Explosive death in coupled oscillators

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Outline

1. Coupled oscillators
2. Effect of interaction: Synchronization and suppression of oscillations
3. Phase transition: First and second order
4. Explosive death
5. Conclusion

Coupled Oscillators

- Provide a useful paradigm for the study of collective behavior of complex systems
- Used to describe a broad class of Self-Organizing phenomena
  - Biological Systems

    Circadian Rhythms, Genetic oscillators
  - Chemical and Physical Systems

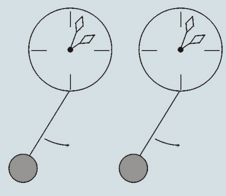
    Coupled Chemically Reacting Cells, Josephson Junction Circuits
- Shows a plethora of complex collective behaviors such as:
  - Synchronization
  - Suppression of oscillation

In Synchronization process collective dynamics of coupled systems changes from incoherence to coherence.

Synchronization depends on nature and strength of the interaction.

Examples:-

- Coupled pendulum
- Clapping of audience
- Fireflies



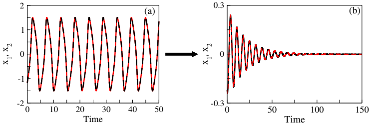
A. S. Pikovsky, M. Rosenblum and J. Kurths. Synchronization: A Universal concept in nonlinear science (2001).

Suppression of Oscillation

Suppression of oscillation can be classified into two classes:

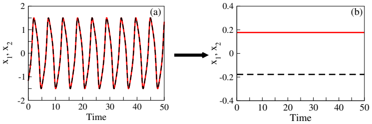
Amplitude death (AD):- In AD, unstable steady state of uncoupled oscillators is stabilized due to interaction.

AD state is important in systems where suppression of unwanted oscillation is necessary, e.g. in laser and mechanical engineering systems.



Oscillation death (OD):- In case of OD, a coupling dependent steady state is stabilized.

OD has significant application in biological systems, e.g.:–Cell differentiation, genetic oscillators etc.



Phase Transition

- A phase transition is a process in which one or more properties of systems change, drastically after a minimal change in one of their control parameter.
- Phase transition can be classified into two classes:
  - First order phase transition: It is characterized by the co–existence of different phases in the proximity of transition point of the control variable.

    This transition exhibits abrupt and discontinuous behavior near the criticality and leads to irreversibility in the system with presence of hysteresis.

    Examples: Melting of ice, boiling of water, super–cooling and super–heating transition.
  - Second order phase transition: Second-order phase transition (also called continuous phase transitions) is reversible, and correspond to order parameters displaying a continuous behavior near criticality.

    Examples: A ferromagnetic transition, superconducting transition etc.

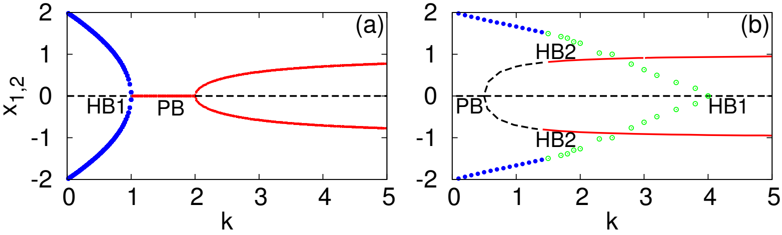
S. Boccaletti, V. Latora, Y. Moreno, M. Chevez, and D. U. Hwang, *Phys. Rep.* **424**, 175 (2006).

Explosive Transition

- Explosive Percolation (EP): Explosive percolation was discovered in 2009, which refers to the sudden formation of an inter-connected giant cluster in a growing network, whose size is comparable with that of the whole network.
- Explosive synchronization (ES): It refers to the abrupt emergence of a collective synchronized state in the coupled oscillators.
- It was first reported by *Gardeñes et.al* (2011), in a generalized Kuramoto model on a scale-free network when the natural frequency and degree of the j^{th} oscillator in the network are correlated, *i.e.*, $\omega_j = k_j$.
- ES is only observed in phase synchronization either in the Kuramoto type model or coupled Rössler system.

J. Gómez–Gardeñes, S. Gómez, A. Arenas, and Y. Moreno, *Phys. Rev. Lett.* **106**, 128701 (2011).
H. Bi, X. Hu, X. Zhang, Y. Zou, Z. Liu, and S. Guan, *Europhys. Lett.* **108**, 50003 (2014).

Generally the transition form oscillatory to steady state is second order.



A first order transition from oscillatory to steady state (termed as Explosive oscillation death (ED)) is first observed in coupled Stuart–Landau oscillators, with frequency weighted coupling (2014).

H. Bi, X. Hu, X. Zhang, Y. Zou, Z. Liu, and S. Guan, *Europhys. Lett.* **108**, 50003 (2014).

Explosive oscillation death in coupled Stuart–Landau oscillators

The dynamics of N Stuart–Landau oscillators with frequency weighted coupling,

$$\dot{z}_j = (a + i\omega_j - |z_j|^2)z_j + \frac{K|\omega_j|}{N}\sum_{i=1}^{N}(z_i - z_j).$$

Here $j = 1, 2..., N$ is the index of oscillators. $z_j = x_j + iy_j$ is the complex amplitude of j^{th} oscillators.

ω_j is the natural frequency of the j^{th} oscillators and K is the uniform coupling strength.

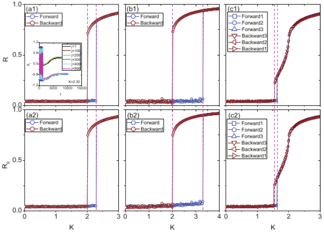
H. Bi, X. Hu, X. Zhang, Y. Zou, Z. Liu, and S. Guan, *Europhys. Lett.* **108**, 50003 (2014).

The Two order parameters $R (0 \leq R \leq 1)$ and $R_\theta (0 \leq R_\theta \leq 1)$ are defined as:

$$Re^{i\psi} = \sum_{j=1}^{N} Z(t)/N$$

$$R_\theta e^{i\phi} = \sum_{j=1}^{N} e^{i\theta_j}/N$$

Here, R represents the amplitude and R_θ characterizes the phase coherence of the system.



Characterization of *ED* for three typical frequency distribution, triangle (left column), Lorentzian (middle column), and uniform (right column), respectively.

This work suggest that apart from the Kuramoto model, an explosive transition might occur in a certain dynamical system involving amplitudes.

Van der Pol (VdP) oscillators

Consider N Van der Pol oscillators coupled through mean–field diffusion:

$$\begin{aligned}\dot{x}_i &= y_i + k(Q\bar{x} - x_i),\\ \dot{y}_i &= b(1 - x_i^2)y_i - x_i,\end{aligned}$$

where:–

- $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$.
- b is the system parameter
- k is the strength and Q with $0 \leq Q < 1$ is the intensity of the mean–field interaction of the coupled system .

U. Verma, A. Sharma, N. Kamal, J. Kurths, and, M. Shrimali, **Scientific Reports 7**, 7936 (2017).

Order Parameter

We define an order parameter $A(k)$, which is the normalized average amplitude of all the oscillators:

$$A(k) = \frac{a(k)}{a(0)}$$

where $a(k) = (\sum_{i=1}^{N} \langle x_{i,max} \rangle_t - \langle x_{i,min} \rangle_t)/N$, is the average amplitude.

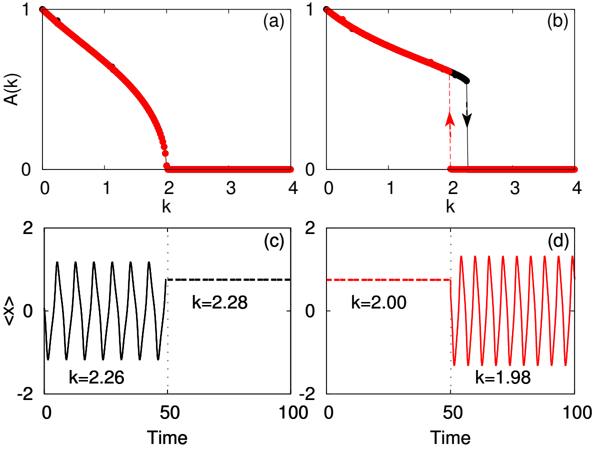
$A(k)$ measure the state of all the oscillators in the coupled systems:

- For oscillatory state $A(k) > 0$ and
- For death state $A(k) = 0$.

Continuation diagram

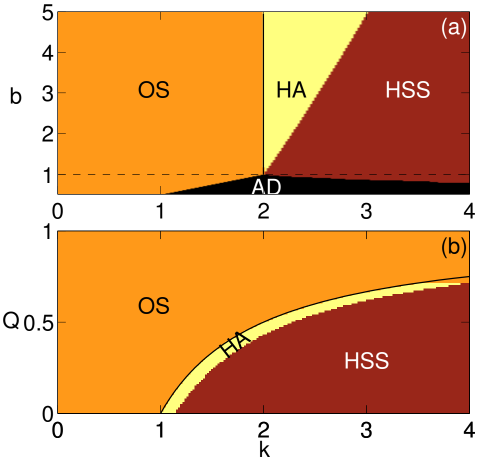
$Q = 0.5$ and $N = 100$

$b = 1$ $b = 2$



U. Verma, A. Sharma, N. Kamal, J. Kurths, and, M. Shrimali, **Scientific Reports 7**, 7936 (2017).

(a) $Q = 0.5$



(b) $b = 2.0$

U. Verma, A. Sharma, N. Kamal, J. Kurths, and, M. Shrimali, **Scientific Reports 7**, 7936 (2017).

Analytical results

The HSS for this system are, $(x_i^* = x^*, y_i^* = y^*, \forall i = 1, \ldots, N)$, where

$$x^* = \sqrt{1 - \tfrac{1}{bk(1-Q)}} \text{ and } y^* = \sqrt{k^2(1-Q)^2 - \tfrac{k(1-Q)}{b}}.$$

The condition for stabilization of HSS is,

$$k = \frac{1}{1-Q}.$$

This equation give the backward transition point.

Lorenz System

Consider N Lorenz system coupled through mean–field diffusion:

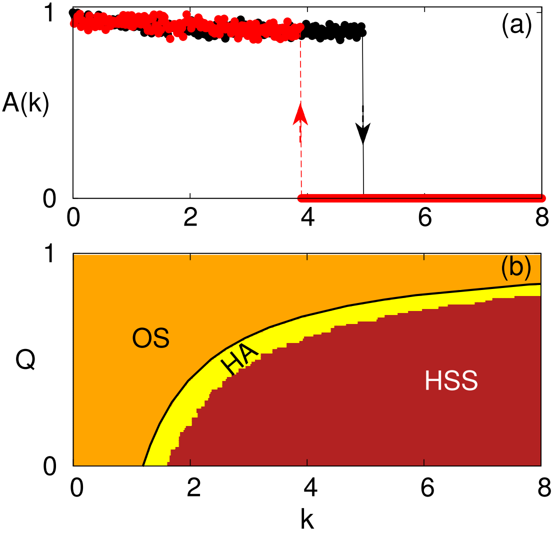
$$\begin{aligned}
\dot{x}_i &= \sigma(y_i - x_i) + k(Q\bar{x} - x_i), \\
\dot{y}_i &= (r - z_i)x_i - y_i, \\
\dot{z}_i &= x_i y_i - b z_i,
\end{aligned}$$

where:–

- $i = 1, 2, ..., N$.
- x_i, y_i and z_i are the state variables of the coupled Lorenz oscillators.
- $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$ are the system parameters.
- k is the strength and Q with $0 \leq Q < 1$ is the intensity of the mean–field interaction.

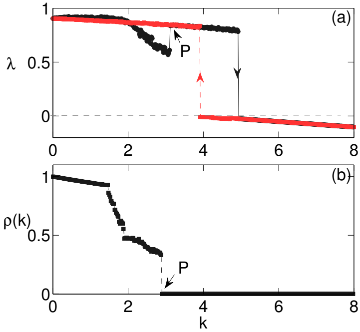
Continuation diagram and phase diagram

$Q = 0.7$ and $N = 100$



Lyapunov exponent and synchronization error

$Q = 0.7$



$\rho(k) = \frac{S(k)}{S(0)}$, as a function of k, where,

$$S(k)=\left\langle \sqrt{\frac{1}{3N}\sum_{i=1}^{N}\left[(x_i-\bar{x})^2+(y_i-\bar{y})^2+(z_i-\bar{z})^2\right]} \right\rangle_t.$$

Homogeneous steady state(HSS) of the coupled Lorenz systems are ($x_i^* = x^*, y_i^* = y^*, z_i^* = z^*, \forall \quad i = 1, \ldots, N$), where,

$x^* = \pm\sqrt{\frac{b\sigma(r-1)-bk(1-Q)}{\sigma+k(1-Q)}}$, $y^* = \frac{\{\sigma+k(1-Q)\}x^*}{\sigma}$ and $z^* = \frac{x^* y^*}{b}$.

The backward transition point of the coupled systems is

$$k = \frac{u^2 + 2u\sigma - v - \sqrt{m}}{2u(Q-1)},$$

where, $m = b^4 - 4u^2 x^{*2} + 4u\sigma x^* y^* + (1+v)^2 - 2b^2(1+v)$.

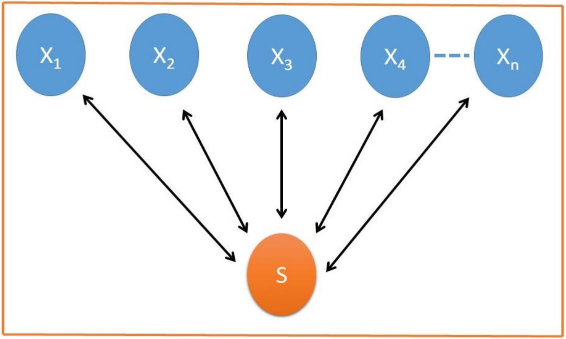
Results

- In the case of Van der Pol oscillators, Explosive transition crucially depends on the damping coefficient.
- Both forward and backward transitions are independent of the size of networks.
- Linear stability analysis correctly predict the backward transition point.

Explosive Death in Environmentally Coupled Systems

Environmental Coupling

Real world systems with many interacting units operating at various label of complexity, but such systems are rarely isolated and they interact with their environment.



$X_{1,2}$ and S represent the oscillators and the environment respectively.

Environmental Coupling

- *S* represents particle species that can freely diffuse in the environment and allows individual oscillators to communicate with each other
- The specific realizations of *S* depends on the context
  - In the BZ reaction, it represents chemical species that diffuse between autocatalytic beads
  - In metabolic oscillations, it represents common metabolites that diffuse between cells
  - In synthetic bacteria, it is the concentration of the autoinducer signaling molecules in the medium
- Synchronization, phase-flip transition, suppression of oscillations, and revival of oscillations are some basic phenomena that occurs in the coupled systems due to environmental coupling.

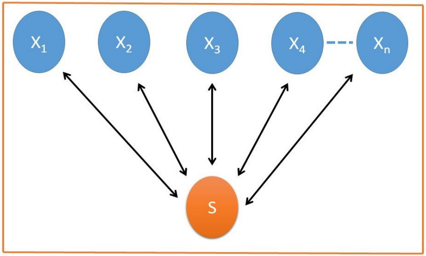
Explosive Death in Environmentally Coupled Systems

Consider N Stuart–Landau oscillators coupled through a common environment,

$$
\begin{aligned}
\dot{x}_i &= (1 - x_i^2 - y_i^2)x_i - \omega y_i + k(s_i - \alpha x_i), \\
\dot{y}_i &= (1 - x_i^2 - y_i^2)y_i + \omega x_i, \\
\dot{s}_i &= -\gamma s_i - k\bar{x}_i,
\end{aligned}
$$

where:–

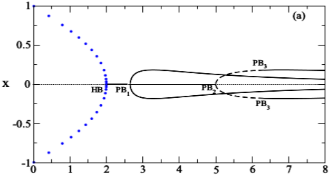
- ω is the frequency of oscillators
- s_i is the state variable of the environment with decay rate γ
- α is the feedback term with $0 \leq \alpha < 1$ which controls the diffusion rate.



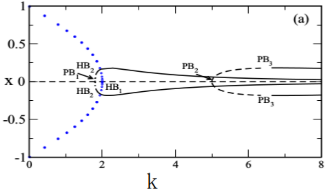
U. K. Verma, A. Sharma, N. K. Kamal, and, M. D. Shrimali, **physics letter A 382**, 2122 (2018).

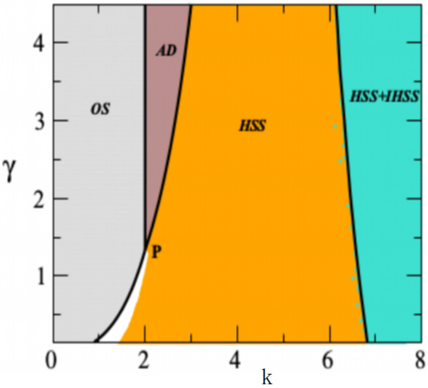
$\alpha = 1.0$ and $\omega = 2$

$\gamma = 3$

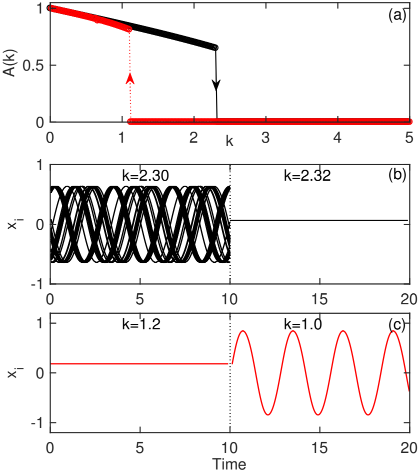


$\gamma = 1$





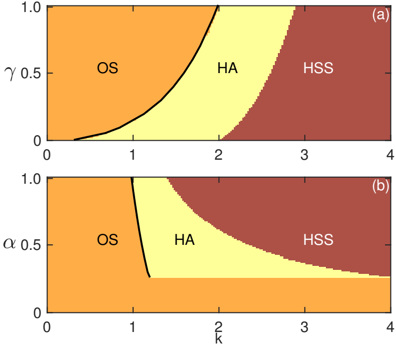
Continuation Diagram and Time Series



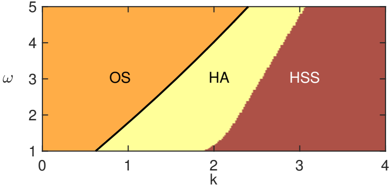
$\alpha = 0.5$, $\gamma = 0.2$ and $\omega = 2$.

Phase Diagram

(a)$\alpha = 0.5$, (b) $\gamma = 0.2$ and $\omega = 2$



$\alpha = 0.5, \gamma = 0.2$



The condition of backward transition point of the coupled system is,

$$\gamma = \frac{-n + \sqrt{n^2 - 4m(4\beta + \alpha k - 2)}}{2(4\beta + \alpha k - 2)}$$

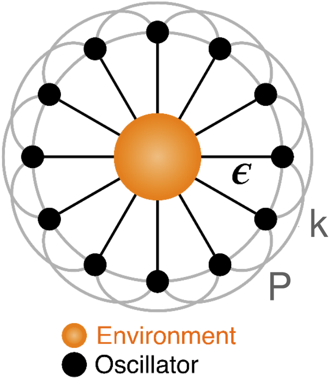
Results

- The phenomena of explosive death occurs due to co-existence of stable oscillatory state along with stable steady state over a range of parameter.
- Hysteresis area crucially depends on the decay rate of the environment and feedback factor in the diffusion term.
- Linear stability analysis correctly predict the backward transition point.

Explosive Death in Complex Network

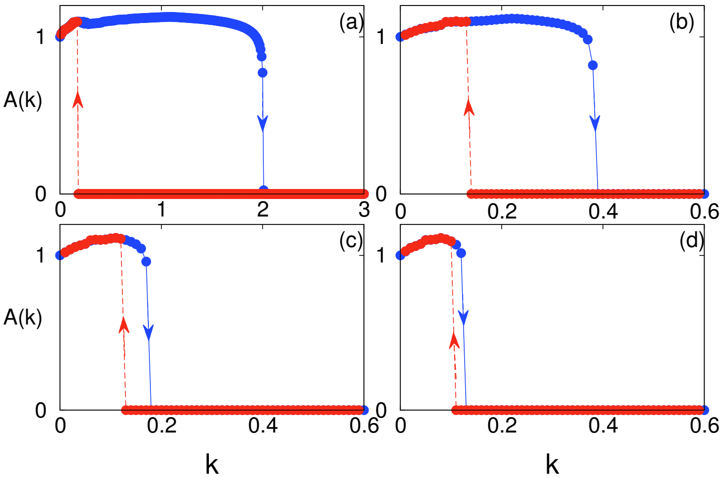
Consider a network of N identical chaotic Rössler oscillators, which are interacting directly with $2P$ neighbours as well as indirectly through a common environment.

$$
\begin{aligned}
\dot{x}_i &= -y_i - z_i + \frac{k}{2P}\sum_{j=i-P}^{i+P}(x_j - x_i) + \epsilon s,\\
\dot{y}_i &= x_i + r y_i,\\
\dot{z}_i &= b + z_i(x_i - c),\\
\dot{s} &= -\gamma s - \frac{\epsilon}{N}\sum_{i}^{N} x_i,
\end{aligned}
$$



P= number of nearest neighbors in each direction

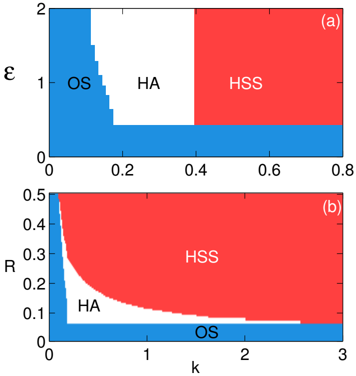
ϵ=environmental coupling strength, k= diffusive coupling strength



(a) $P = 8$, (b) $P = 20$, (c) $P = 32$, and (d) $P = 40$

Phase Diagram

Coupling range $R = \frac{P}{N}$



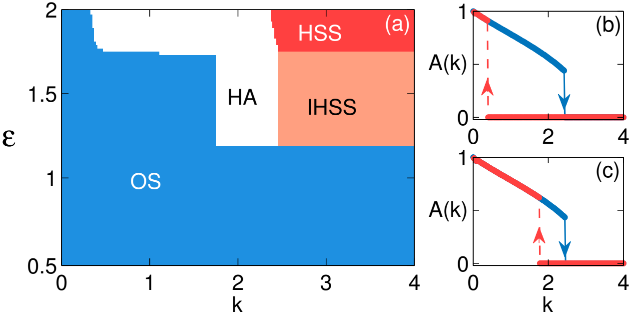
(a) $P = 20$ (b) $\epsilon = 1.5$

Consider a network of N identical Van der Pol oscillators, which are interacting directly with $2P$ neighbours as well as indirectly through a common environment.

$$\begin{aligned}
\dot{x}_i &= y_i + \frac{k}{2P}\sum_{j=i-P}^{i+P}(x_j - x_i) + \epsilon s, \\
\dot{y}_i &= a(1 - x_i^2)y_i - x_i, \\
\dot{s} &= -\gamma s - \frac{\epsilon}{N}\sum_{i}^{N} x_i,
\end{aligned}$$

where $a = 0.8$ is a system parameter.

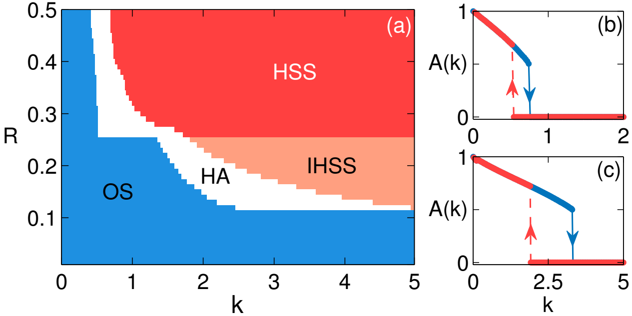
$P = 20$



(b) $\epsilon = 1.8$, (c) $\epsilon = 1.5$

Phase Diagram and Continuation Diagram

$\epsilon = 1.5$



(b) $P = 40$, (c) $P = 16$

Results

- The explosive death transition crucially depends on the environmental coupling strength and architecture of the network.
- In both chaotic and periodic systems, for weak environmental coupling strength coupled systems show only oscillatory behavior, but for strong coupling strength coupled systems exhibit a first-order transition from oscillatory state to death state.
- For small coupling range R, system exhibit only oscillatory behavior but explosive death transition is observed for $R > 0.08$.

Explosive death in network of Van der Pol oscillators with conjugate coupling

Van der Pol oscillators

Consider the network of N coupled Van der Pol (VdP) oscillators with conjugate coupling:

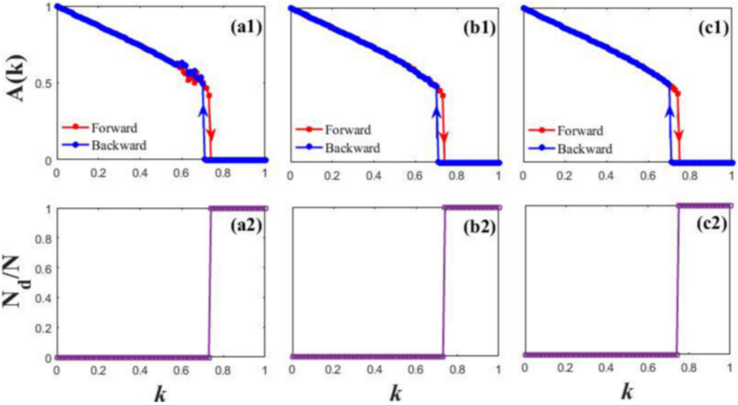
$$\begin{aligned}
\dot{x}_i &= y_i + \frac{k}{d_i} \sum_{j=1, j\neq i}^{N} g_{ij}(y_j - x_i), \\
\dot{y}_i &= a(1 - x_i^2)y_i - x_i + \frac{k}{d_i} \sum_{j=1, j\neq i}^{N} g_{ij}(x_j - y_i),
\end{aligned}$$

where b is a system parameter.
g_{ij} is the the connection matrix.
$\sum_{j=1}^{N} d_{i,j}$ accounts for the degree of the i^{th} oscillator.

N. Zhao, Z. Sun, X. Yang, and W. Xu, **Physical Review E 97**, 062203 (2018).

Continuation Diagram

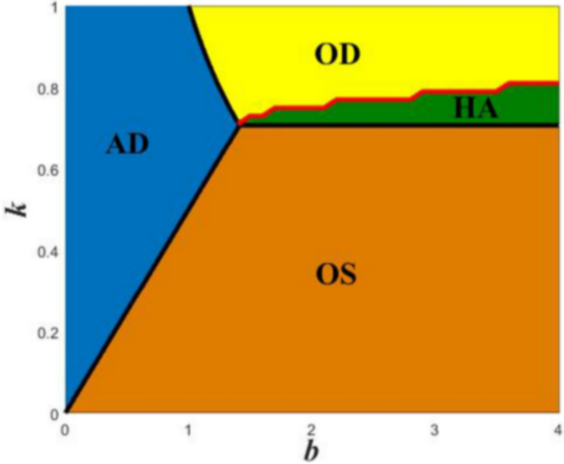
star Network | nearest-neighbor network | all-to-all network



$b = 2$ and $N = 100$

Phase Diagram

For Nearest-neighbor network



Results

Network structure	Forward transition (k_f) point (k_f)	Backward transition (k_b) point (k_b)
Star	$k_f \approx 0.74$	$k_b \approx 0.71$
Nearest-neighbor	$k_f \approx 0.74$	$k_b \approx 0.71$
All-to-all	$k_f \approx 0.75$	$k_b \approx 0.71$

- Explosive transition crucially depends on the damping coefficient.
- Forward transition point depends on the network topologies.
- Backward transition point is independent of the network topologies and the damping coefficient.

Conclusion

- Suppression of oscillations in coupled limit cycle and chaotic oscillator is studied from phase transition point of view.
- The transition from an oscillatory to death state or vice versa can be first order in an identical systems of oscillators.
- Coupled limit cycle and chaotic oscillators are studied with direct and indirect coupling with different network topology.
- Linear stability analysis correctly predict the backward transition point.
- The transition is general and has been found in coupled systems where oscillatory solution co–exist with coupling dependent homogeneous steady state.


THANK
YOU!