Light-cone spreading of perturbations in the Heisenberg spin chain

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Outline

- Chaos in classical systems.
- Propagation of chaos in many-body systems.
- The out-of-time-ordered-correlator (OTOC) as a probe of quantum chaos and of ballistic propagation of information.
- Numerical results for classical Heisenberg spin-chain.

Chaos in classical systems

A butterfly fluttering its wings in Beijing causes a tornado in Kansas!!

Classical Chaos - Sensitivity to initial conditions

Consider a dynamical system with deterministic time-evolution

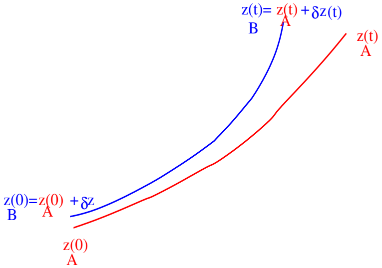
$$\frac{dz_x}{dt} = f_x(\mathbf{z}) \ , \quad x = 1, 2, \ldots, N \ .$$

Consider two different initial conditions

$$z^A(0) \ \text{and} \ \ z^B(0) = z^A(0) + \delta z(0) \ .$$

After time t let

$$\delta z(t) = z^B(t) - z^A(t)$$



Definition of chaos — at large t

$$\lim_{\delta z(0) \to 0} \frac{|\delta z(t)|}{|\delta z(0)|} \sim e^{\lambda(\mathbf{z})t}$$

with $\lambda > 0$ — LYAPUNOV EXPONENT

Propagation of chaos

Suppose that the initial perturbation is localized in space

$$\delta z_x(0) = \epsilon \delta_{x,0} \ .$$

We expect that the differences in phase-space variables at any point in space should eventually diverge with time. Thus

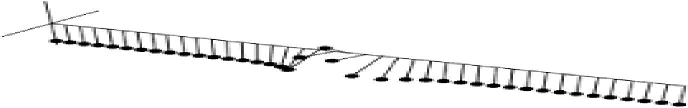
$$\lim_{\delta z_0(0)\to 0} \frac{|\delta z_x(t)|}{|\delta z_0(0)|} \sim \phi(x,t) e^{\lambda(\mathbf{z})t} \ .$$

Some questions:

- How long does it take before the perturbation is felt at point x ?
- How does the spatio-temporal evolution of the perturbation take place?

Spread of perturbations in a Hamiltonian system

Consider a chain of coupled oscillators (e.g coupled pendula).



$z_x = (q_x,\ p_x)$ — (position, momentum) of x^{th} particle. $[x = -N/2, \ldots, N/2]$.

- Disturb system locally.
- The spread of the perturbation can be expressed as a Poisson bracket:

$$\frac{\partial q_x(t)}{\partial q_0(0)} = \{q_x(t), p_0(0)\} \qquad \{A, B\} = \sum_x \frac{\partial A}{\partial q_x}\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}\frac{\partial B}{\partial q_x}$$

- Linear response $\langle \partial q_x(t)/\partial q_0(0)\rangle \sim \langle q_x(t) q_0(0)\rangle$ —- correlation functions
- Chaos and non-linear response — $\left\langle \left(\frac{\partial q_x(t)}{\partial q_0(0)}\right)^2 \right\rangle$

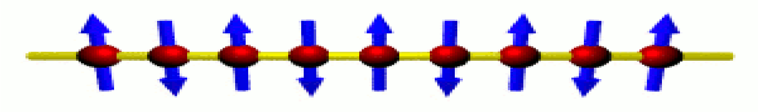
Chaos in quantum systems

- $z \to \psi$ — No chaos since linear dynamics.
- Usual characterization of quantum chaos: level-spacing statistics
- Maldacena,Shenker,Stanford — replace Poisson bracket by commutator. Hence look at

$$-\langle [q_x(t), p_0(0)]^2 \rangle / \hbar^2 ,$$

  $< ... >$ represents expectation over pure or thermal state.
- No long time divergence but perhaps the short-time growth shows a exponential growth regime and one can extract a Lyapunov exponent.
- Ehrenfest time and scrambling time.
- Ballistic spread characterized by butterfly velocity, light-cone velocity, Lieb-Robinson velocity.

Classical Heisenberg spin chains



Heisenberg spins $\mathbf{S} = (S^x, S^y, S^z)$ — unit three-dimensional vectors.
Hamiltonian given by

$$H = -\sum_{\ell=1}^{N} \mathbf{S}_\ell . \mathbf{S}_{\ell+1} \ .$$

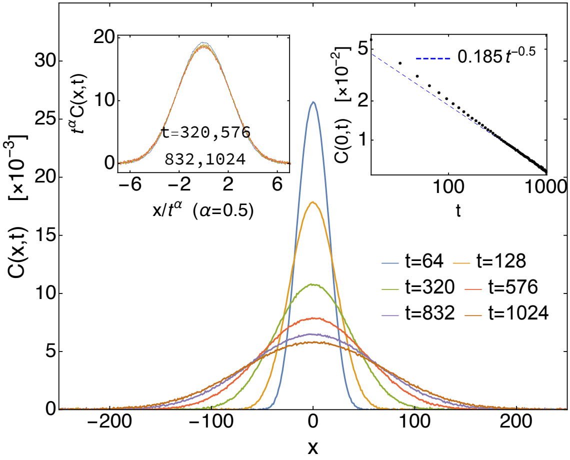
Equations of motion [Structure of usual Hamiltonian dynamics]

$$\dot{\mathbf{S}}_\ell = \mathbf{S}_\ell \times \mathbf{B}_\ell^{\text{eff}} = \{\mathbf{S}_\ell, H\} \ .$$

Linear response at high temperatures

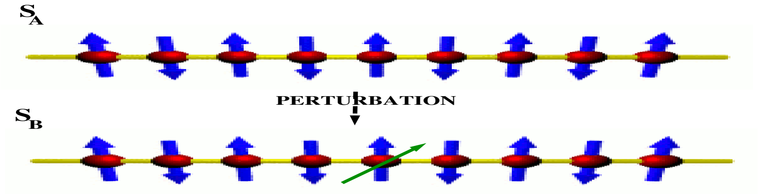
Expect diffusive hydrodynamic equations.

Spreading of a spin perturbation $\quad C_{ss}(x,t) = \langle \mathbf{S}_x(t).\mathbf{S}_0(0)\rangle_{eq}$.



Energy correlations are similar.

Chaos and ballistic propagation in the *XXX* Heisenberg chain at high temerature



- Consider finite perturbation $\delta \mathbf{S}_0(0) = \epsilon \hat{\mathbf{n}}$ and look at

$$D(x,t) = \langle [\delta \mathbf{S}_x(t)]^2 \rangle / 2 = 1 - \langle \mathbf{S}_x^A(t) . \mathbf{S}_x^B(t) \rangle$$

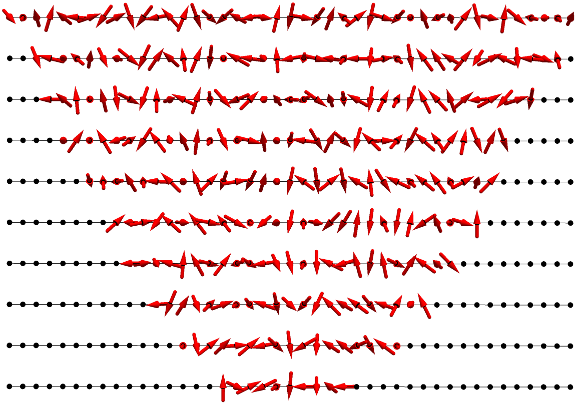
.

- Clearly $D(x,t) \to 1$ at large times
- However in the limit $\epsilon \to 0$

$$\left\langle \left(\frac{\partial \mathbf{S}_x(t)}{\partial \mathbf{S}_0(0)} \right)^2 \right\rangle = \frac{D(x,t)}{\epsilon^2} \sim e^{2\lambda t}$$

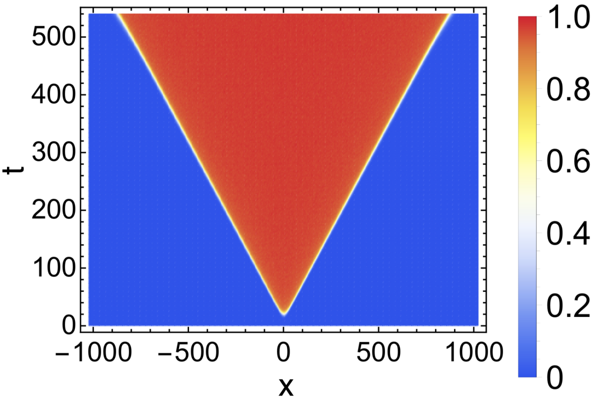
Spreading of perturbation for single realization

Space-time evolution of localized perturbation.



Light-cone evolution of $D(x, t)$

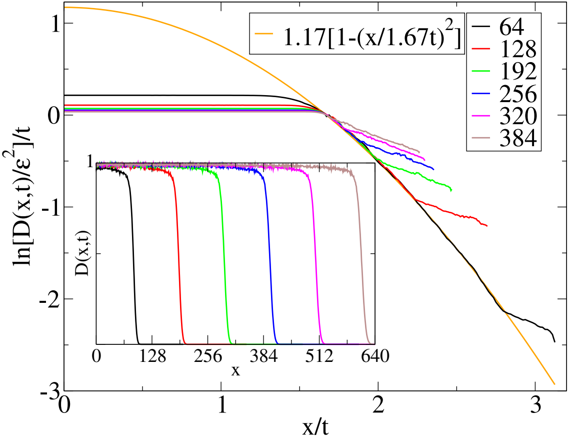
System of $N = 2048$ spins, average over random initial conditions.



The signal decays exponentially with distance away from the light cone $x = v_b t$, with $v \approx 1.67$.

Evolution of $D(x, t)$

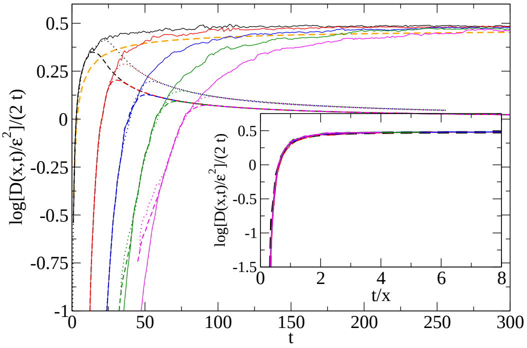
Plot of $D(x, t)$ profile at different times



From numerics, the front behaves as $D(x, t) \sim \epsilon^2 \; e^{2\lambda t[1-(x/v_b t)^2]}$

Behaviour of $D(x, t)$

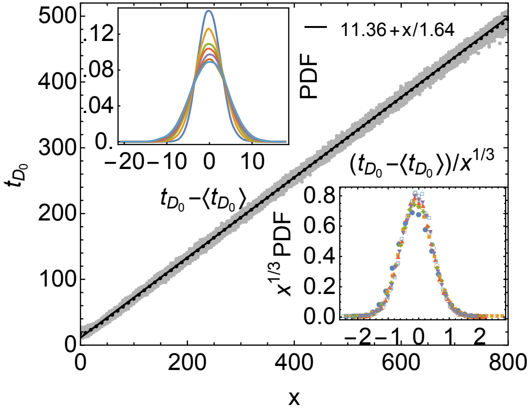
Growth of $D(x, t)$ profile at different positions – from linear and non-linear dynamics.



From numerics, the signal grows as $D(x, t) \sim \epsilon^2 \; e^{2\lambda t[1-(x/v_b t)^2]}$, with $\lambda \approx 0.5$, $v_b \approx 1.67$.

Behaviour of $D(x,t)$

Estimating the butterfly speed - At what time does the signal reach some threshold value D_o at a given point x.



$x^{1/3}$ broadening of the front.

Connection to KPZ surface growth

- Exponential growth at early times (Butterfly effect)
- Ballistic front (Butterfly velocity)
- Saturation at long times
- Broadening of growth front - possily related to KPZ

Define

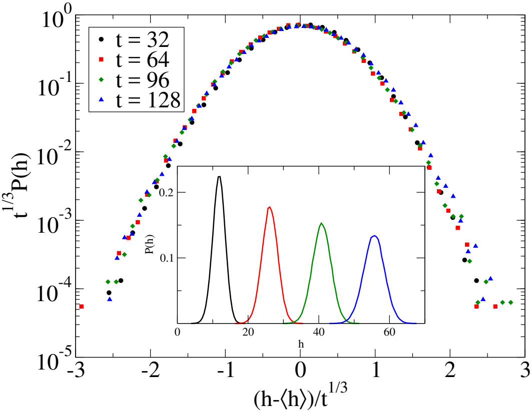
$$h(x,t) = \lim_{\epsilon \to 0} \log[\delta \mathbf{S}^2(x,t)/2\epsilon^2]/2$$

Our observations:

$$h(x,t) = at + bx^2/t + t^{1/3}\eta$$

Numerically $a \approx \lambda_L$ and $b \approx -\lambda_L/v_b^2$.
This suggests that $h(x,t)$ is like the KPZ height field.
Distribution of η — Tracy-Widom ?

Connections to KPZ



Clear $t^{1/3}$-scaling but no signs yet of Tracy-Widom.

Conclusions

- Investigated the classical version of “Out-of-Time-Ordered-Correlator” in the Heisenberg spin chain.

  —Ballistic front propagation in a diffusive system, exponential growth, broadening of front and possible connections to KPZ universality.

- $D(x,t) \sim e^{2\mu(x/t)t}, \quad \mu(v) = \lambda[1-(v/v_b)^2]$ — velocity-dependent Lyapunov exponent (R. J. Deissler and K. Kaneko, 1987)
  Integrable systems: $\mu(v) \sim -(1-v/v_b)^{3/2}$.

- What physical observable ? Information propagation, Loschmidt echo,...

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