Ordinal Networks

Using networks to detect dynamical regime change in time series

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Preamble

A *dynamical system* is a triple $(T, \mathcal{M}, \phi)$ where

$$\phi : U \subset T \times \mathcal{M} \to \mathcal{M},$$

T is an additive monoid and $\mathcal{M}$ is a set.

Think of T as *time* and $\mathcal{M}$ as *space*. The operator $\phi_t(z)$ defines the evolution of a point $z \in \mathcal{M}$ forward by time t.

To make T sufficiently "time-like" we require:

- $\phi_0(z) = z$ for all $z \in \mathcal{M}$
- $\phi_{t+s}(z) = \phi_t \circ \phi_s(z)$ for all $z \in \mathcal{M}$ and $t, s \in T$

Hence, $\phi_t(\cdot)$ is the dynamical *evolution operator*.

Typically, $\mathcal{M}$ will be a smooth manifold that is locally diffeomorphic to $\mathbb{R}^n$.

An Experiment

Let $\Phi(z_0) = \{\phi_t(z_0) | t \in T\}$ be the *trajectory* for an initial condition z_0[1]

Let $h : \mathcal{M} \to \mathbb{R}$ be a *measurement function* and suppose that we can only observe

$$x_i = h(\phi_{i\kappa}(z_0))$$

for $i \in \mathbb{Z}^+ \cup \{0\}$. We call κ the *sampling rate* of our experiment.

What can the *time series* $\{x_i\}_{i=1}^N$ tell us about ϕ?

[1] We can, of course, think about trajectories backwards in time as well — and to do so is more typical. Moreover, we have yet to prove that $\Phi(z_0)$ is well defined, but suppose for now that it is.

Takens' Embedding Theorem (1981)

The map

$$x_i \mapsto (x_i, x_{i+1}, \ldots x_{(i+m-1)}) =: v_i$$

is an embedding of a compact manifold with dimension $d \in \mathbb{Z}^+$ ($m = 2d + 1$) if $h : \mathcal{M} \to \mathbb{R}$ is C^2 and "generic"[2]. Moreover, evolution of v_i are diffeomorphic to the dynamics of ϕ.

Corollary (Sauer, York, and Casdagli, 1981)

Takens' embedding theorem also holds for $d \in \mathbb{R}^+$ with the condition that $m \geq 2d + 1$.

Hence, a fractal attractor $\mathcal{A} \subset \mathcal{M}$ can be *reconstructed* from a time series generated from a trajectory lying on that attractor.

[2] Sufficiently well coupled between the *d*-dimensional variables and the theorem is then true almost always.

Delay reconstruction of experimental data

Suppose that a time series, as defined above, is the output of a deterministic and stationary (autonomous) dynamical system.

Techniques to reconstruct the underlying attractor are now well established[3] and widely used.

This is usually achieved by estimating two parameters: *embedding dimension* m, and *embedding lag* τ and applying the map

$$x_i \mapsto (x_i, x_{i-\tau}, \ldots x_{(i-(m-1)\tau)}) =: v_i$$

Can we do better?

[3] See M. Small Applied Nonlinear Time Series Analysis, World Scientific, 2003.

Table of contents

Recurrence networks

Poincaré Recurrence Theorem (Poincaré, 1890; Carathéodory, 1919)

For a volume preserving dynamical system with continuous evolution operator ϕ and bounded trajectories, then for any open set $V \subset \mathcal{M}$ there exists orbits that come arbitrarily close to V infinitely often.

More generally, and *attractor* $\mathcal{A} \subset \mathcal{M}$ of a dynamical system is a set of points to which trajectories converge — and in particular, *strange attractors* are the objects of central interest in the study of chaotic dynamics.

The *recurrence matrix* $R(\epsilon)$ is an $N \times N$ matrix with

$$R_{ij} = \begin{cases} 1 & d(v_i, v_j) < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

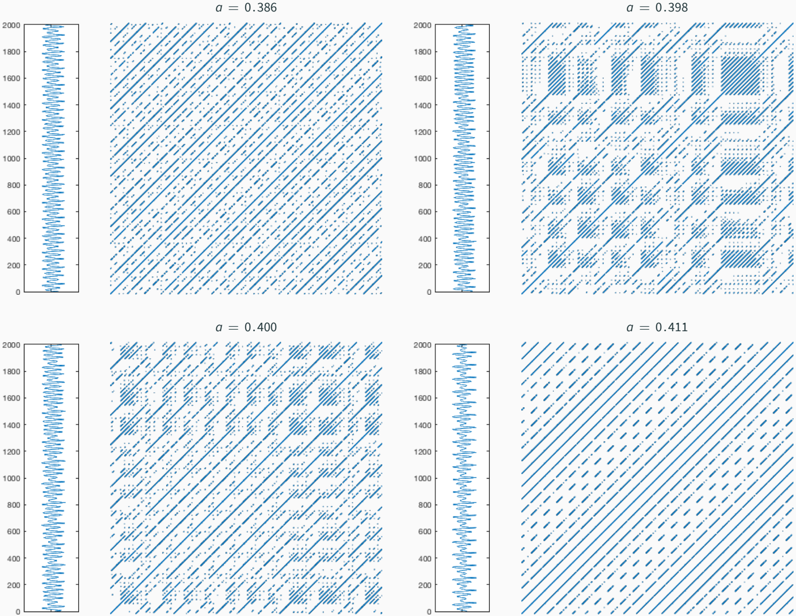
where $\epsilon > 0$ is a threshold and $d(\cdot, \cdot)$ is the usual Euclidean metric[4].

An alternative definition[5] of the recurrence matrix $R(k)$ is to set

$$R_{ij} = \begin{cases} 1 & \sum_{\ell \neq i,j} \mathbb{I}\left(d(v_i, v_\ell) < d(v_i, v_j)\right) < k \\ 0 & \text{otherwise} \end{cases}$$

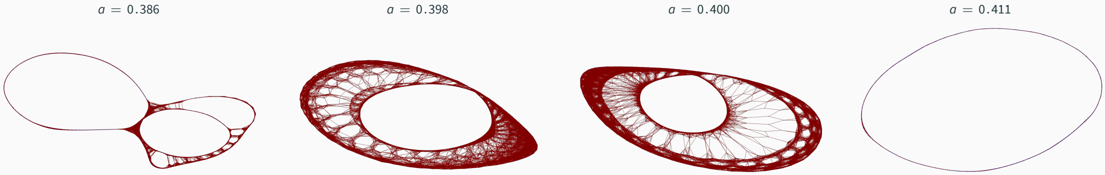
That is, we either choose neighbours within an ϵ-ball, or we select the k-nearest neighbours.

[4] Marwan, Romano, Thiel, Kurths *Physics Reports* (2007) **438**: 237
[5] Eckmann, Kamphorst, Ruelle *Europhysics Letters* (1987) 5: 973–977.

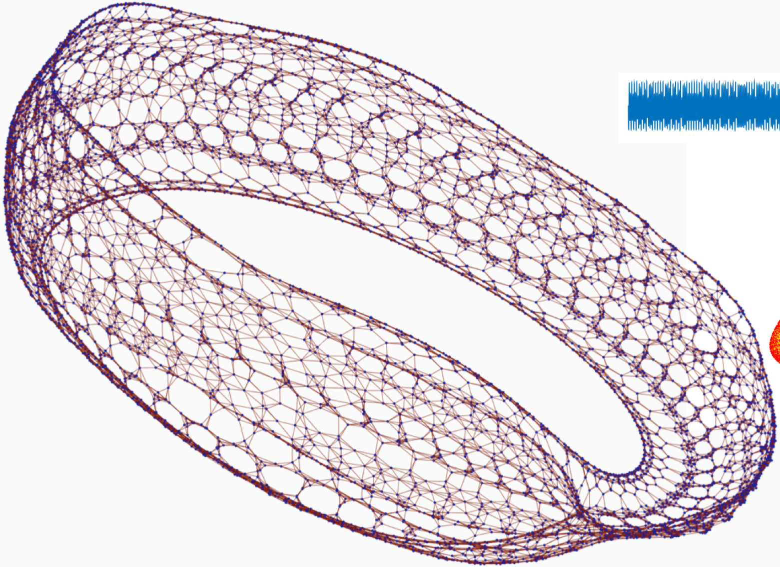


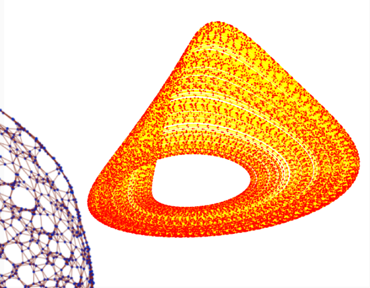
a = 0.386
a = 0.398
a = 0.400
a = 0.411
2000
1800
1600
1400
1200
1000
800
600
400
200
0

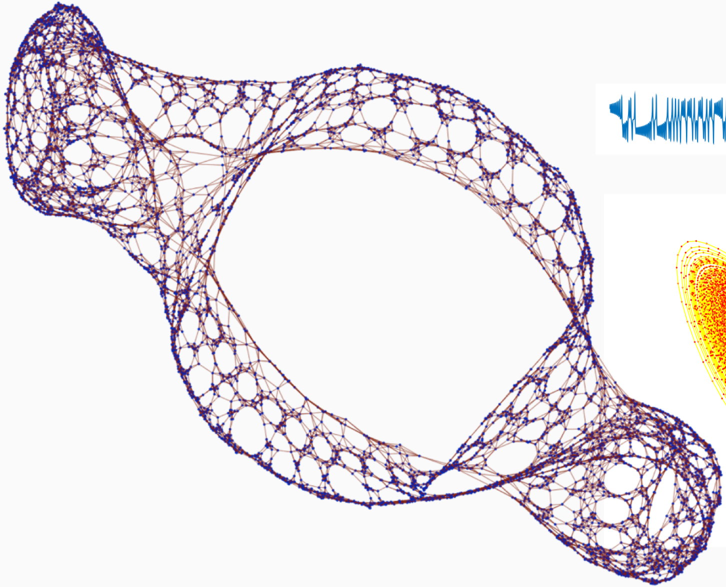
How can we bring *networks*[6] in on the act? Clearly, we could treat $A \equiv R(k)$ (or $R(\epsilon)$) as an adjacency matrix:

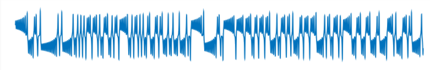


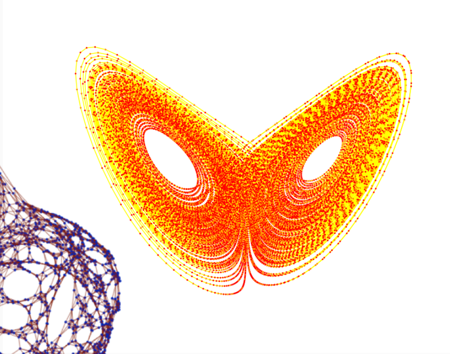
[6]Apologies to all graph theorists: in what follows graph, network, node, vertex, edge and link are pairwise synonymous.

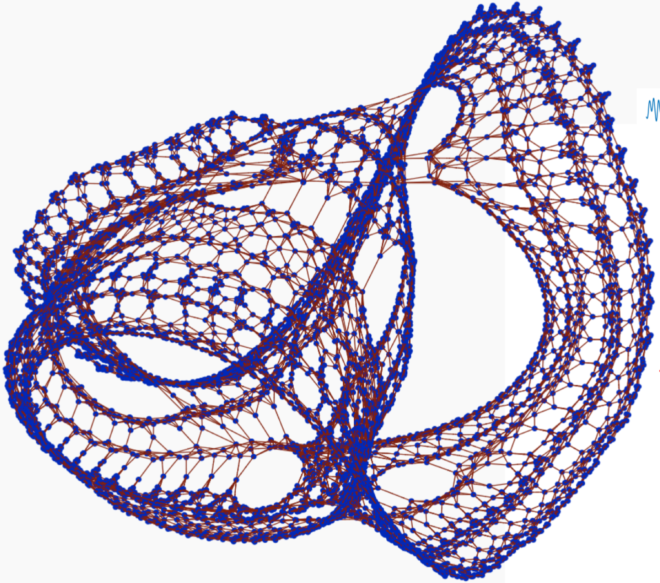


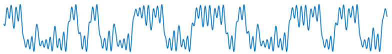


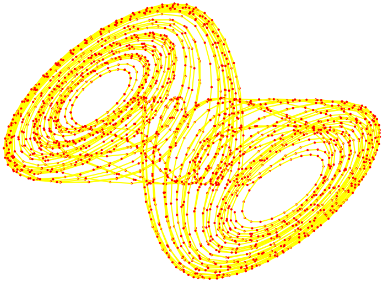












Motif superfamily

periodic flow

chaotic flow

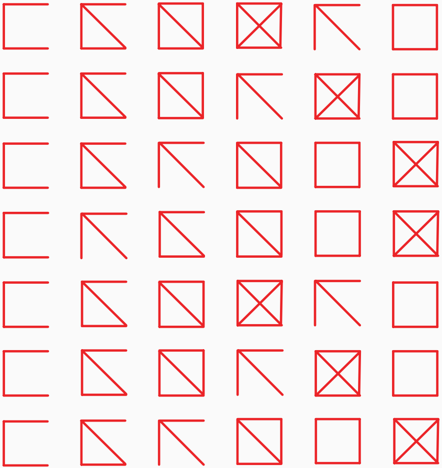
hyper-chaotic flow

noisy periodic flow

chaotic map

hyper-chaotic map

noise

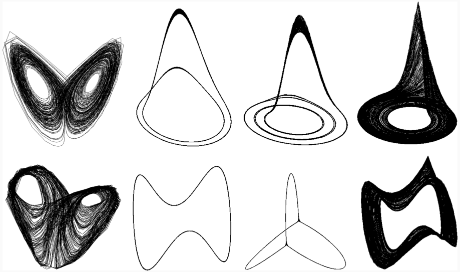


Compute the relative frequency of subgraphs of size *n* to create an order sequence of connected graphs of *n* nodes.[7]

But, can we do better?

[7] Xu, Zhang, Small, *Proceedings National Academy of Sciences USA* (2008) **105**: 19601-19605

Why does this work?



Observe that certain motifs are less likely to occur in a 1-dimensional manifold — these motifs become increasingly common as the local dimension of the dynamics increases.

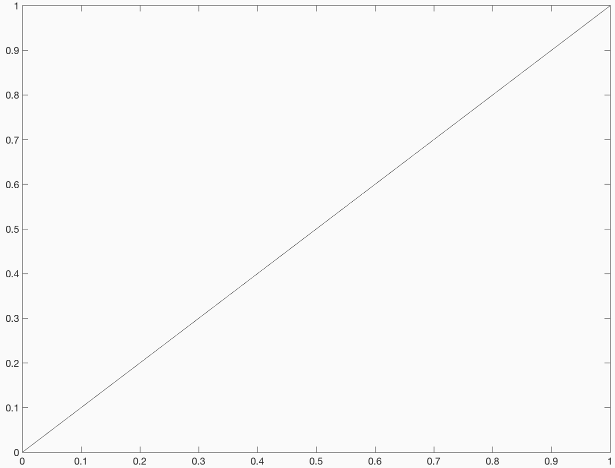
Hirata *at al.* [8] provided a constructive proof that a distance matrix (diffeomorphic to the original) could be obtained from a ϵ-ball recurrence plot. Time information added trajectories. Khor and Small[9] extended this to k-neighbour recurrence networks showing there is a metric which would map k-neighbours to ϵ-balls.

[8] Hirata, Horai and Aihara *Eur. Phys. J.: Spec. Top.* (2008) **164** 13–22

[9] Khor and Small *Chaos* (2016) **26** 043101.

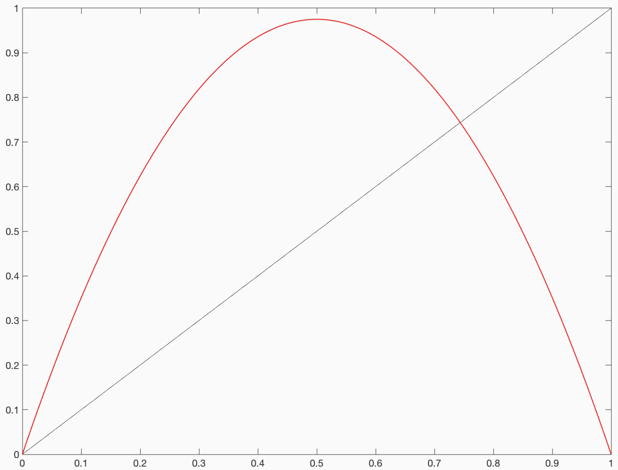
Forbidden sequences

Logistic map



Consider the logistic map $x_{i+1} = F(x_i) = \lambda x_i(1 - x_i)$ for $x_i \in [0, 1]$ and $0 < \lambda \leq 4$. What is the possible ordering (i.e. sorted into relative magnitude) of x, $F(x)$, $F^2(x)$ and $F^3(x)$?

Logistic map



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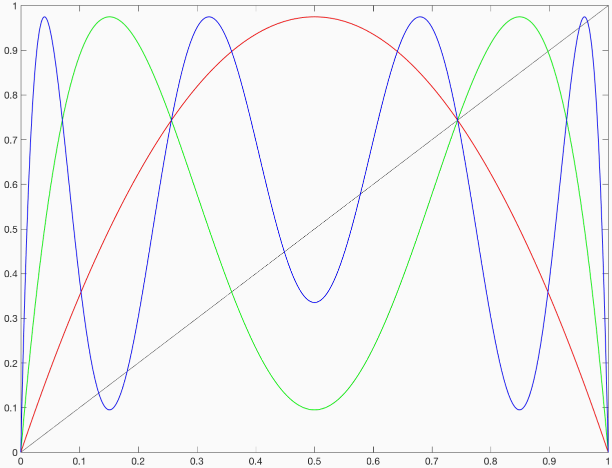

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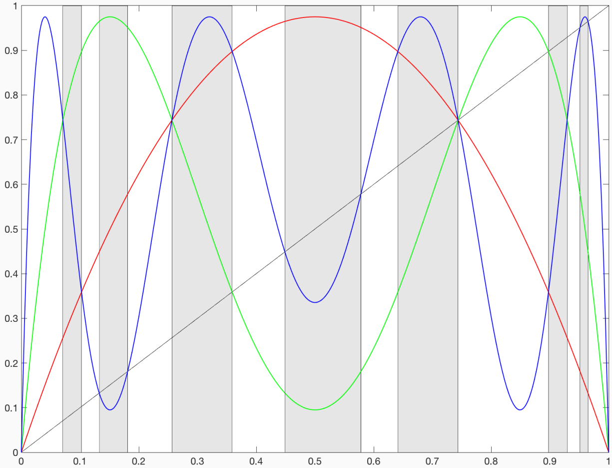

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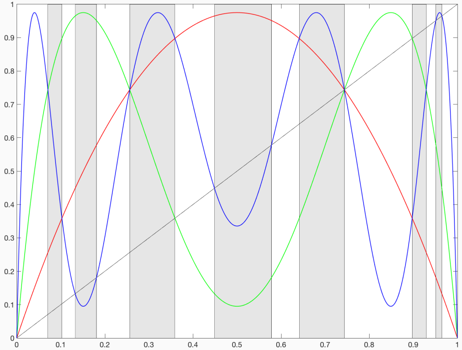
Logistic map

π_i	x_t	$\mu(Q_i)$[10]
(4321)	$(0, 0.071)$	0.1714
(4312)	$(0.071, 0.103)$	0.0361
(4213)	$(0.103, 0.133)$	0.0298
(3214)	$(0.133, 0.181)$	0.0424
(4213)	$(0.181, 0.256)$	0.0583
(4231)	$(0.256, 0.359)$	0.0710
(3241)	$(0.359, 0.449)$	0.0583
(2143)	$(0.449, 0.578)$	0.0826
(3142)	$(0.578, 0.641)$	0.0411
(3241)	$(0.641, 0.744)$	0.0710
(2314)	$(0.744, 0.897)$	0.1305
(1423)	$(0.897, 0.929)$	0.0361
(1432)	$(0.929, 0.951)$	0.0296
(2431)	$(0.951, 0.964)$	0.0215
(1432)	$(0.964, 1)$	0.1203



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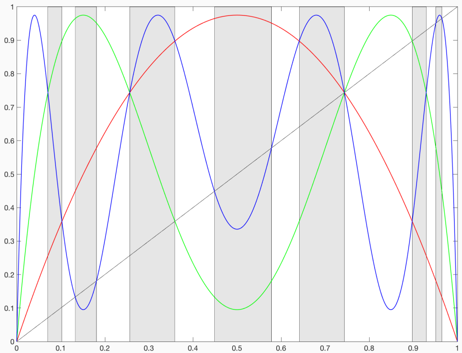


[10] $\mu(x) = \frac{1}{\pi\sqrt{(x(1-x))}}$ see Hall and Wolff *J R Stat Soc B* (1995) 57 439–452.

Ordinal Networks

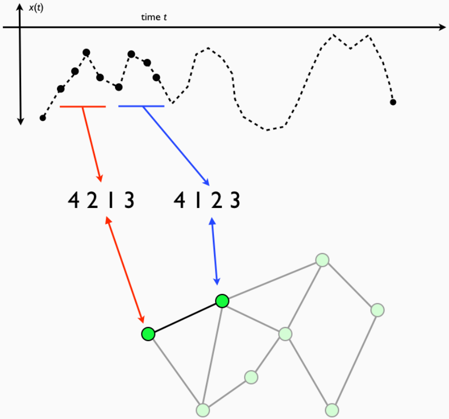
Let $\{x_i\}_{i=1}^{N}$ denote the scale time series and choose a window size $w \in \mathbb{Z}^+$. As before

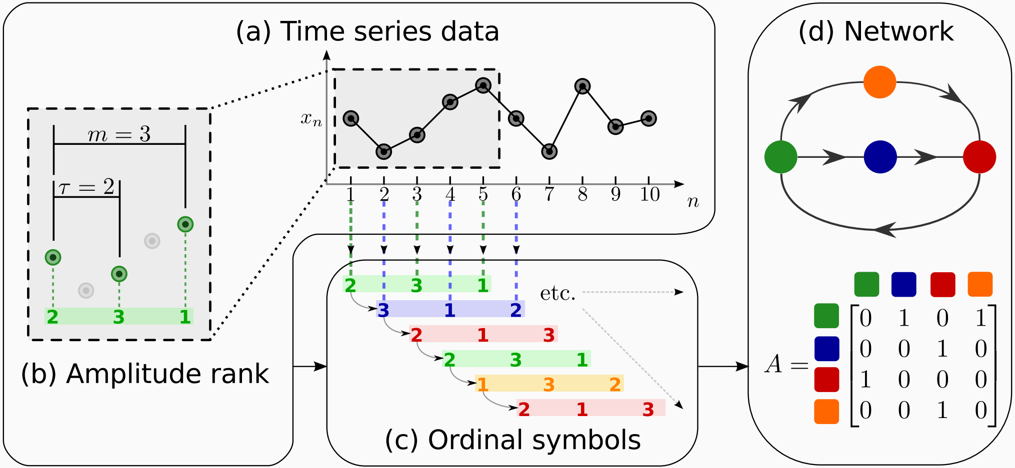
$$v_i := (x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+w-1})$$

and let $\pi_i \in S_w$ a permutation of the integers $1, 2, \ldots, w$ with the property that x_{i+k-1} (the k-th element of v_i) is the $pi_i(k)$-th largest element of $\{x_i, x_{i+1}, \ldots, x_{i+w-1}\}$.

π_i encodes the rank ordering of the coordinates of v_i.

Construct a graph with each vertex associated with a permutation, and connect vertices if the corresponding windows occur in succession (non-overlapping).





(a) Time series data
x_n
1 2 3 4 5 6 7 8 9 10
n
(b) Amplitude rank
$m = 3$
$\tau = 2$
2 3 1
(c) Ordinal symbols
2 3 1
3 1 2
2 1 3
2 3 1
1 3 2
2 1 3
etc.
(d) Network
$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Each ordinal sequence is a wedge of space bounded by hyperplanes bounded by $x_i = x_j$

Associate with each ordinal label π_i the set $Q_i = \{(x_1, x_2, \dots x_w) | x_j \text{ is the } (\pi_i)_j\text{-th largest}\}$. Then

$$\bigcup_i^{w!} Q_i = \mathcal{M} \quad \text{and} \quad Q_j^\circ \cap Q_i^\circ = 0 \text{ iff } i \neq j.$$

Recall that the trajectory of z_0 is observed via an experiment as

$$\dots, \phi_{-2\kappa}(z_0), \phi_{-\kappa}(z_0), \phi_0(z_0), \phi_\kappa(z_0), \phi_{-2\kappa}(z_0), \dots$$

and so we can build the bi-infinite sequence

$$\{s_n\}_{-\infty}^{\infty} = \{s_n \in S_w | \phi_{n\kappa} \in Q_{s_n}\}$$

and then

$$W_{ij} \quad = \quad \mu\left(Q_i \cap \phi_{-\kappa} Q_j\right)$$

for measure μ. W is the adjacency matrix of the weighted network, A the binary version (i.e. $A_{ij} = 1$ iff $W_{ij} \neq 0$ and $A_{ij} = 0$ otherwise).

Entropies

Knowing that $W_{ij} = \mu\left(Q_i \cap \phi_{-\kappa}(Q_j)\right)$ it now reasonable to define the entropy (in one of several ways):

- Permutation entropy (*per symbol*)[11]

$$h_\mu^{\mathrm{PE}} = -\frac{1}{w-1} \sum_{\pi \in S_w} \mathrm{Prob}\,(\pi) \log \mathrm{Prob}\,(\pi)$$

- Conditional permutation entropy

$$h_\mu^{\mathrm{CE}} = -\sum_{\pi \in S_w} \sum_{\xi \in S_w} \mathrm{Prob}\,(\xi = \phi_\kappa(\pi)) \log \frac{\mathrm{Prob}\,(\xi = \phi_\kappa(\pi))}{\mathrm{Prob}\ \pi}$$

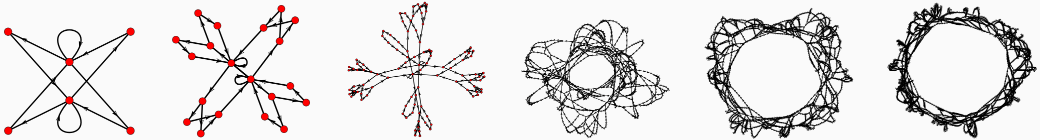
- Topological entropy

$$\log\left[\lambda_{\max}(A)\right] \to h^{\mathrm{TP}}$$

[11] The normalisation here is a convention in the literature.

"Clearly" periodic dynamics of maps of period p will lead to cyclic chain graphs of length (at most) p.

Periodic dynamics of flows is complicated by sampling rate and the necessarily much larger w and hence $w!$ possible symbols.[12]

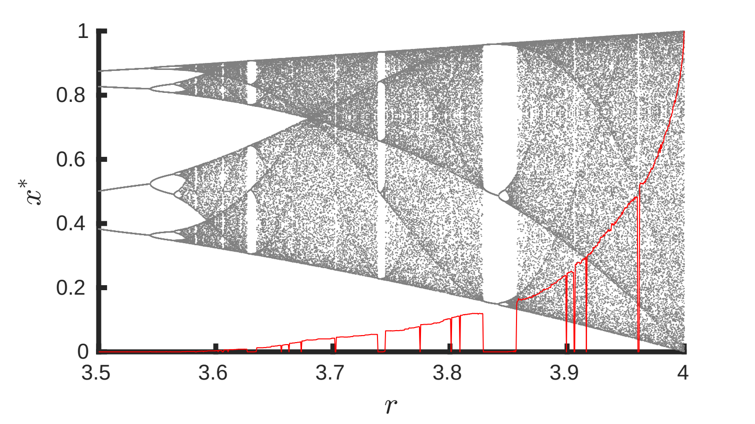


What about chaotic dynamics? Apart from entropy, what more can be said?

[12] Although, typically a much larger number of forbidden patterns too.

Nullity of A

Computation of $\log \mathrm{Null}(A)$ from time series as a function of the bifurcation parameter.



Logistic map $x_{n+1} = rx_n(1 - x_n)$

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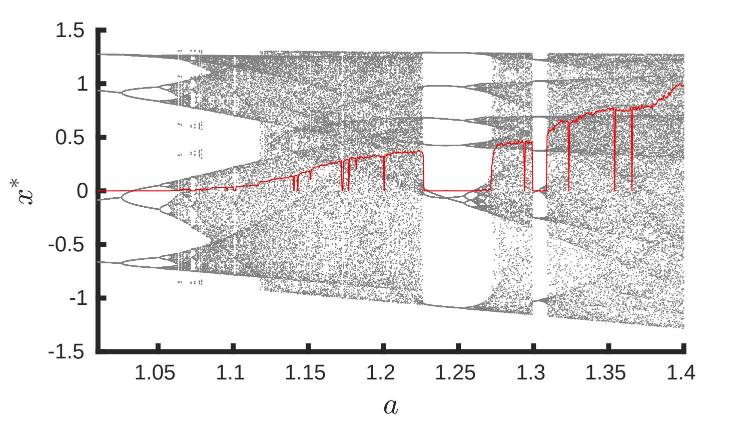

$$\text{Hénon map} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a x_n^2 + y_n \\ 0.3 x_n \end{pmatrix}$$

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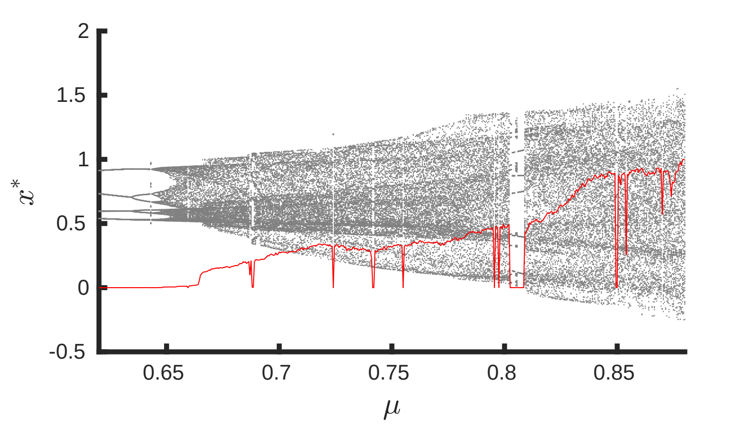


Ikeda Map $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + \mu(x_n \cos\theta_n - y_n \sin\theta_n) \\ \mu(x_n \sin\theta_n + y_n \cos\theta_n) \end{pmatrix}$ where $\theta_n = 0.4 - \frac{6}{1+x_n^2+y_n^2}$

Maximal Lyapunov Exponent

Let $u = (u_1, u_2, \ldots)$ be the right eigenvector of the largest non-negative eigenvalue of the transpose of the transition matrix (i.e. normalise by incoming neighbours) and define the matrix P by

$$P_{ij} = \frac{\mu(Q_i \cap \phi_{-\kappa} Q_j)}{\mu(Q_j)} \cdot \frac{u_j}{u_i}$$

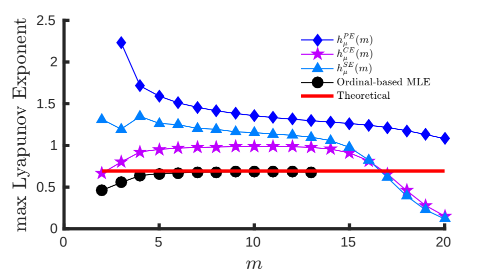
with associated left eigenvector v_p.

The largest Lyapunov exponent[13] is[14]

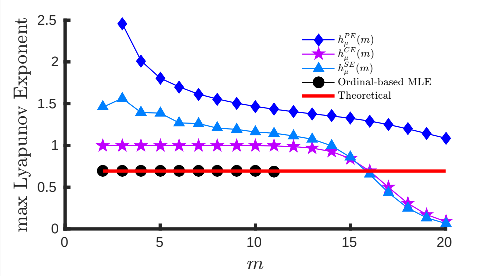
$$\mathrm{LLE} \quad = \quad - \sum_{i,j=1}^{w!} v_{P,i} P_{ij} \log \left(\frac{\mu(Q_i \cap \phi_{-\kappa} Q_j)}{\mu(Q_j)} \right)$$

[13] Erstwhile, the sum of positive Lyapunov exponents.
[14] Froyland. *Nonlinearity* (1999) 12 79-101

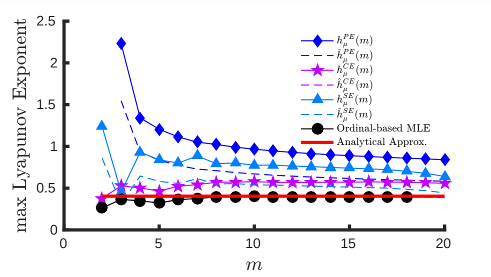
logistic



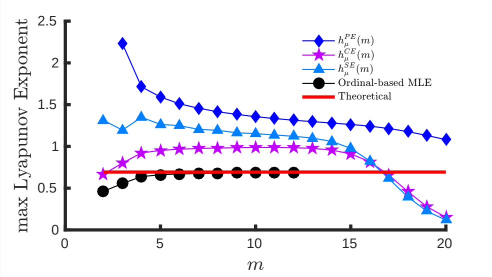
shift



cubic

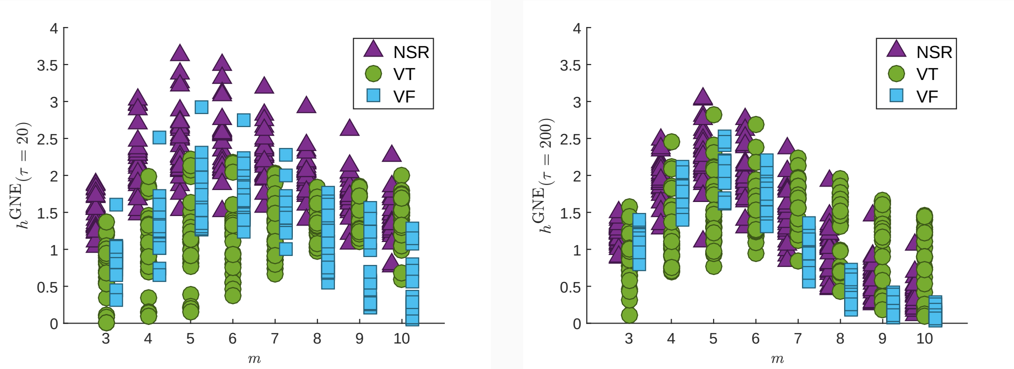


tent



Experiments

Global node entropy for different time scale ($\kappa \equiv \tau$) and window ($w \equiv m$).



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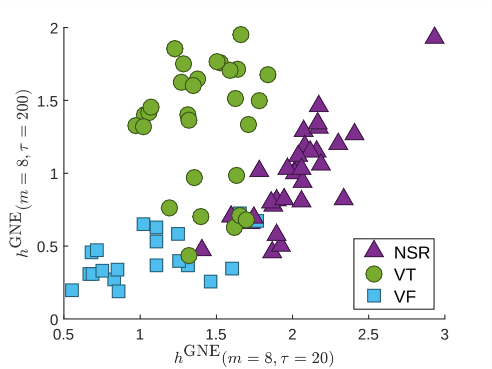
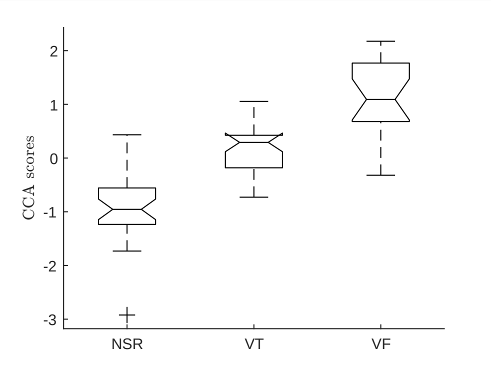
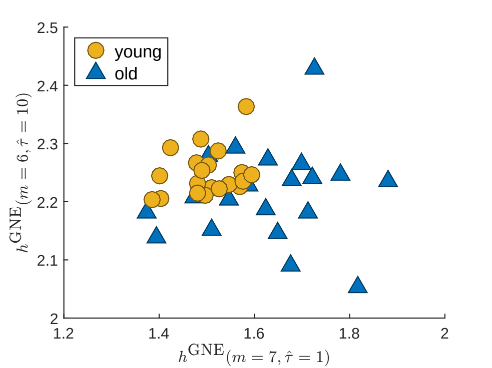
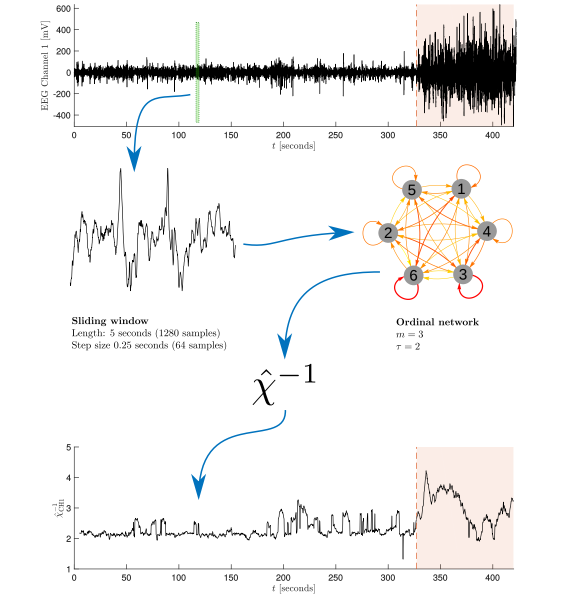


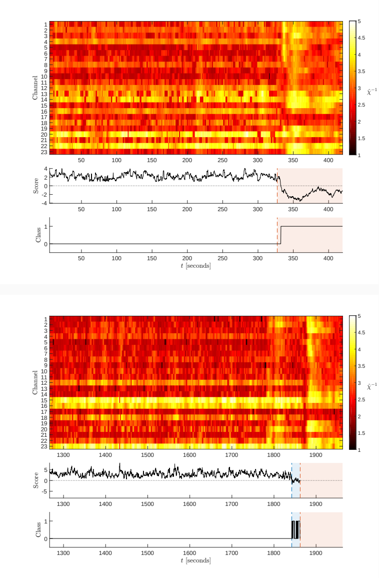


Global node entropy for different time scale ($\kappa \equiv \hat{\tau}$) and window ($w \equiv m$).



Next ... epileptic seizure detection and prediction (Electroencephalogram)





Detecting structural transitions

Likelihood, trajectories and forbidden sequences

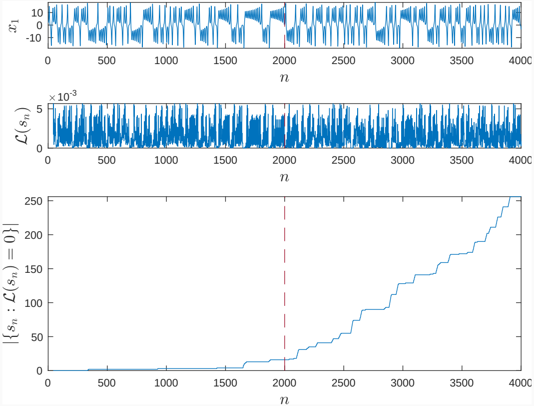
Structural transition and global bifurcation are often preceded by more subtle signatures of change (critical slowing down, broadening of attractor).

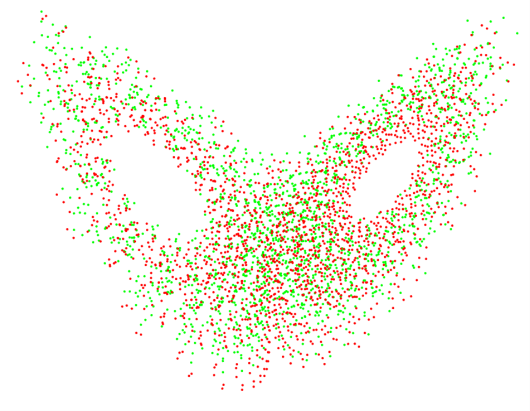
How do we detect these changes?

Can we do more than apply a sliding window and compare invariants?

1. Compute the Likelihood of a newly observed trajectories, given the past. Likelihood should decrease at transitions...
2. Observe rate of forbidden sequences or new symbols. *Rate* of discovery should increase.

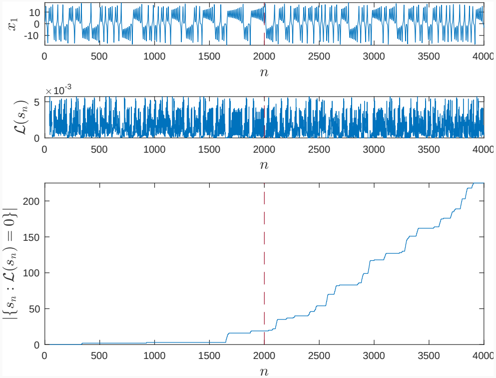
Examples



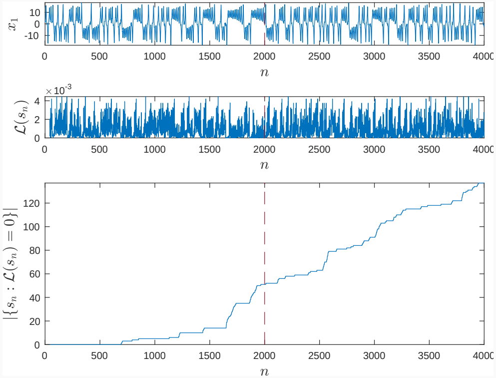


Clean Lorenz, *x* component, $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho_1 = 28$, $\rho_2 = 29$

Examples

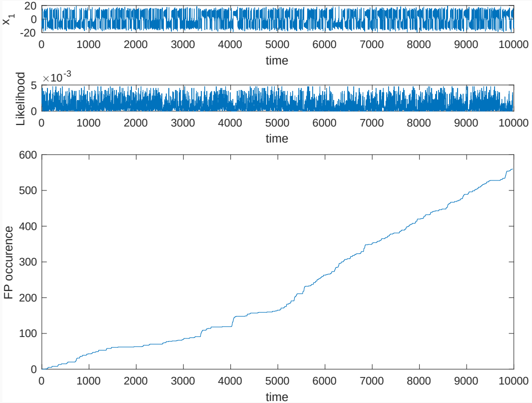


SNR= 30db



SNR=20db

Examples



SNR= 10db

Postamble

References

- M. McCullough, “Nonlinear time series analysis using ordinal networks with select applications in biomedical signal processing”, PhD Thesis, UWA, 2017.
- K. Sakellariou, “Ordinal symbolic network methodologies for nonlinear time series analysis”, PhD Thesis, UWA, 2017.
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- A. Khor and M. Small. “Examining *k*-nearest neighbour networks: superfamily phenomena and inversion” *Chaos* 26 (2016): 043101.
- M. McCullough, M. Small, T. Stemler and H.H.C. Iu. “Time lagged ordinal partition networks for capturing dynamics of continuous dynamical systems” *Chaos* 25 (2015): 053101.
- M. Small. ”Complex Networks from Time Series: Capturing Dynamics”, *IEEE International Symposium on Circuits and Systems Proceedings* (2013): 2509-2512

Recommended reading

- M. McCullough, K. Sakellariou, T. Stemler, and M. Small. “Regenerating time series from ordinal networks” *Chaos* 27 (2017) 035814
- M. McCullough, K. Sakellariou, T. Stemler, and M. Small. “Counting forbidden patterns in irregularly sampled time series Part I - The effects of under-sampling, random depletion and timing jitter” *Chaos* 26 (2016) 123103.
- K. Sakellariou, M. McCullough, T. Stemler, and M. Small. “Counting forbidden patterns in irregularly sampled time series Part II - Reliability in the presence of highly irregular sampling” *Chaos* 26 (2016) 123104.

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