_q^t[(x_p^t)^2 + (x_q^t)^2]}{\lambda^t}$$

Analytically Derived Expression

- Our assumptions stand valid for small IPR values (r1 region) and the regime where IPR values increases rapidly (initial part of r2 region)
- During the further evolution, for which the IPR value approaches close to the saturation zone, our assumptions do not hold good
- We achieve 60% of the optimal IPR value within 500 iterations, whereas optimization process takes approx. 20,000 iterations

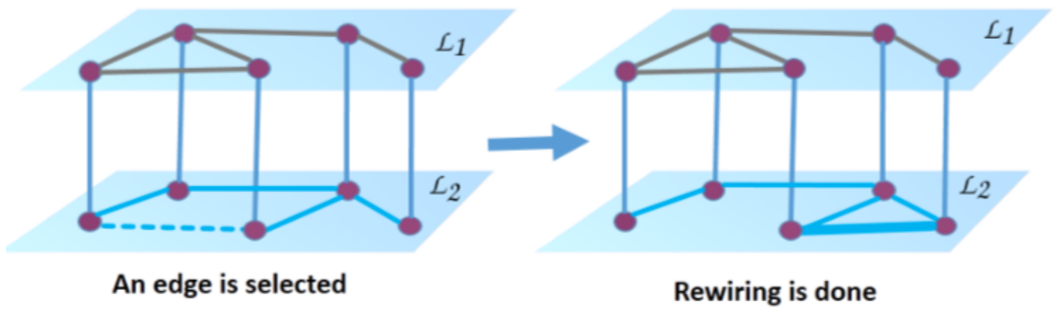






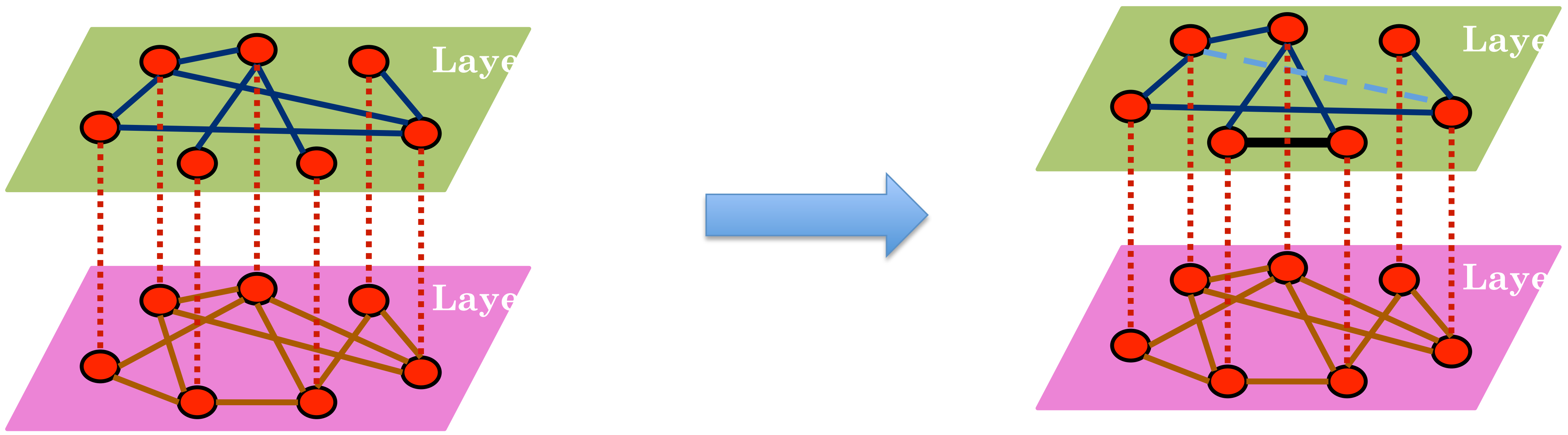
$$\begin{pmatrix} A_1 & I \\ I & A_2 \end{pmatrix}$$

Optimized evolution of multiplex network by *one layer* rewiring

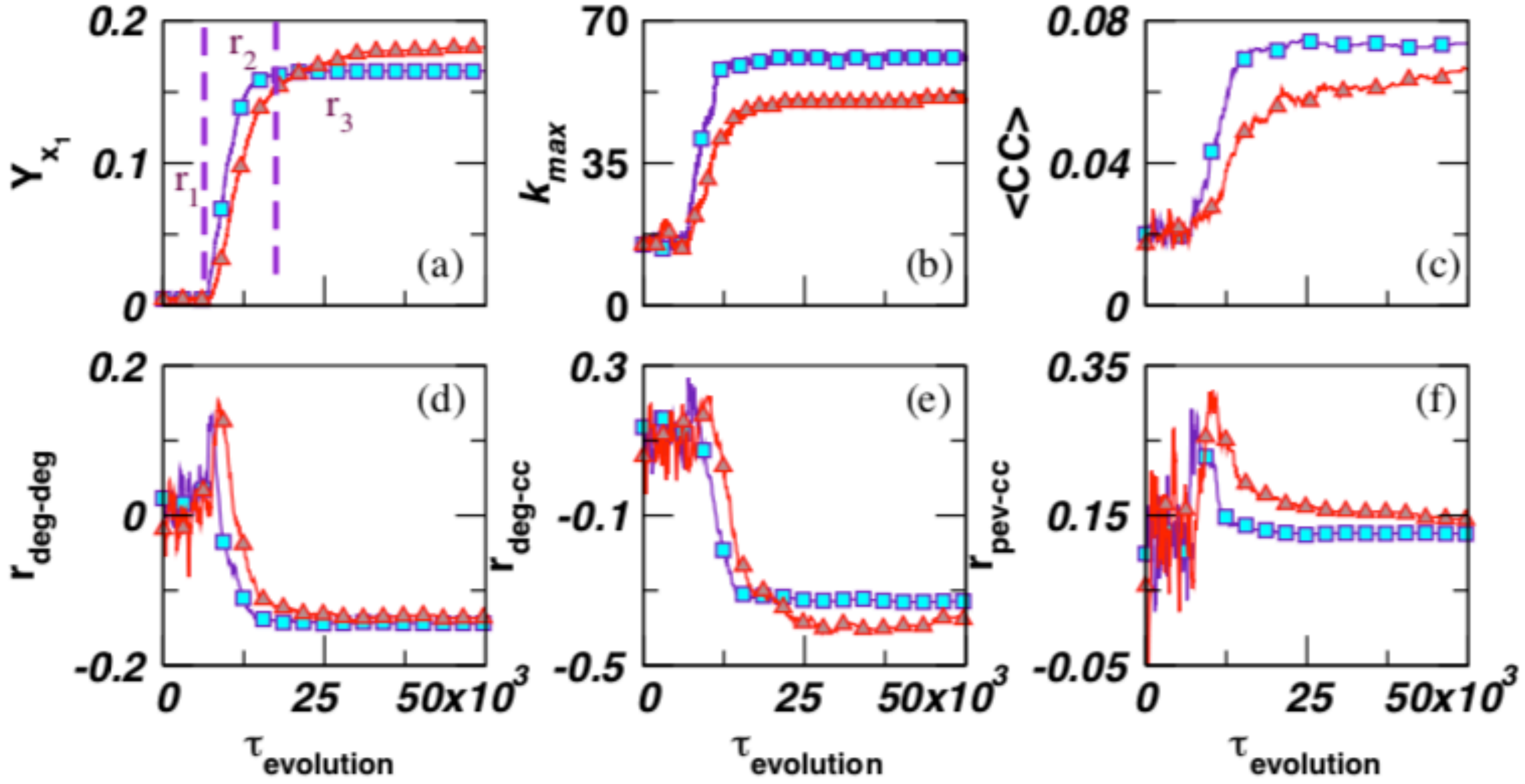


Aim: To control a desired property of systems represented by Multiplex networks

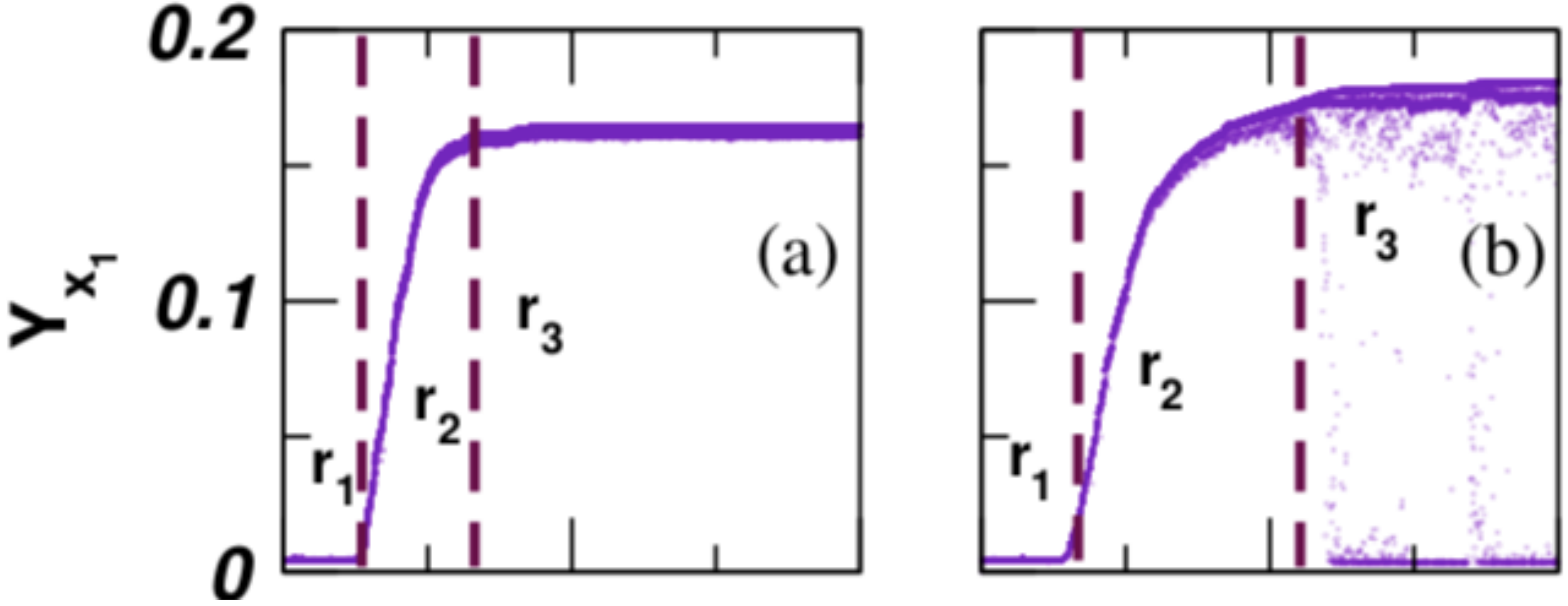
Restriction: only one layer can be altered or in our control



Changes in networks properties during the optimized evolution of multilayer network:



Optimized evolution of multiplex network



Left: Single layer rewiring Right: Both layers rewiring

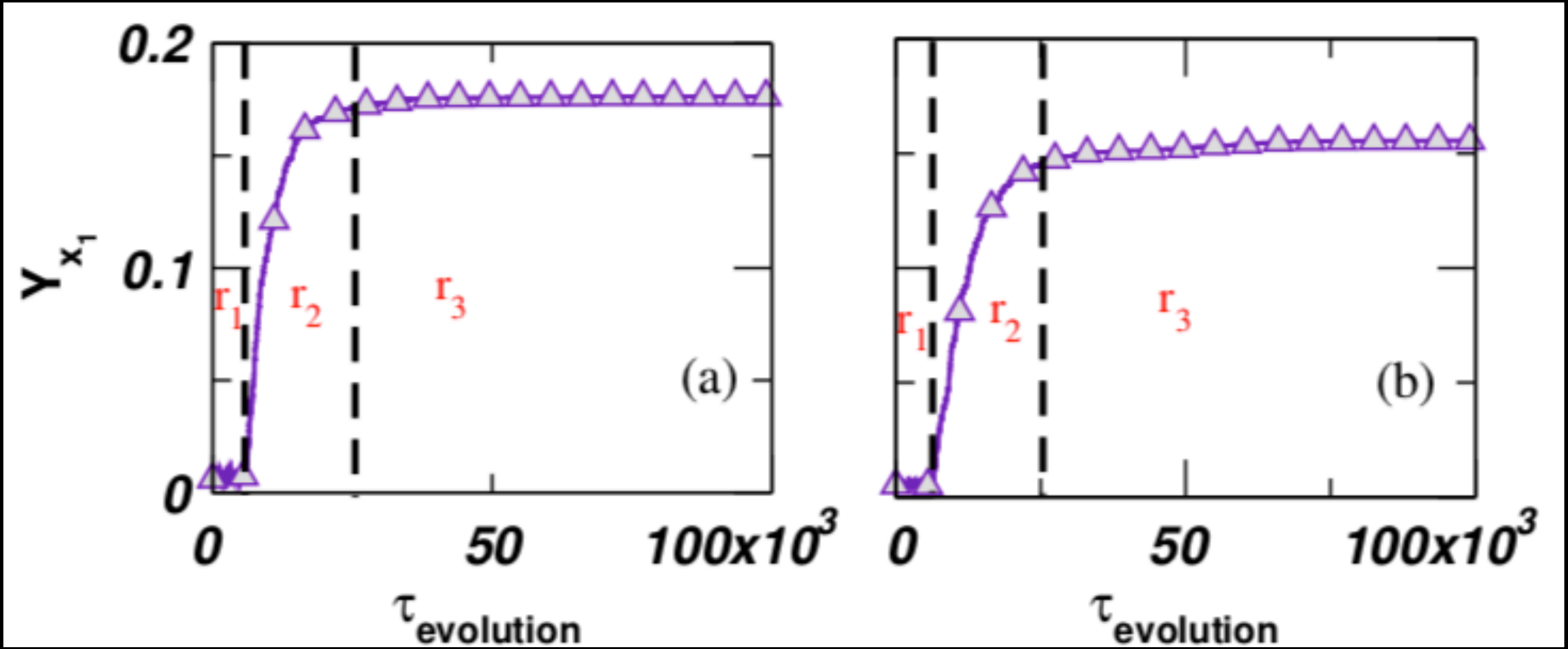
Single layer rewiring leads to almost the same value of IPR as the both layer rewiring

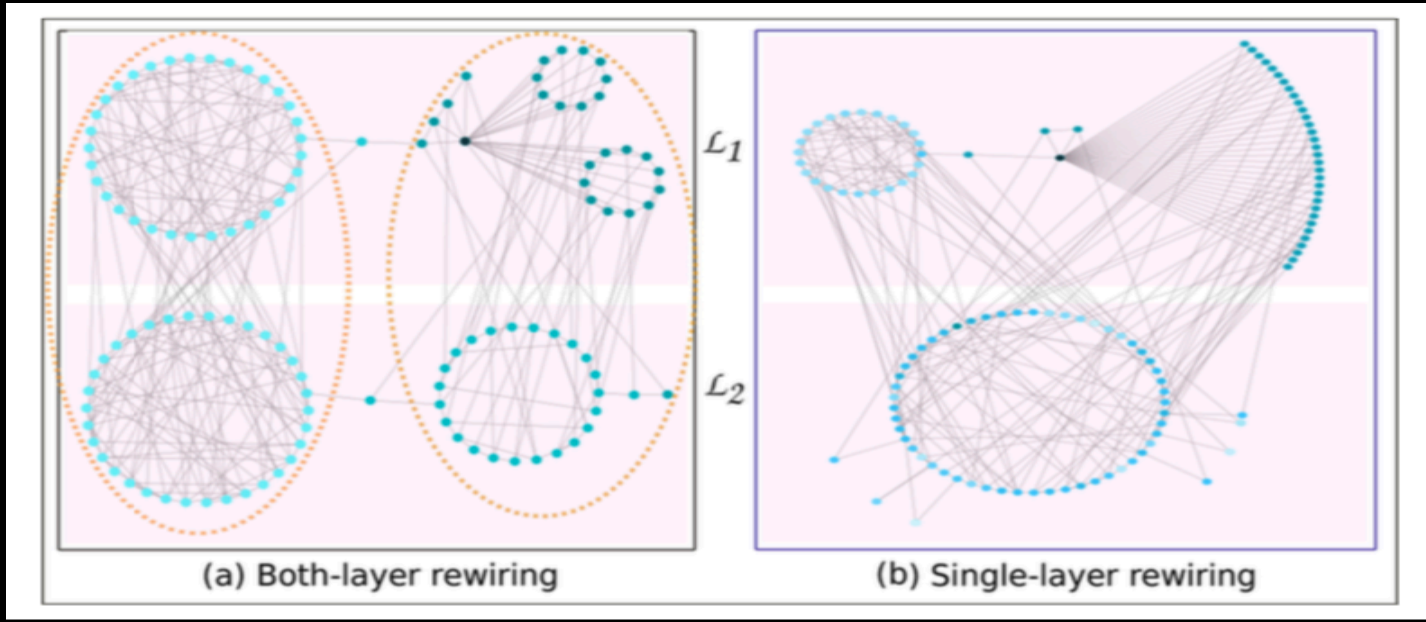
Importance of Optimization

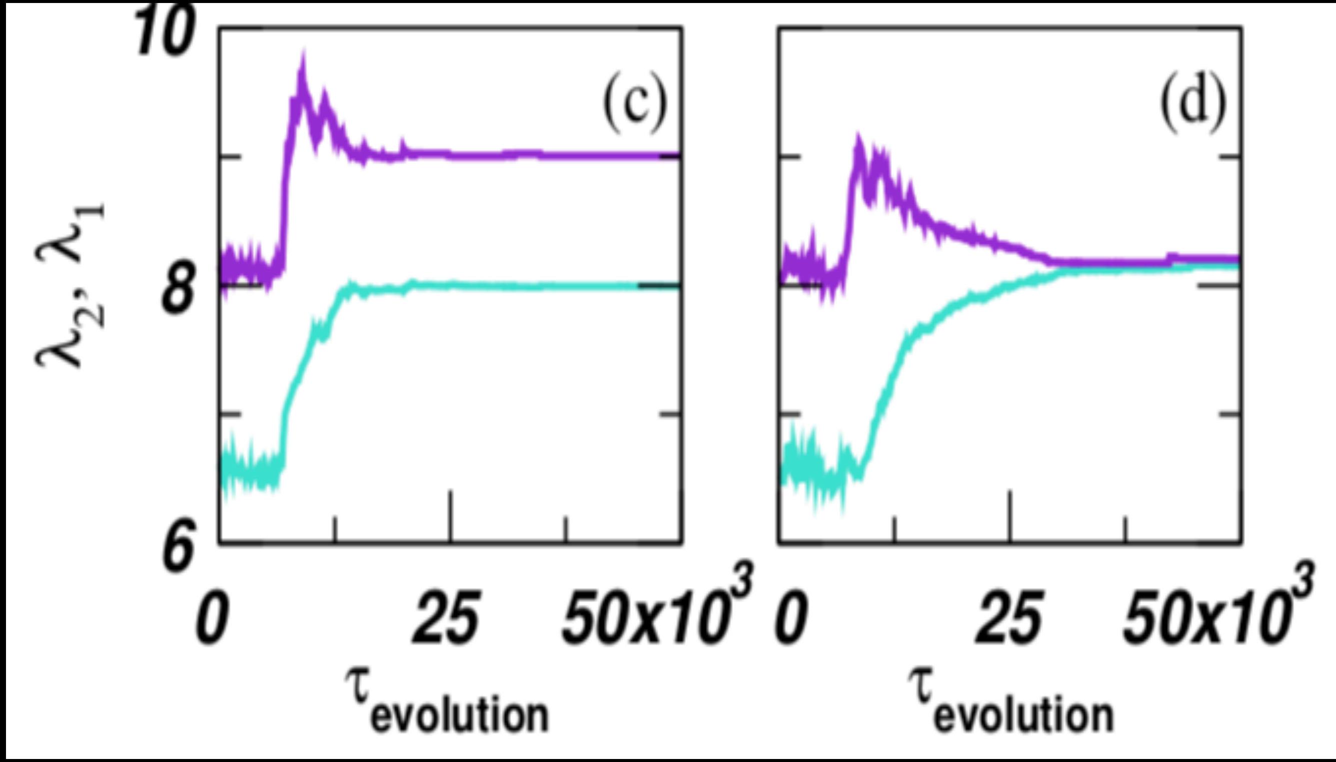
- Optimization of complex networks is behind the success of technological as well as natural adaptive processes
- The brain learns by rewiring its synaptic connections. Deep learning machines changes internal structures of neural network to optimize its logical outputs
- The difficulty lies in the fact that optimization complexity increases exponentially by the size of the system
- We show that localization behavior of a whole multiplex network can be optimized by only rewiring a single network layer

Optimization complexity can be drastically reduced

Three layers and four layer MN: single layer rewiring







Both layer rewiring:

- No degenerate eigenvalues
- We get rid of the critical region

Various Real World Multiplex Networks

Network	l	N	$\langle k \rangle$	$Y_{x_1^{\mathcal{M}}}$	k_{max}	$\langle CC \rangle$	$r_{deg-deg}$	λ_1	λ_2
Moscow Athl.	3	124423	4.01	0.03	4840	0.07	-0.12	75.22	71.5
NYClimate	3	148936	5.39	0.07	9742	0.08	-0.10	118.5	99.2
MLKing2013	3	318962	2.51	0.08	8689	0.02	-0.11	93.2	85.5
Cannes2013	3	573353	3.98	0.2	8675	0.05	-0.09	94.26	86.9
Higgs mux	2	886744	31.09	0.003	51387	0.09	-0.09	653.5	436.7
ObamaIsrael	3	2258678	3.55	0.15	21650	0.07	-0.04	151.77	139.9
Drosophila	4	10255	7.62	0.008	175	0.09	0.07	46.96	31.0
Homo	4	34363	10.22	0.09	9570	0.16	-0.05	118.76	67.2

IPR of the corresponding random networks ~ N/3 ~ 0.00001

- ➢IPR of these real world MNs are much higher than the corresponding random networks
- ➢Largest two eigenvalues are not that close or well separated

GIST

- ✓ We develop a learning framework to explore localization of eigenvector through an optimization method
- ✓ Localized networks possess several structual properties, incorporating only one of them does not lead to localization
- ✓ The most localized network corresponds to the critical state: degenerate eigenvalues
- ✓ Localization property is robust against the edge rewiring
- ✓ Localization of entire system represented by multiplex network by single layer rewiring

- Localization of principal eigenvector localization
  - Localization of disease, computer virus on smaller section
  - Faster spread of information: e.g. awareness of vaccination

Pradhan, Yadav, Dwivedi and SJ, Phys. Rev. E (2017)

- Controlling PEV Localization of one layer: controlling propagation of perturbation in a network by appropriate multiplexing

SJ and Pradhan, Phys. Rev. E (2018)

Future Direction…

- So far we restricted to undirected and un-weighted networks, the approach can be extended to obtain a comprehensive picture of PEV localization on directed and weighted networks

- The present work is restricted to PEV of adjacency matrices, it will be interesting to study other lower order eigenvectors localization on emerging network properties

- Multi-point localization

RELATION WITH CONTROL THEORY: FUTURE ASPECTS

- Networks theory has proven its aptness in providing insights into controllability at a fundamental level

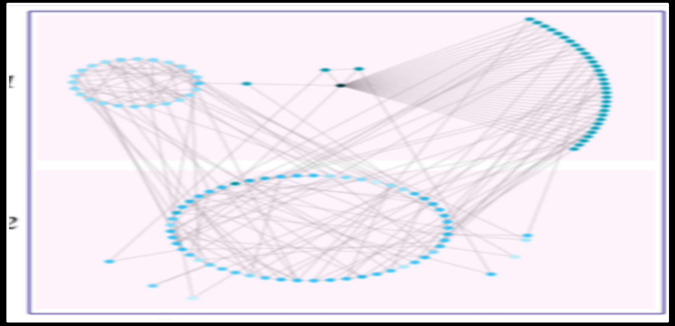
- In traditional approaches, external inputs are imposed to affect the dynamics of few nodes causing control of the entire system

  Y. Y. Liu et. al. Nature (2011)

- Our work refines the concept of controllability: by addition of a new system (layer) we can change the dynamical evolution of the entire system (multiplex) to a desired behavior

CONTROLLING ONE LAYER BY ANOTHER

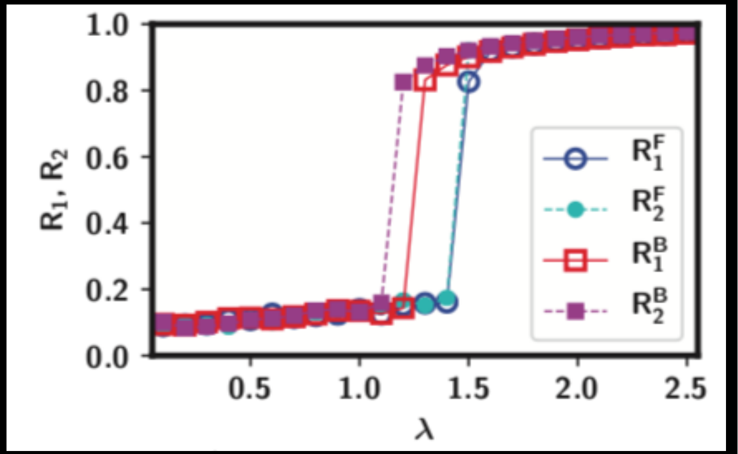
- ➢ Regulating localization of Eigenvector of adjacency matrix of *one layer*



<u>Chimera: Symmetry breaking resulting in coexistence of regular and irregular behavior</u>

- ✓ Multiplexing affects Chimeras: Controlling chimeras in one layer by changing another layer

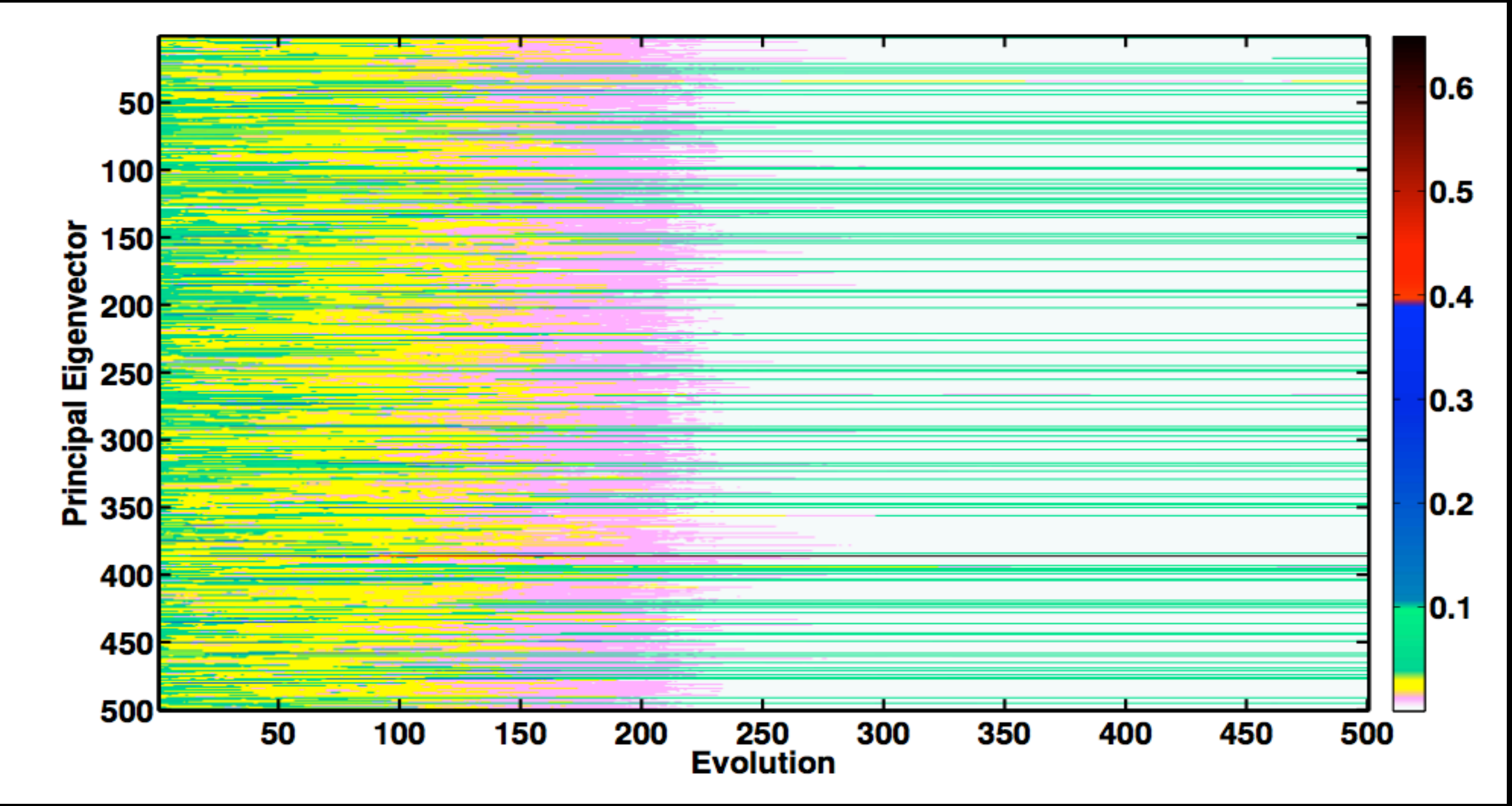
- ➢ Synchronizability in multiplex networks by rewiring one layer
- ➢ Multiplexing induced explosive synchronization*: First order phase transition* to synchronization



Thank you !!

Acknowledgements:DST and CSIR

Principal Eigenvectors during evolution

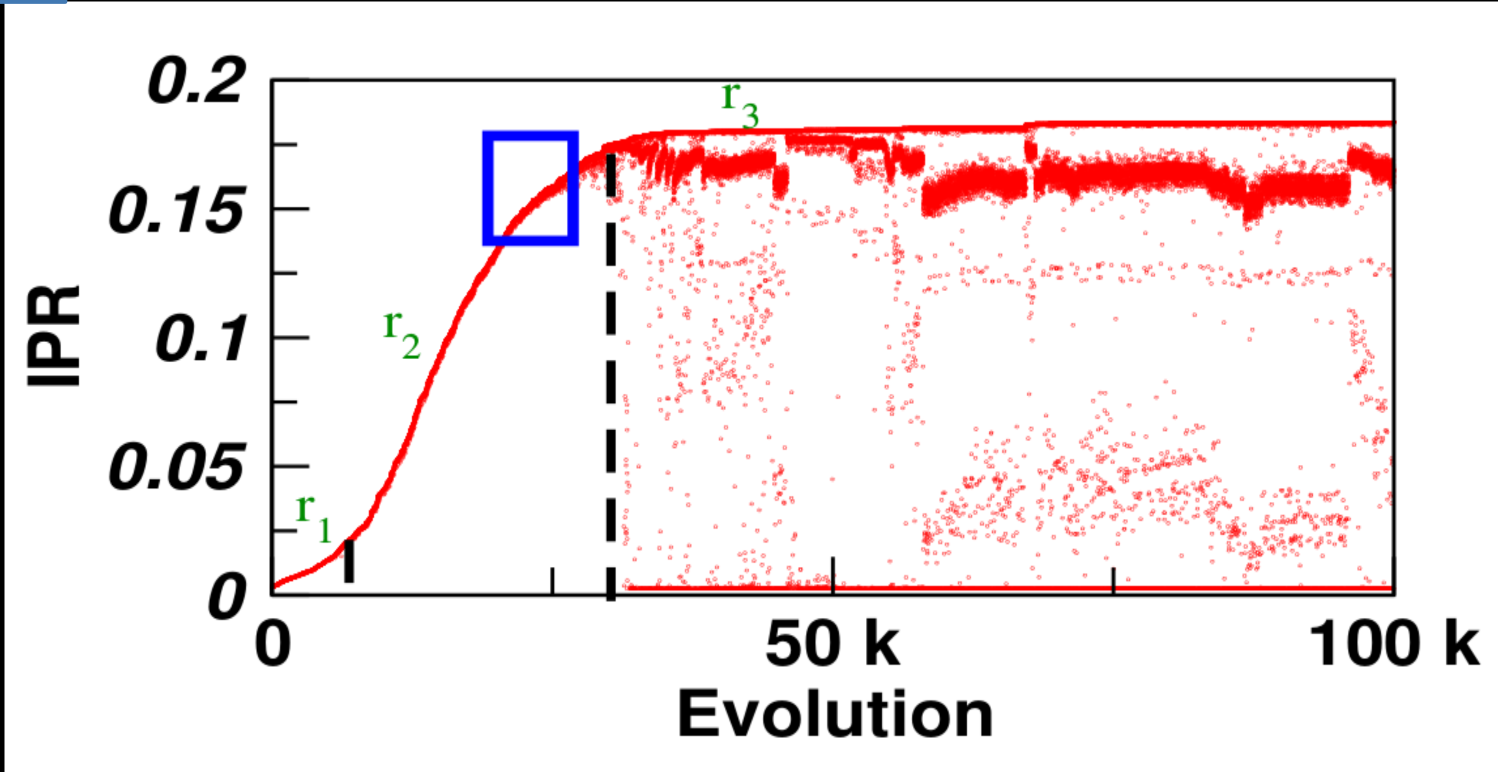


- Black line indicates the hub node weight

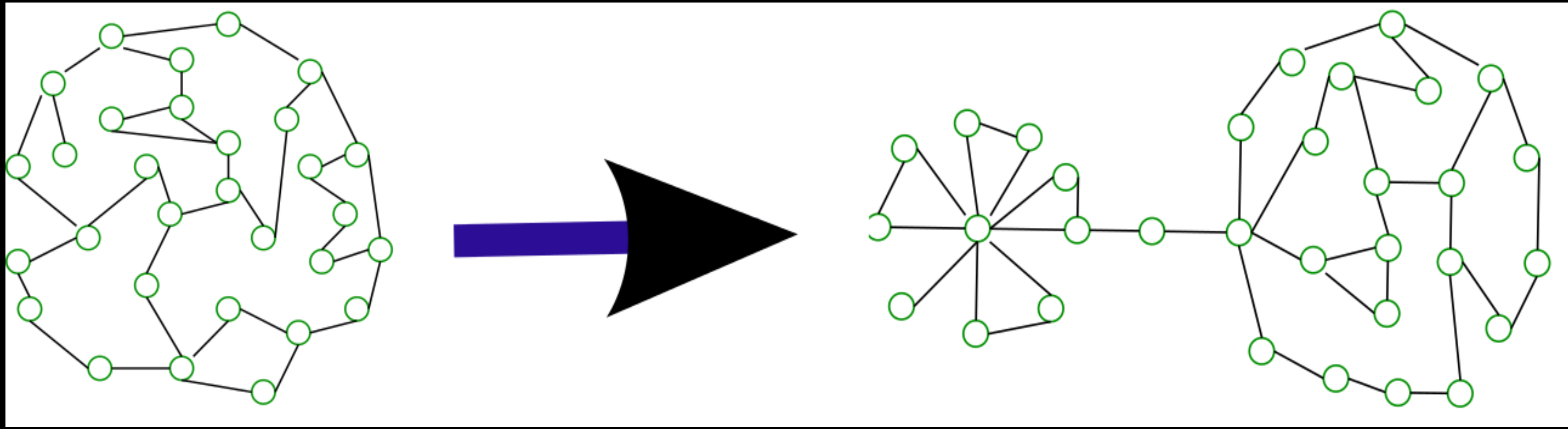
- During the evolution, formation of the hub node happens much before the IPR value gets saturated

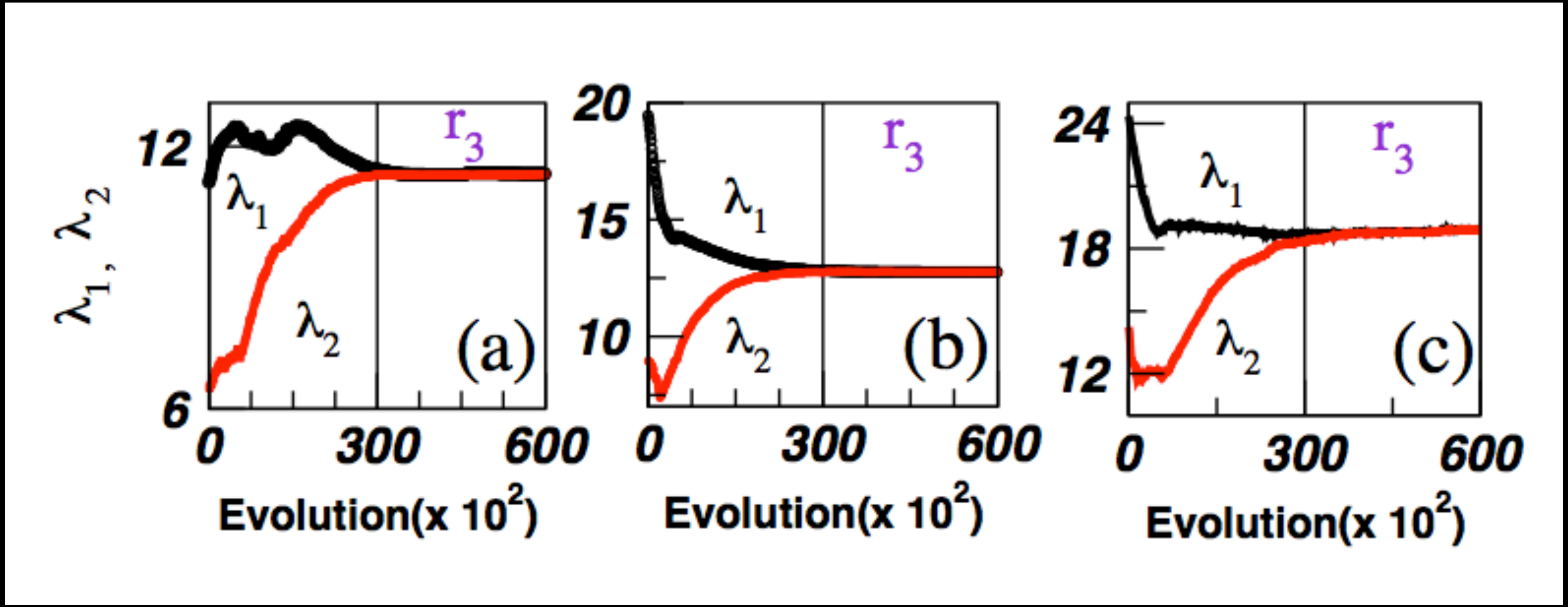
Critical Regime

- As IPR gets saturated, there exists few edge rewiring that leads to a complete delocalization of PEV



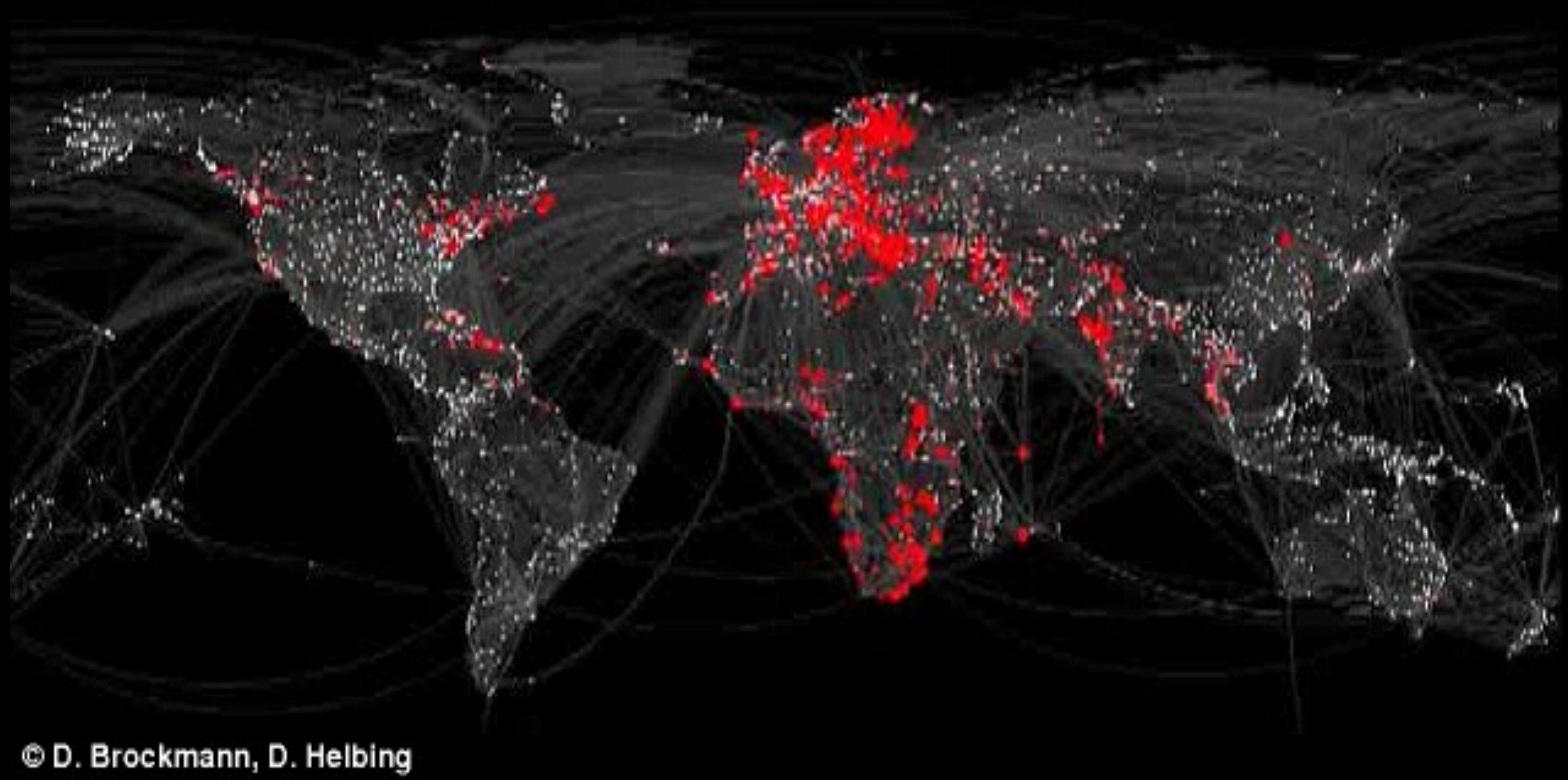
- r3 region is very sensitive to edge rewiring
- In r2 region, IPR is close to r3, but this region is robust against edge removal/rewiring





λ_1, λ_2
12
6
λ_1
λ_2
r_3
(a)
0
300
600
Evolution(x 10^2)
20
15
10
λ_1
λ_2
r_3
(b)
0
300
600
Evolution(x 10^2)
24
18
12
λ_1
λ_2
r_3
(c)
0
300
600
Evolution(x 10^2)

How London Heathrow can spread a pandemic



http://www.dailymail.co.uk/sciencetech/article-2523302/Terrifying-video-reveals-London-Heathrow-spread-pandemic-DAYS.html