Dynamics of Rewired Networks

Interplay of Nodal Dynamics and Switching Links

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What is the spatiotemporal behaviour of a collection of elemental dynamical units with varying degrees of randomness in their connections?

How does switching the underlying connectivity network influence the spatiotemporal patterns?

What emerges from the interplay of the nodal dynamics and the changing links?

- Dynamically changing links can induce spatiotemporal order
- Time-varying networks can prevent catastrophic blow-ups
- Random rewiring enhances the sensitivity of networks to heterogeneity

Consider a one-dimensional ring of coupled nonlinear maps

The Sites (Nodes, Vertices) are denoted by integers $i = 1, \ldots, N$ where N is the size of the Lattice (Network)

On each site is defined a state variable denoted by $x_n(i)$

Corresponds to the physical variable of interest

The evolution of this lattice in discrete time n under standard nearest neighbour interactions :

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{2}\{x_n(i+1) + x_n(i-1)\}$$

Strength of Coupling : ϵ

The Local On-Site Map could, for instance, be the fully Chaotic Logistic Map:

$$f(x) = 4x(1-x)$$

Now consider the system with its coupling connections rewired randomly in varying degrees

At every update we will connect a fraction p of randomly chosen sites in the lattice to two random sites

That is, we will replace a fraction p of nearest neighbour links by random connections

$p = 0$: corresponds to the usual nearest neighbour interaction

$p = 1$: corresponds to completely random coupling

- Dynamic Rewiring : random links are switched
- Static Rewiring (Quenched) : fixed random links

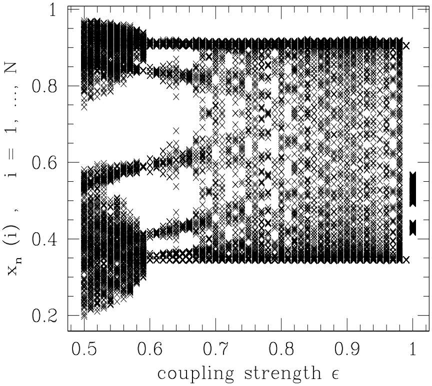
Switched Random Links brings a new relevant time-scale to the problem:

Time-scale at which underlying web of connections changes

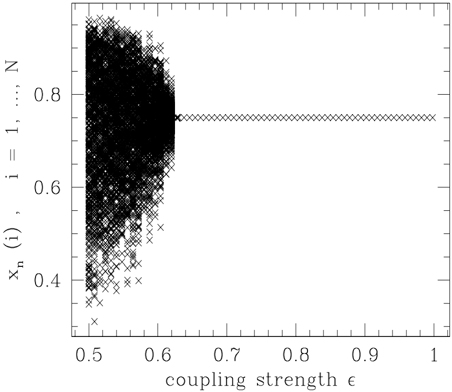
Dynamically Rewired Connections :

First, consider fast network changes

Specifically, at every dynamical update of the nodal state, the links are switched



Coupled logistic maps with regular nearest neighbour connections



Coupled strongly chaotic logistic maps with dynamically changing random connections

(Sinha, 2002)

Dynamic Random Links create a large window in coupling parameter space where a spatiotemporal fixed point state gains stability

Onset of spatiotemporal fixed point : ϵ^*

For completely random coupling $p = 1 : \epsilon^* \sim 0.6$

For all $p > 0$: there is a stable region of synchronized fixed points

i.e. for all finite p, $\epsilon^* < 1$

In the stable region of synchronized fixed points

namely, in the parameter interval $\epsilon^* \leq \epsilon \leq 1$:

All sites are stabilzed at x^*

where x^* is the strongly unstable in the local chaotic map

In order to analyse the stability of the network with all sites at x^*, we will construct an average probabilistic evolution rule for the sites

Mean Field version of the dynamics:

All sites have probability p of being coupled to random sites, and probability $(1-p)$ of being wired to nearest neighbours

Then the averaged evolution equation of a site j is

$$x_{n+1}(j) = (1-\epsilon)f(x_n(j)) + (1-p)\frac{\epsilon}{2}\{x_n(j+1) + x_n(j-1)\} \\ +p\frac{\epsilon}{2}\{x_n(\xi) + x_n(\eta)\}$$

where ξ and η are random integers between 1 and N

Analysis of the probabilistic evolution equation for the fully chaotic logistic map gives the stability condition:

$$\frac{1}{1+p} < \epsilon < 1$$

which yields:

$$\epsilon^* = \frac{1}{1+p}$$

So the range of stability $\mathcal{R} = 1 - \epsilon^*$ is

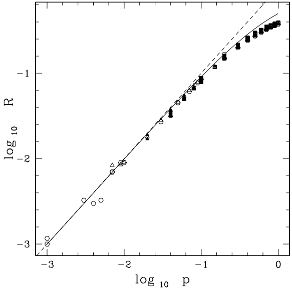
$$\mathcal{R} = 1 - \frac{1}{1+p} = \frac{p}{1+p}$$

For small p ($p << 1$) standard expansion gives

$$\mathcal{R} \sim p$$

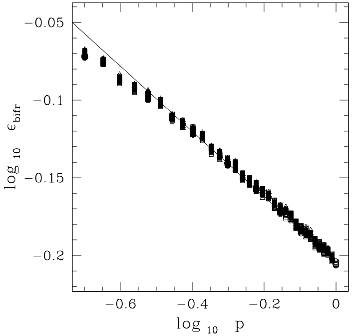
- Regular nearest neighbour coupling ($p = 0$) gives a null range
- Fully random connections ($p = 1$) yields the largest stable range : $R \sim 1/2$

Stable synchronized fixed point range $\mathcal{R}$ vs. random rewiring probability p



Solid line : analytical result; Dotted line: $\mathcal{R} = p$
Points : Simulations (with different lattice sizes $N = 10, 50, 100, 500$)

(Sinha, 2002)



Coupled Tent Maps (squares); Circle Maps (triangles); Logistic Maps (circles)

Similar effects of switched random links were found in networks of coupled maps, where the local maps the exhibited the interesting and potentially useful property of robust chaos

S De and SS, Nonlinear Dynamics (2015)

Static Rewiring

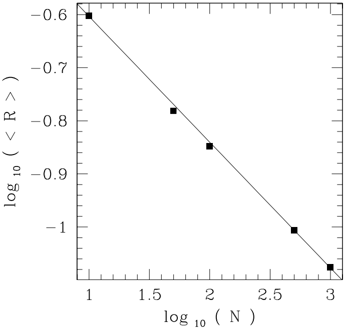
For dynamical rewiring : the synchronization range $\mathcal{R}$ is independent of the size and initial network connections

On the other hand, for static rewiring there is a spread in the values of $\mathcal{R}$ obtained from different realisations of the random connections

The distribution of $\mathcal{R}$ is dependent on the size of network, with average $\mathcal{R}$ scaling with N as :

$$< \mathcal{R} > \sim N^{-\nu}$$

with $\nu \sim 0.24$



Average range $<\mathcal{R}>$ of spatiotemporal synchronization obtained in the case of static random connections with respect to network size N ($p = 1$)

- This behaviour can be understood by a examining the linear stability of the homogeneous solution: $x_n(j) = x^*$ for all sites j at all times n
- Considering the dynamics of small perturbations over the network, one obtains the transfer matrix connecting the perturbation vectors at successive times to be a sum of :
  - $N \times N$ diagonal matrix : with entries $(1 - \epsilon) f'(x^*)$
  - $\epsilon/2 \times \mathbf{C}$: where $\mathbf{C}$ is the connectivity matrix

For example: for $p = 1$, $\mathbf{C}$ is a $N \times N$ sparse non-symmetric matrix with two random entries of 1 on each row

- The minimum of the real part of the eigenvalues of **C**, $\lambda_{\min}$, crucially governs the stability
- Typically $\epsilon^* = 2/\{\lambda_{\min} + 4\}$ when $f'(x^*) = -2$
- Now the values of $\lambda_{\min}$ obtained from different static realisations of the connectivity matrix **C** are distributed differently for different sizes N
- For small N this distribution is broad and has less negative averages (~ -1)
- On the other hand for large N the distribution gets narrower and tends towards the limiting value of -2
- This implies that the range of stability tends to zero for large enough networks under static rewiring

Regular Lattice : Spatiotemporal Chaos

Static Rewiring (Frozen Random Links) : Spatiotemporal Chaos

Dynamic Rewiring (Switched Random Links) : Spatiotemporal Fixed Point

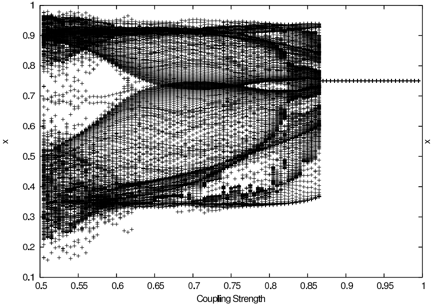
Static to Dynamic Transition

How fast does a network have to rewire in order to induce spatiotemporal order?

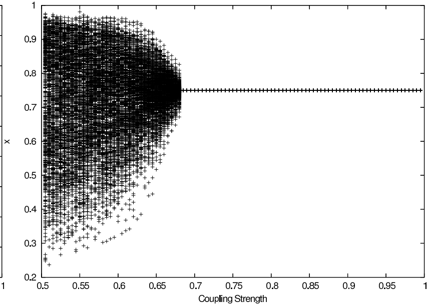
A Mondal, SS, J Kurths, Phys. Rev. E (2008)

V. Kohar, P Ji, A. Choudhary, SS, J. Kurths, Phys. Rev. E (2014)

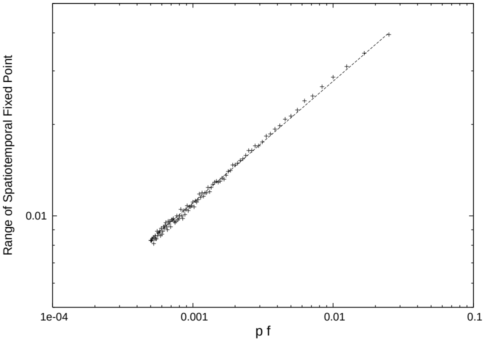
A. Kumar, V. Agrawal and SS, Euro. Phys. J. B (2015) r = 100



r = 1



Range of the spatiotemporal fixed point R vs $p\ f$, where p is the rewiring probability and $f = 1/r$ is the network rewiring frequency



Effective randomness : $\sim p \times f$

When the random connections are quite static, the dynamics of the network is spatiotemporally chaotic

However, when these random links are switched around fast, namely the network is rewired frequently, one obtains a spatiotemporal fixed point over a large range of coupling strengths

Namely, rapidly switched random links enhance spatiotemporal regularity

Evidence of a sharp transition from a globally attracting spatiotemporal fixed point to spatiotemporal chaos as the rewiring frequency is decreased

Ring of Coupled Cells

The single cell consists of a minimal biochemical pathway of three-step reaction sequence

It is a general model of a large variety of functional dynamics observed in cellular systems

with Somdatta Sinha & S. Rajesh

Here, a substrate S1 is converted to another substrate S2 and it is then converted to S3 through a reaction mediated by an enzyme

It is assumed that two different feedback loops governs the regulatory process of the pathway

The first one is a negative feedback, which is provided by the end-product inhibition of S2 by S3 and the other is due to the autocatalytic production of S2 by S3 by the enzyme

$$\frac{dx}{dt} = F(z) - kx$$

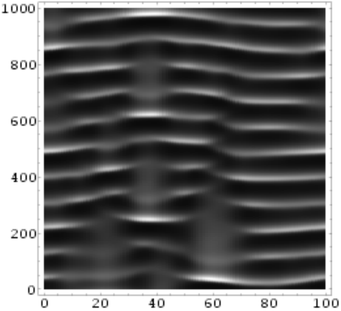
$$\frac{dy}{dt} = x - G(y, z)$$

$$\frac{dz}{dt} = G(y, z) - qz$$

$F(z) = \frac{1}{1+z^4}$; $\qquad G(y, z) = \frac{Ty(1+y)(1+z)^2}{L+(1+y)^2(1+z)^2}$

Coupling : Diffusion of end-product z

$p = 0$ $p = 0.05$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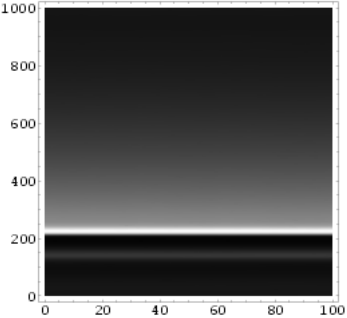
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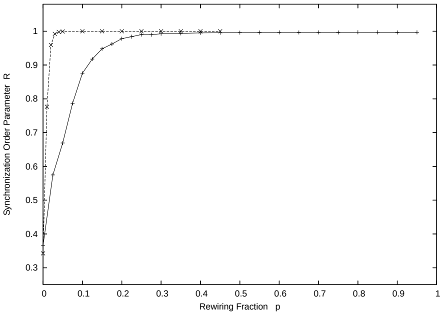
Efficiency of Dynamic Rewiring vis-a-vis Static Disorder

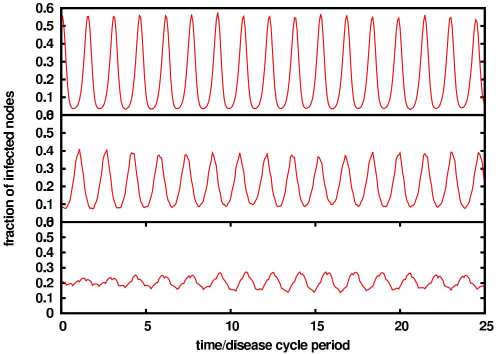


$p \rightarrow 0$ transition to synchronization for Dynamic Rewiring

Infection Spreading on a Dynamically Changing Network

with Vivek Kohar (2013)

Evolution of the number of infected sites



Network rewiring periods $r = 1$ (top), $r = 3$ (middle) and $r = 5$ (bottom)

Central observation:

- Transition from low quasi-fixed state (analogous to endemic infection) to self-sustained oscillations (analogous to periodic outbreak of disease) as links switch more frequently
- Namely, disease cycles get more synchronized, indicating the onset of epidemics, as the underlying network changes more rapidly
- What matters is, not just the topology of the connectivity matrix, but how fast (if at all) the links change

Multiplex Networks

- A large class of engineered and natural systems, ranging from transportation networks to neuronal networks, are best represented by multiplex network architectures, namely a network composed of two or more different layers where the mutual interaction in each layer may differ from other layers.
- We considered a multiplex network where the intralayer coupling interactions are switched stochastically with a characteristic frequency.
- We explored the intralayer and interlayer synchronization of such a time-varying multiplex network.
- To quantify the local stability of complete synchronous states we use the Master Stability Function approach, and for global stability we employ the concept of Basin Stability.

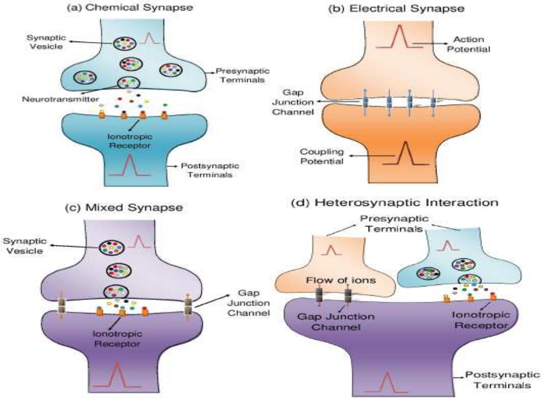
- Interestingly, we clearly find that the higher frequency of switching links in the layers enhances both intralayer and interlayer synchrony, yielding larger windows of synchronization.
- We investigated the robustness of interlayer synchronization against a progressive demultiplexing of the multiplex structure, and we find that for rapid switching of intralayer links, the interlayer synchronization persists even when a large number of interlayer nodes are disconnected.

Time-varying multiplex network: Intralayer and interlayer synchronization,
S. Rakshit, S. Majhi, B.K. Bera, Sudeshna Sinha, and D. Ghosh,

Physical Review E December 2017

Hyper-Networks

- We study the firing patterns and synchronization of coupled Hindmarsh-Rose model neurons with hyper-network architecture, involving two distinct types of networks arising from interactions that occur through the electrical gap junctions and the chemical synapses.
- Specifically, we consider the connections corresponding to the electrical gap junctions to form a small world network, while the chemical synaptic interactions form a unidirectional random network.



(a) The chemical transmission happen through unidirectionally by chemical synapses. (b) Bidirectionally electrical transmission is mediated through the electrical gap junctional channel. (c) Mixed synaptic interaction: simultaneously electric and chemical synaptic interaction is happen between two neuron. (d) Heterosynaptic interaction: one neuron is interacted with two other neuron simultaneously, one connected through gap junction channel and other with chemical synapses.

- All the connections in the hyper-network are allowed to vary in time, by rewiring the links stochastically with a characteristic rewiring frequency.
- We find that the coupling strength necessary to achieve complete neuronal synchrony is lower when the links are switched rapidly.
- Further, the average time required to reach the synchronized state decreases as synaptic coupling strength and/or rewiring frequency increases.

- Further, we investigate the resilience of the synchronous states with respect to increasing network size, and find that synchrony can be maintained up to larger network sizes by increasing rewiring frequency.

- Lastly we find that time-varying links not only promote complete synchronization, but also have the capacity to change the local dynamics of each single neuron.
- Specifically, in a window of rewiring frequency and synaptic coupling strength, we observe that the spiking behavior becomes more regular.

Emergence of synchronization and regularity in firing patterns in time-varying neural hyper-networks
S. Rakshit , B.K. Bera, D. Ghosh and Sudeshna Sinha

Physical Review E May, 2018

Used the *Wu-Chua Conjecture*, which states that for a network of linearly coupled dynamical systems, if the synchronization threshold for a network of size N is ϵ_N^*, and the synchronization threshold for a network of size n is ϵ_n^*, then

$$\epsilon_N^* \gamma_2(N) = \epsilon_n^* \gamma_2(n)$$

where $\gamma_2(N)$ and $\gamma_2(n)$ are the smallest non-zero eigenvalues of the Laplacian matrix for network size N and n respectively.

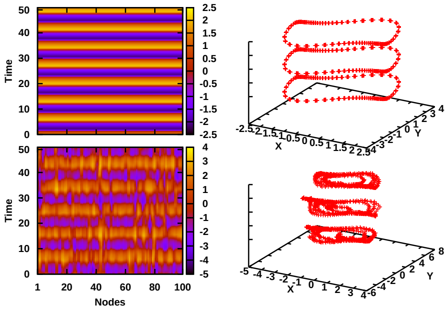
The analytically derived necessary condition based on this conjecture for synchrony was in excellent agreement with numerical results.

This lends further support to the *Wu-Chua Conjecture* as well.

Taming Blow-ups through Dynamic Random Links

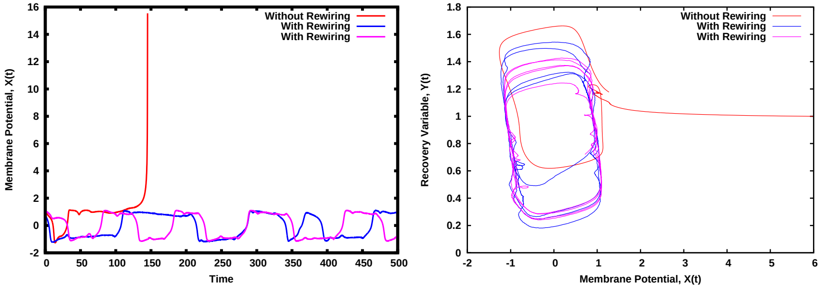
with Anshul Chowdhary & Vivek Kohar,

Scientific Reports (Nature) (2014)

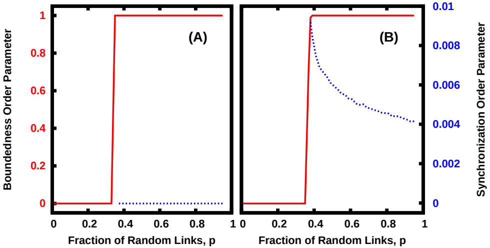


Here coupling strength is significantly greater than critical coupling strength

Blow-up is prevented in both the cases, but faster rewiring leads to a synchronized state (top row) whereas slower rewiring leads to de-synchronized distorted limit cycles (bottom row).



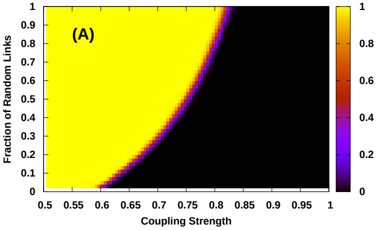
Explosive Growth Prevented

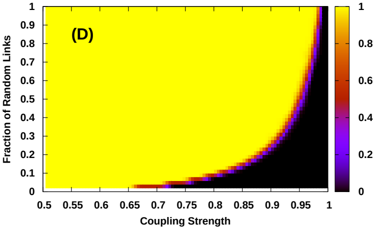


Variation of the Boundedness Order Parameter (solid red curve) and the Synchronization Order Parameter (dotted blue curve), for varying fraction of random links p.

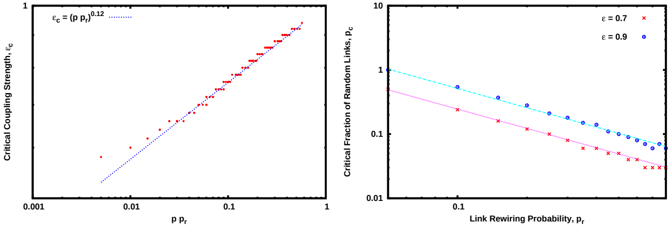
Network rewiring time period (A) $r = 0.01$ and (B) $r = 0.1$.

p_r : Link Rewiring Probability





"Boundedness Order Parameter" : for $p_r = 0.1$ and $p_r = 0.9$



(Left) Variation of the critical coupling strength at which blow-up occurs, ε_c, with respect to the product of the link rewiring probability p_r and fraction of random links p
The points are data from a wide range of p and p_r, and they collapse to the scaling form shown by the line

(Right) Variation of the minimum fraction of random links necessary to prevent blow-ups vs link rewiring probability p_r

Similar phenomena was also found in networks of piecewise linear maps, displaying robust chaos

De & SS, 2015

Random links enhance the sensitivity of networks to heterogeneity

Ultrasensitivity to Non-Uniformity in Networks of Bi-Stable Dynamical Elements

Scalable ultra-sensitive detection of heterogeneity via coupled bistable dynamics,
K P Singh, R Kapri & SS, *Europhysics Letters*, 2012

Verification of scalable ultra-sensitive detection of heterogeneity in an electronic circuit,
V Kohar, A Choudhary, K P Singh and SS, *Eur. Phys. J. ST*, 2013

Random links enhance the sensitivity of networks to heterogeneity,
Pranay Deep Rungta & SS, *Europhysics Letters*, Dec 2015

Probe the evolution of N bistable elements, coupled to k neighbours, given by:

$$\dot{x}_i = G(x_i) + a_i + C \left[\frac{1}{k} \Sigma_{j=1}^{k} (x_j - x_i) \right] + b$$

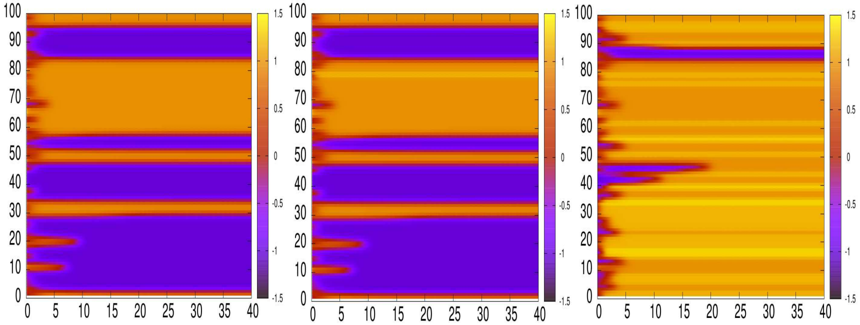
- Interaction term is a sum over the k neighbors of the i^{th} node, and gives the local field experienced by each element
- C : strength of coupling
- Local dynamics is determined by $G(x_i)$ which is a generic nonlinear function giving rise to a bistable potential
- Parameter b is a global bias, common to the elements, and can be used as a "control lever" to tip the collective behaviour of the system to different patterns

- Local parameter a_i may differ from element to element, leading to heterogeneity in the system
- This local parameter determines the location and depth of the stable states of the nodal dynamics in the uncoupled case
- N_0 is the number of elements with $a_i = 0$, and $N_1 = N - N_0$ is the number of elements with $a_i = 1$.

The principal question is: how sensitive are collective dynamical features, which can be considered as the output of the system, to small inhomogeneity?

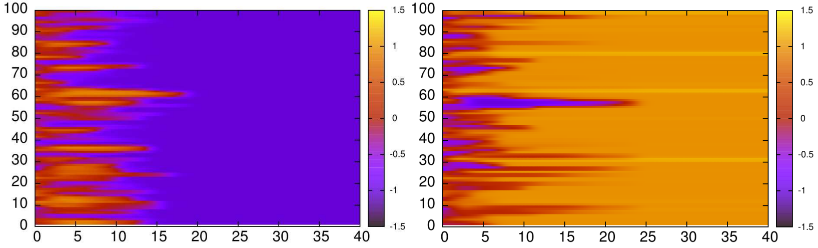
Namely can the collective response, such as the ensemble average $\langle x \rangle$, detect and reflect small differences in a_i?

Regular Networks



Number of sites N_1 with $a_i = 1$ is (left to right) 0, 1, 30

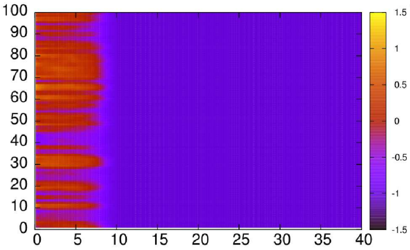
Network with Static Random Links

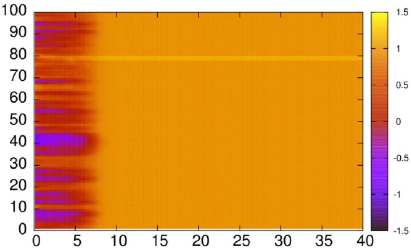


Number of sites N_1 with $a_i = 1$ is (left) 0 and (right) 1

Fraction of random links $p = 0.8$

Network with Dynamically Changing Random Links

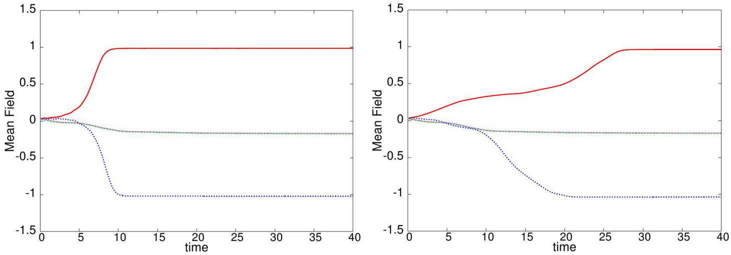




Number of sites N_1 with $a_i = 1$ is (left) 0 and (right) 1

Fraction of random links $p = 0.8$

Evolution of the Mean-Field : ultrasensitivity to heterogeneity



For the case of random links the presence of just one distinct element in a network of size 100 swings the collective field from around -1 to 1, while for a regular ring of bistable elements the presence of 1 distinct element changes the mean field incrementally.
Response time of dynamically changing network is more rapid

Outlook

- Dynamically changing links can induce spatiotemporal order
- Time-varying networks can prevent catastrophic blow-ups
- Random rewiring enhances the sensitivity of networks heterogeneity