Dynamics of piecewise smooth maps
Lecture 5

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Robust Chaos

The intersection of stable and unstable manifolds of a fixed point (a homoclinic tangle) is one of the classic mechanisms to create chaotic solutions in smooth systems. The mechanism also applies to piecewise smooth systems, and Banerjee et al (1998) use this idea to show that there are robust chaotic attractors in the BCNF, a phenomenon they dubbed 'robust chaos'. Banerjee et al provide a brief plausibility argument for the proof of the chaotic attractor, here we will use results of Misieurewicz which provide a more direct demonstration of the phenomenon.

Fixed points of Lozi

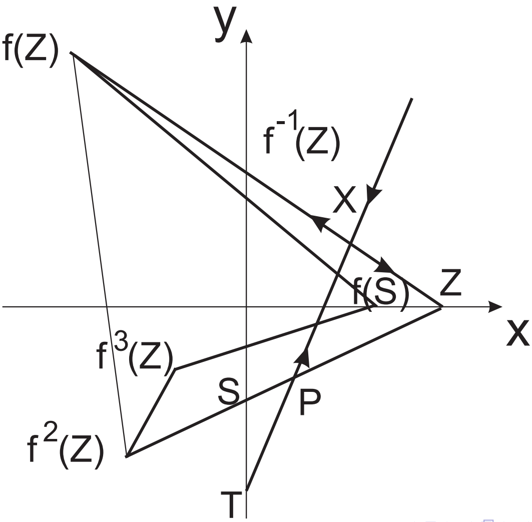
$$x \to 1 - a|x| + y, \quad y \to bx, \quad a, b > 0.$$

Note that the map is a homeomorphism and the left half plane maps to the lower half plane whilst the right half plane maps to the upper half plane. The y-axis, $x = 0$, maps to the x-axis, $y = 0$.
If $a + b > 1$ then the system has two fixed points,

$$Y = \left(-\frac{1}{b+a-1}, -\frac{b}{b+a-1} \right), \quad X = \left(\frac{1}{1-b+a}, \frac{b}{1-b+a} \right)$$

Both are saddles; the Jacobian at Y has an stable negative eigenvalue with an eigenvector of negative slope, and an unstable positive eigenvalue with an eigenvector of positive slope whilst the Jacobian at X has an unstable negative eigenvalue with an eigenvector of negative slope, and a stable positive eigenvalue with an eigenvector of positive slope. The stable and unstable manifolds of X will be particularly important.

Local invariant manifolds



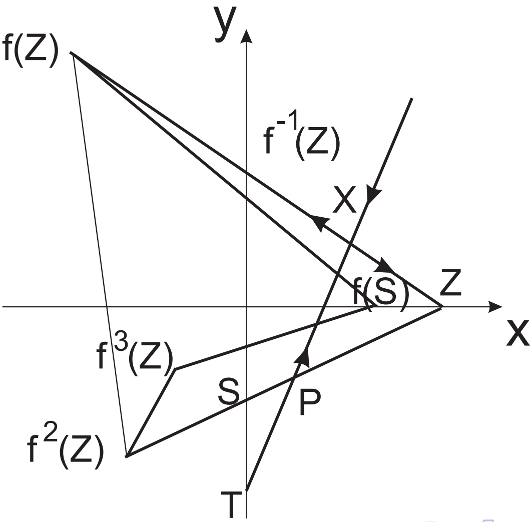
Intersections...

A great deal of the argument used to show the existence of a strange attractor for the Lozi map relies on brute force calculation. By a little elementary geometry the stable direction of X (with eigenvalue s_+) intersects the y-axis at T where

$$T = \left(0, \frac{2b - a - \sqrt{a^2 + 4b}}{2(1 + a - b)} \right).$$

Since T is on the y-axis $f(T)$ is on the x-axis and since T is on the stable manifold of X, $f(T)$ will be the intersection of TX with the x-axis.

T and f(T)



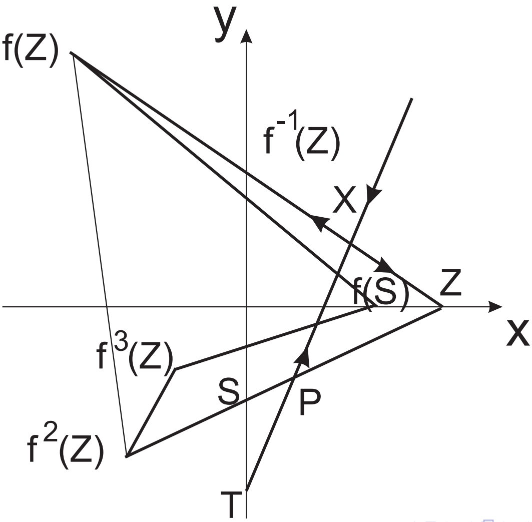
More intersections

Similarly, the unstable direction of X intersects the (positive) x-axis at Z where

$$Z = \left(\frac{2 + a + \sqrt{a^2 + 4b}}{2(1 + a - b)}, 0 \right).$$

The local unstable manifold of X thus contains the line segment $f(Z)Z$. Now (calculation) $f(Z)$ is in $x < 0$, so $f^2(Z)$ lies in the lower half plane. There are thus two cases depending on whether $f^2(Z)$ lies on the left or right of the y-axis. In what follows below we consider only the case for which $f^2(Z)$ is on the left of the y-axis.

Z and its images

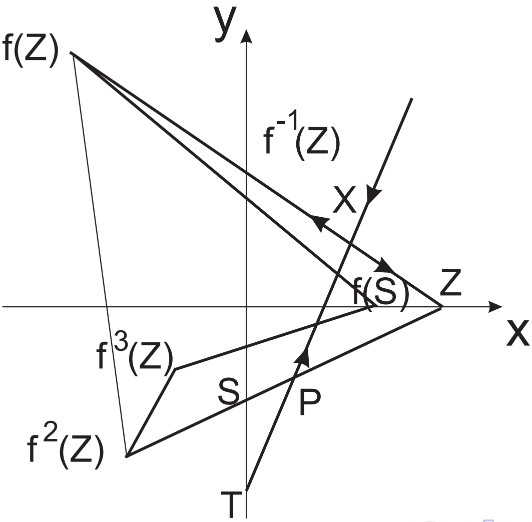


A Trapping region

So, by assumption (restricting the cases being considered) $f^2(Z)$ lies in the lower half plane with $x < 0$, and so $f^3(Z)$ can be calculated explicitly. This calculation can be used to show the following lemma.

Consider the Lozi map with parameters as described above. If $f^3(Z)$ lies in the triangle $\triangle = Zf(Z)f^2(Z)$ then $f(\triangle) \subset \triangle$.

S and a triangle



Trapping

$\triangle$ is composed of two parts:

$$\triangle_1 = f^{-1}(Z)ZS \text{ in } x \geq 0 \quad \text{and} \quad \triangle_2 = Sf^{-1}(Z)f(Z)f^2(Z) \text{ in } x \leq 0$$

(note that $\triangle_2$ is not a triangle!). Thus

$$f(\triangle_1) = Zf(Z)f(S) \subset \triangle$$

and

$$f(\triangle_2) = Zf^2(Z)f^3(Z)f(S) \subset \triangle$$

and so $f(\triangle) \subset \triangle$ as required.

Trapping Lemma

Thus $\triangle$ is a compact invariant set and hence contains an attractor provided $f^3(Z)$ is contained in $\triangle$. Brute calculation establishes that this is true provided a further condition is put on a and b.

If $a > 0$, $0 < b < 1$, $a > b + 1$ *and* $2a + b < 4$ *then* $f(\triangle) \subset \triangle$.

But what about that strange attractor?

Towards Strange attractors

Banerjee et al provide a plausibility argument for the existence of strange attractors (albeit at different parameters of the border collision normal form, though they also discuss the case here) based on (a) the existence of transverse homoclinic intersections; and (b) the existence of heteroclinic connections between the unstable manifold of Y and the stable manifold of X. Misieurewicz takes a more direct route, and whilst this is more transparent we should say something about the ideas of Banerjee et al before continuing.

Banerjee et al: the Lambda Lemma

Since X is a saddle it has stable and unstable manifolds. Suppose that C is curve segment that crosses a part of the stable manifold of X, $W^s(X)$, transversely, then under iteration the intersection point will converge on X and the part of the remainder of the curve near the intersection point will move close to X and then expand close to the unstable manifold of X, $W^u(X)$. The Lambda Lemma states that this idea can be stated precisely: in any neighbourhood of any point in $W^u(X)$ there exist a point in the image of C.

Banerjee et al: strange attractors

In particular, if C is itself a part of $W^u(X)$, so the intersection is a point in $W^u(X) \cap W^s(X)$, i.e. a transverse homoclinic point, then images of $W^u(X)$ lie arbitrarily close to any point in $W^u(X)$, giving a form of recurrence. Similarly, if there is a transverse intersection between $W^u(Y)$ and $W^s(X)$ then images of $W^u(Y)$ also lie arbitrarily close to any point in $W^u(X)$. Banerjee et al use this, together with the fact that in $x < 0$ iterates are attracted to $W^u(Y)$ and in $x > 0$ they are attracted to $W^u(X)$ to deduce that the clox sure of $W^u(X)$ is a chaotic invariant set.

Homoclinic points

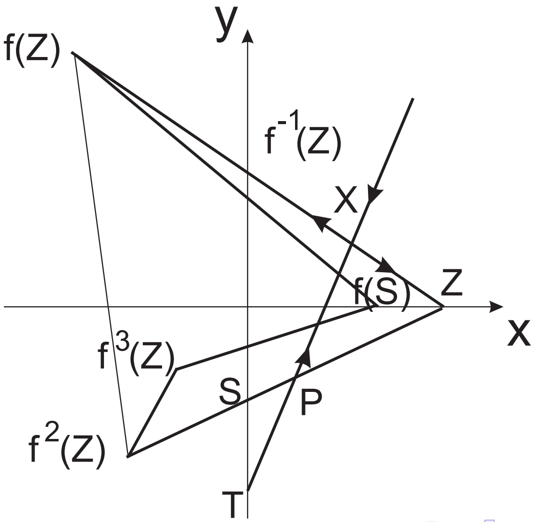
In the case considered here we can have a transverse homoclinic point.

If S lies above T on the y-axis than the Lozi map has a transverse homoclinic point.

Why? If S lies above T then there exists an intersection point P between XT (part of the stable manifold of X) and $f^2(Z)Z$ (part of the unstable manifold of X as it is the image of $f^{-1}(Z)f(Z)$).

The precise condition is messy.

The homoclinic point P



Misieurewicz, 1980

Theorem *Suppose that* $a > 0$, $2a + b < 4$, $a\sqrt{2} - 2 - b > 0$ *and* $b < \frac{a^2-1}{2a+1}$. *Then the attractor of the Lozi map is the closure of* $W^u(X)$ *and the map is topologically transitive on this set.*

Remark: A subset $\mathcal{A}$ of $\mathbb{R}^2$ is topologically transitive if for all open U_k $k = 0, 1$ with $U_k \cap \mathcal{A} \neq \emptyset$ there exists n such that $f^n(U_0) \cap U_1 \neq \emptyset$.

Sketch Proof: The proof is split into a number of stages.

No detail

Step 1: $\triangle$ contains an attracting invariant set. Messy to construct a closed set G such that $\triangle$ is contained in the interior of G and such that the attracting set

$$\tilde{G} = \cap_0^\infty f^n(G) = \cap_0^\infty f^n(\triangle) = \tilde{\triangle}.$$

So for any $x \in \triangle$ (and in particular, for any x in the attractor) there is an open neighbourhood of x in G.

Step 2: Let $H_0 = XZP$ and $H = \cup_0^\infty f^n(H_0)$. Then the boundary of H, ∂H is contained in $XP \cup W^u(X)$, $f(H) \subset H$, and $\tilde{H} = \cap f^n(H) = \tilde{\triangle}$.

Step 3: That $\tilde{\triangle}$ is the closure of the unstable manifold of X is shown by using G and $\tilde{G}$ to show that $c\ell(W^u(X)) \subseteq \tilde{\triangle}$ and H and $\tilde{H}$ to show that $\tilde{\triangle} \subseteq c\ell(W^u(X))$.

Step 4: Finally a hyperbolicity argument for expansion on the unstable manifold is used to show that f is topologically mixing on $\tilde{\triangle}$.

Young's Theorem

Young's Theorem (1985) provides an alternative approach to the chaotic attractors of border collision normal forms and their generalizations using invariant measures. This is not the place to give a detailed technical description of the theorem, but it is nonetheless useful to know that such techniques exist and can be applied to examples.

A measure μ on a space is essentially a way of assigning size or probability to subsets (strictly speaking, measureable subsets) of the space and invariance means $\mu(F^{-1}(U)) = \mu(U)$; ergodic property

$$\frac{1}{n}\sum_{0}^{n-1} g(f^n(x)) \to \int_X g d\mu$$

as $n \to \infty$ (for μ almost all x: spatial averages equal time averages.

Hypotheses

Let $R=[0,1]\times[0,1]$ and let $S=\{a_1,\dots,a_k\}\times[0,1]$ be a set of vertical switching surfaces with $0<a_1<\cdots<a_k<1$. Then $f\colon R\to R$ is a Young map if f is continuous, f and its inverse are C^2 on $R\backslash S$ and $f=(f_1,f_2)^T$ satisfies the expansion properties (H1)-(H3) below on $R\backslash S$.

$$(H1)\quad \inf\left\{\left(\left|\frac{\partial f_1}{\partial x}\right|-\left|\frac{\partial f_1}{\partial y}\right|\right)-\left(\left|\frac{\partial f_2}{\partial x}\right|-\left|\frac{\partial f_2}{\partial y}\right|\right)\right\}\geq 0,$$

$$(H2)\quad \inf\left(\left|\frac{\partial f_1}{\partial x}\right|-\left|\frac{\partial f_1}{\partial y}\right|\right)=u>1,\quad \text{and}$$

$$(H3)\quad \sup\left\{\left(\left|\frac{\partial f_1}{\partial y}\right|+\left|\frac{\partial f_2}{\partial y}\right|\right)\left(\left|\frac{\partial f_1}{\partial x}\right|-\left|\frac{\partial f_1}{\partial y}\right|\right)^{-2}\right\}<1.$$

The Theorem

Young's Theorem describes measures that project nicely onto one-dimensions. Technically this is expressed as having absolutely continuous conditional measures on unstable manifolds. Intuitively this means that the measure projects nicely onto on dimension.

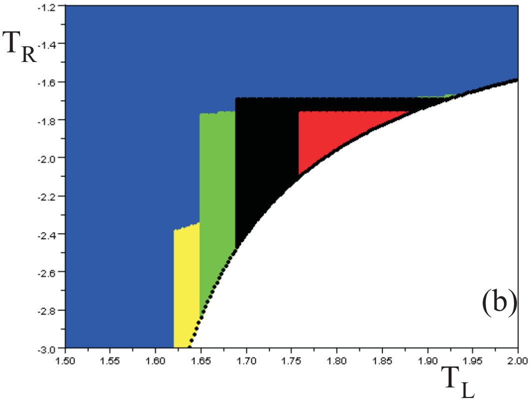
Let $Jac(f)$ denote the Jacobian matrix of f and recall that u is defined in (H2).

If f is a Young map, $|Jac(f)| < 1$ for $x \in R \backslash S$, and there exists $N \geq 1$ s.t. $u^N > 2$ and if $N > 1$ then $f^k(S) \cap S = \emptyset$, $1 \leq k < N$, then f has an invariant probability measure that has absolutely continuous conditional measures on unstable manifolds.

Since the result is for piecewise C^2 maps and the conditions only depend on derivatives this result has the important corollary that results for the piecewise linear border collision normal form, which should more correctly be called a truncated normal form, persist when small nonlinear terms are added.

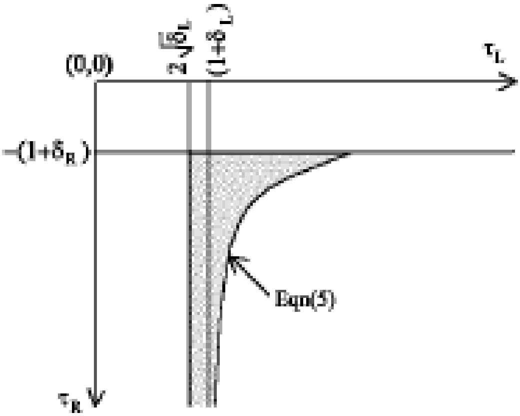
Numerical Confirmation

I have used this theorem to obtain numerical confirmation of the existence of strange attractors for the BCNF.



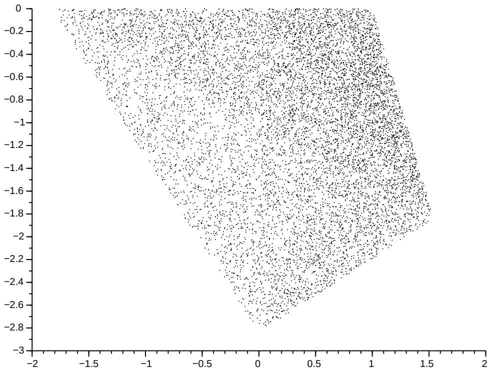
Banerjee et al

Banerjee et als region:



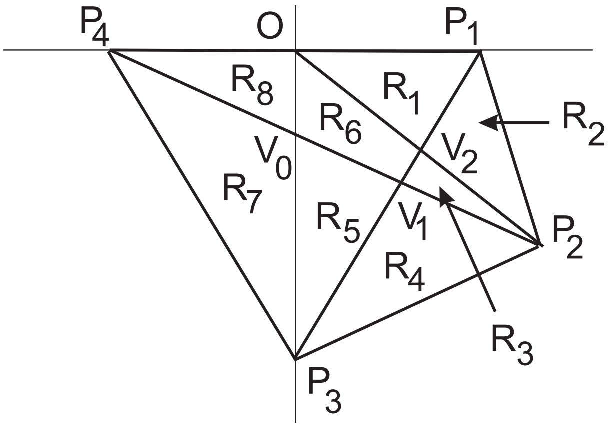
Two dimensional Attractors

These were quasi-one-dimensional attractors (Hausdorff dimension less than two). Can also observe apparently two-dimensional attractors. Do they really exist?



Markov partitions...

In an early example I used Markov partitions (with Ivan Wong)



Buzzi and Tsujii

But then learned of work by Buzzi (1999,2001) and Tsujii (2001) which made it possible to do much better (e.g. Glendinning 2016)

There exist open parameter regions of the BCNF such that if $\mu < 0$ *the map has a stable fixed point and if* $\mu > 0$ *the map has an attractor of topological dimension two (i.e. contains open sets).*

Note that technical results depend crucially on linearity!

(Constucted on bridge by harbour at Barcelona Mathematical Days, 2014.)

dimension n?

But can do higher dimensions too (Glendinning, 2015)

There exist open parameter regions of the n-dimensional BCNF such that if $\mu < 0$ *the map has a stable fixed point and if* $\mu > 0$ *the map has an attractor of Hausdorff dimension n and generically this has topological dimension n.*

Challenges

- Too many to list
- Nonlinearity???
- Avoiding lists (except this one!)
- What level of description appropriate – or is it about technique?
- Connection with flows.
- Lots of beautiful more detailed questions (genesis of infinitely many sinks and role of area preservation), what do I mean by quasi-m-dimensional attractor?
- ....