Dynamics of piecewise smooth maps: Lecture 4

Paul Glendinning

School of Mathematics, Manchester

Summer School, ICTS Bangalore, June 2018

Maps of the Plane: Fixed points (non-boundary case)

Given

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n)$$

with $f: \mathbb{R}^2 \to BR^2$ a fixed point satisfies

$$x = f(x)$$

and assuming this is not on the boundary stability is determined (via standard small perturbation argument) via the eigenvalues of the Jacobiann matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

with partial derivatives eva,uated at the fixed point.

Stability triangle

Eigenvalue equation

$$s^2 - Ts + D = 0$$

where T is the trace of the Jacobean and D the determinant. Stable if eigenvalues

$$s_{\pm} = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$$

lie inside the unit circle. Boundary in (T, D) space has three components:

Complex conjugates

If $T^2 < 4D$ then $s_\pm$ are complex conjugates and

$$|s_\pm|^2 = s_+ s_- = D$$

as

$$s^2 - Ts + D = 0$$

so first stability criterion is

$$T^2 < 4D, \quad D < 1.$$

Real: positive trace

If $T^2 > 4D$ then $s_\pm$ are real and if $T > 0$ the $|s_+| > |s_-|$ and so both are less than one if

$$s_+ = \tfrac{1}{2}(T + \sqrt{T^2 - 4D}) < 1,$$

i.e.

$$\sqrt{T^2 - 4D} < 2 - T$$

so $T < 2$ and

$$T^2 - 4D < 4 - 4T + T^2, \quad \text{i.e.} \quad T < 1 + D.$$

So second stability criterion is

$$0 < T < 2, \quad T^2 > 4D, \quad T < 1 + D.$$

Real: negative trace

If $T^2 > 4D$ then $s_\pm$ are real and if $T < 0$ the $|s_-| > |s_+|$ and so both are less than one if

$$s_- = \tfrac{1}{2}(T - \sqrt{T^2 - 4D}) > -1,$$

i.e.

$$\sqrt{T^2 - 4D} < 2 + T$$

so $T > -2$ and

$$T^2 - 4D < 4 + 4T + T^2, \quad \text{i.e.} \quad -T < 1 + D.$$

So third stability criterion is

$$-2 < T < 0, \quad T^2 > 4D, \quad -T < 1 + D.$$

Stability triangle

The three criteria are therefore

$$T^2 < 4D, \quad D < 1,$$
$$T^2 > 4D, \quad 0 < |T| < 2, \quad |T| < 1 + D.$$

which fit together as

$$D < 1, \quad |T| < 1 + D.$$

Generic Linear Types

- stable/unstable foci ($s_{\pm}$ complex conjugate pair)
- stable/unstable node ($s_{\pm}$ real distinct, $|s_{\pm}| < 1$)
- saddle

Note effect of negative s.

Periodic Orbits

As in the one-dimensional case these are fixed points of f^p, and stability via Df^p.

Fixed points of continuous PWS systems

Choose co-ordinates with switching surface $x = 0$ so

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n) = \begin{cases} f_0(x,y) & \text{if } x < 0 \\ f_1(x,y) & \text{if } x \geq 0 \end{cases}$$

and continuity:

$$f_0(0,y) = f_1(0,y).$$

(This mild form of PWS system is already complicated enough without adding discontinuity.)

Fixed points with $x \neq 0$ have same local structure as just described.

Fixed points on $x = 0$ for continuous PWS systems

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n) = \begin{cases} f_0(x,y) & \text{if } x < 0 \\ f_1(x,y) & \text{if } x \geq 0 \end{cases}$$

and continuity:

$$f_0(0,y) = f_1(0,y),$$

and fixed point

$$f_0(0,0) = f_1(0,0) = 0,$$

Codimension one so ignore!

Border collision bifurcation

BUT cannot ignore in one parameter families:

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n, \mu) = \begin{cases} f_0(x, y, \mu) & \text{if } x < 0 \\ f_1(x, y, \mu) & \text{if } x \geq 0 \end{cases}$$

and continuity:

$$f_0(0, y, \mu) = f_1(0, y, \mu),$$

and fixed point on $x = 0$ if $\mu = 0$

$$f_0(0, 0, 0) = f_1(0, 0, 0) = 0,$$

What happens locally?

Border collision normal form

Nusse and Yorke (1992): local behaviour (lowest order terms) can be chosen so that

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} f_0(x_n, y_n) & \text{if } x < 0 \\ f_1(x_n, y_n) & \text{if } x_n > 0 \end{cases}$$

with

$$f_k(x, y) = \begin{pmatrix} T_k & 1 \\ -D_k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \quad k = 0, 1.$$

Border collision normal form (derivation)

By Taylor expansion

$$f_k(x,y) = \begin{pmatrix} a_k & s \\ b_k & t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mu \begin{pmatrix} u \\ v \end{pmatrix}, \quad k = 0, 1.$$

Where we have ignored quadratic terms and higher, t, s, u, v are independent of k by continuity. Then cooridinate transform keeping $x = 0$ invariant:

$$Y = \alpha x + y, \quad X = \beta x$$

with $\beta \neq 0$ and coefficient of y non-zero for independence and unity by scale invariance.

Now just go through the calculation!

Initial remarks

$$f_k(x,y) = \begin{pmatrix} T_k & 1 \\ -D_k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \quad k = 0, 1.$$

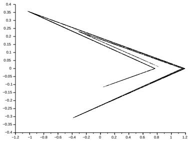
- Change scale by a factor of $|\mu|$: without loss of generality $\mu \in \{-1, 0, 1\}$.
- If $D_1 > 0$ then right half-plane mapped to lower half-plane; upper half-plane if $D_1 < 0$.
- If $D_0 > 0$ then left half-plane mapped to upper half-plane; lower half-plane if $D_0 < 0$.
- BCNF a homoemorphism if $D_0 D_1 > 0$; non-invertible if $D_0 D_1 < 0$.

Example: Lozi map

Recall:

Lozi map (non-smooth) 1978

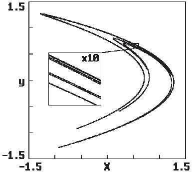
$$\begin{aligned} x_{n+1} &= 1 - a|x_n| + by_n \\ y_{n+1} &= x_n \end{aligned}$$



• first non-trivial 2D example with proved chaotic attractor (1980)

Hénon map (smooth) 1976

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + by_n \\ y_{n+1} &= x_n \end{aligned}$$



• still no explicit parameters with proved chaotic attractor

Lozi map

$$\begin{aligned} x_{n+1} &= 1 - a|x_n| + by_n \\ y_{n+1} &= x_n \end{aligned}$$

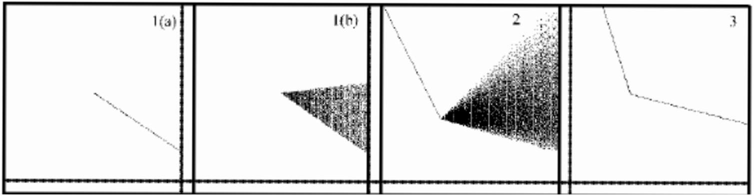
i.e. with $Y = by$

$$f_k(x, Y) = \begin{pmatrix} T_k & 1 \\ -D_k & 0 \end{pmatrix} \begin{pmatrix} x \\ Y \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \quad k = 0, 1.$$

with

$$\mu = 1, \quad T_0 = a, \quad T_1 = -a, \quad D_0 = D_1 = -b.$$

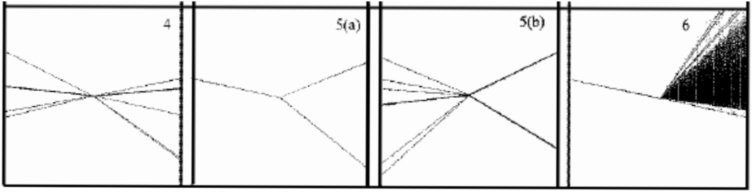
Attractors of the BCNF:Banerjee and Grebogi (1999)



Some bifurcations of the BCNF (only attractors are mentioned): μ is plotted horizontally and the projection of attractors onto the x-axis vertically. 1(a) no attractor to fixed point; 1(b) no attractor to chaos; 2 fixed point to chaos; 3 fixed point to fixed point.

More Attractors of the BCNF

:Banerjee and Grebogi (1999):



Some bifurcations of the BCNF (only attractors are mentioned): μ is plotted horizontally and the projection of attractors onto the x-axis vertically. 4 coexisting fixed point and period 3 to coexisting fixed point and period 4; 5(a) fixed point to period two; 5(b) fixed point and period 11 to period 2; 6 fixed point to period 5 and chaotic attractor.

Fixed Points of the BCNF

Solve

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} T_k & 1 \\ -D_k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \quad k = 0, 1,$$

and then check that $x > 0$ if $k = 1$ and $x < 0$ if $k = 0$ (i.e. fixed point is not virtual).

Fixed Points: $x > 0$

In $x > 0$:

$$T_1 x + y + \mu = x, \quad -D_1 x = y, \quad x > 0.$$

Solving these simple linear equations gives

$$x = \frac{\mu}{1 + D_1 - T_1}, \quad y = -\frac{D_1 \mu}{1 + D_1 - T_1}$$

and so provided $1 + D_1 - T_1 \neq 0$ there is a fixed point for an appropriate sign of μ: $\mu > 0$ if $1 + D_1 - T_1 > 0$ and $\mu < 0$ if $1 + D_1 - T_1 < 0$.

Fixed Points: $x < 0$

A precisely analogous manipulation shows that there is a fixed point in $x < 0$ provided $\mu > 0$ if $1 + D_0 - T_0 < 0$ and $\mu < 0$ if $1 + D_0 - T_0 > 0$.

Stability: $|D_k| < 1$ and $|T_k| < 1 + D_k$ in $x < 0$. This condition can also be written as

$$0 < 1 + D_0 - |T_0|.$$

Non-smooth saddlenode?

Putting the two branches of solutions together we see that if

$$(1 + D_0 - T_0)(1 + D_1 - T_1) > 0$$

then the two fixed points exist for opposite signs of μ whilst if

$$(1 + D_0 - T_0)(1 + D_1 - T_1) < 0$$

then the two fixed points exist for the same sign of μ (rather like a smooth saddle-node bifurcation).

Period two

Except in the degenerate case that a Jocobian has an eigenvalue of -1, in which case there can be a degenerate line of orbits of period two, an orbit of period two has one point on each side of $x = 0$. The equations are a little more messy, but still linear. Going through the detailed calculation period two points are at (x_0, y_0) with $x_0 < 0$ and (x_1, y_1) with $x_1 > 0$ and

$$\begin{aligned} x_0 &= \mu + y_1 + T_1 x_1 \qquad & y_0 = -D_1 x_1 \\ x_1 &= \mu + y_0 + T_0 x_0 \qquad & y_1 = -D_0 x_0 \end{aligned}$$

which imply

$$(x_k, y_k) = \left(\frac{1 + T_{1-k} + D_{1-k}}{(1 + D_0)(1 + D_1) - T_0 T_1} \mu, -D_{1-k} x_{1-k} \right), \quad k = 0, 1.$$

Continue...

$$(x_k, y_k) = \left(\frac{1 + T_{1-k} + D_{1-k}}{(1+D_0)(1+D_1) - T_0 T_1} \mu, -D_{1-k} x_{1-k} \right), \quad k = 0, 1.$$

These lie on the 'correct' (i.e. opposite) sides of the y-axis for one sign of μ provided

$$(1 + T_0 + D_0)(1 + T_1 + D_1) < 0$$

and if this inequality does not hold then there are no non-degenerate points of period two.

Stability

Stability is determined by the trace and determinant of the product of the linear parts of the BCNF:

$$\begin{pmatrix} T_0 & 1 \\ -D_0 & 0 \end{pmatrix} \begin{pmatrix} T_1 & 1 \\ -D_1 & 0 \end{pmatrix} = \begin{pmatrix} T_0 T_1 - D_1 & T_1 \\ -D_0 T_1 & -D_0 \end{pmatrix}$$

and the period two orbit is stable if the modulus of the trace and the modulus of the determinant satisfy equivalent conditions as for the fixed points; i.e. it is stable if

$$|D_0 + D_1 - T_0 T_1| < 1 + D_0 D_1, \quad |D_0 D_1| < 1.$$

So what?

So much for the equations – but what combinations of fixed points and periodic orbits can be involved in bifurcations? This is not obvious from the equations. We leave this question as an exercise for the moment, it turns out (see last section) that non-smooth saddlenode cannot produce period two on the 'other' side of the bifurcation...

But can other attractors exist after a non-smooth saddlenode (Stommel?).

Other periodic orbits

Although a great deal was known about periodic orbits and the regions of parameter space for which they exist (and may coexist) from the works of Gardini and others , the more recent approach of Simpson and Meiss makes a systematic approach possible.

Positives: their work makes it possible to calculate and describe general features of resonance regions (locking) for BCNF.

Negatives: the methods rely on the piecewise affine nature of the maps and not all results remain true with nonlinear terms.

Formalism

Let $s_1, \ldots s_n$ be a sequence of 0s and 1s, and suppose we wish to look for a periodic orbit of period n such that the k^{th} point of the periodic orbit lies in $x < 0$ if $s_k = 0$ and in $x > 0$ if $s_k = 1$. To find such an orbit it is necessary to solve the fixed point equation for the n^{th} iterate of the map, taking into account the required sequece (s_k), and then to determine whether the fixed point (ia periodic orbit of f) is *real*, i.e. its orbit passes through the regions $x \leq 0$ and $x \geq 0$ in the prescribed order, or *virtual*, otherwise, in which case the solution does not correspond to an orbit of the BCNF.

Equations

At each iteration $f(\mathbf{x}) = A_{s_k}\mathbf{x} + \mu\mathbf{e}$ and so by induction

$$f^n(\mathbf{x}) = M_s\mathbf{x} + \mu P_s\mathbf{e}$$

where

$$M_s = A_{s_n}\dots A_{s_1}, \quad P_s = I + A_{s_n} + A_{s_n}A_{s_{n-1}} + \cdots + A_{s_n}\dots A_{s_2}.$$

The point calculated on the orbit of period n in the half plane determined by s_1 is a solution of the fixed point equation $\mathbf{x}_1 = f^n(\mathbf{x}_1)$, i.e.

$$\mathbf{x}_1 = \mu(I - M_s)^{-1}P_s\mathbf{e}.$$

Of course, this exists and is unique if $I - M_s$ is non-singular, or eqivalently if $\det(I - M_s) \neq 0$.

x-coordinates

The same process can be repeated for each point on the orbit: the image of $\mathbf{x}_1$ is $\mathbf{x}_2$ which satisfies a similar equation but with s replaced by $s_1 s_n \ldots s_2$. Define the shift σ on these periodic sequences so that

$$\sigma(s_n \ldots s_2 s_1) = s_1 s_n \ldots s_2$$

then the n points on the orbit of period n corresponding to s are

$$\mathbf{x}_{k+1} = \mu(I - M_{\sigma^k s})^{-1} P_{\sigma^k s}\mathbf{e}, \quad k = 0, 1, \ldots, n-1.$$

Simpson and Meiss (2010) show that the x coordinate can be written as

$$x_{k+1} = \mu \frac{\det P_{\sigma^k s}}{\det(I - M_s)}, \quad k = 0, 1, \ldots, n-1,$$

where we have used the fact that $\det(I - M_{\sigma^k s})$ is independent of k (to see this simply note that $A_{s_1}(I - M_s)A_{s_1}^{-1} = I - M_{\sigma s}$). The remainder of the derivation is far from trivial linear analysis and details can be found in their paper.

Real or virtual?

So far, so much manipulation. But is this solution real or virtual? The answer is very similar to that in the case of the orbit of period two:

Fix $s = s_1 \dots s_n \in \{0,1\}^n$ and suppose that $\det(I - M_s) \neq 0$ and $\det P_{\sigma^k s} \neq 0$, $k = 0, 1, \dots, n-1$. If there exists $g \in \{-1, 1\}$ such that

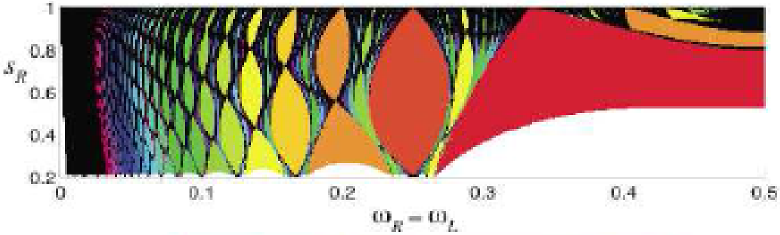
$$\begin{array}{ll} \mathrm{sign}\,(\det P_{\sigma^{k-1} s}) = -g & \text{if } s_k = 0 \\ \mathrm{sign}\,(\det P_{\sigma^{k-1} s}) = g & \text{if } s_k = 1 \end{array}$$

then the periodic orbit corresponding to s exists for $\mu > 0$ if $g\det(I - M_s) > 0$ and for $\mu < 0$ if $g\det(I - M_s) < 0$.

The proof is straightforward from the definitions and equations for the x-coordinate. Note that at this stage we have not used the assumption that the map is two-dimensional.

Resonance tongues and pinching

Figure shows regions of parameter space with different periodic orbits...
from Simpson and Meiss (2010)



Parameters

The parameters are chosen so that

$$T_L = 2r_L \cos(2\pi\omega_L), \quad D_L = r_L^2, \quad T_R = \frac{2}{s_R}\cos(2\pi\omega_R), \quad D_R = \frac{1}{s_R^2}$$

and in the Figure,

$$r_L = 0.2, \quad \mu = 1,$$

and $\omega_R = \omega_L$ is the parameter on the horizontal axis, and s_R is the parameter on the vertical axis. The resonant tongues in which the periodic orbits exist have a 'sausage' shaped pinched structure which can be understood using the methods devemopped above.

Two ingredients: first m/n

The analysis of these bifurcations involves two ingredients. First, the rotation order of the periodic points implies that the points on the periodic orbit can be arranged on a circle (with no self-intersections) so that the order on the circle is $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and the effect of the map f is

$$f(\mathbf{x}_k) = \mathbf{x}_{k+m}$$

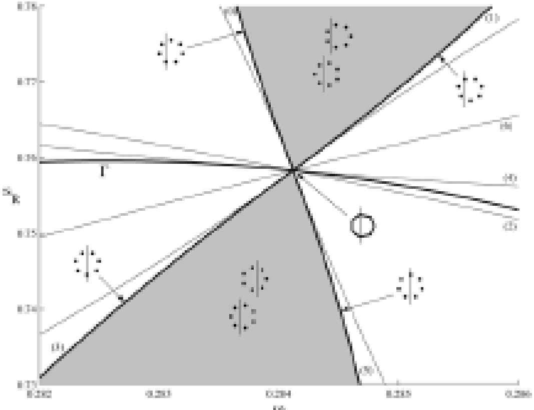
where the index $k + m$ is interpreted modulo n with the convention that $0 \equiv n$. However, this order does not indicate where the switching surface lies, so the second ingredient specifies the position of the switching surface with respect to the periodic points.

Two ingredients: ... then ℓ

Assume that the switching surface separates the periodic points into two consecutive sets of points on the circle with ℓ points in $x < 0$ and $n - \ell$ in $x > 0$. The orbit is therefore specified by three positive integers: n, m and ℓ, and the labelling can be chosen so that $\mathbf{x}_1, \dots, \mathbf{x}_{n-\ell}$ lie in $x > 0$ and $\mathbf{x}_{n-\ell+1}, \dots, \mathbf{x}_n$ lie in $x < 0$. This information is enough to specify the symbolic description s of the orbit (note that it is NOT the rotation-compatible sequences as the position of the switching surface which determines s is not the same as the coding of the rotations). We shall refer to these orbits as (n, m, ℓ)-orbits.

Schematic view

If this periodic orbit undergoes a border collision bifurcation itself, then one point intersects the switching surface and by continuity this must be either $\mathbf{x}_1$ or $\mathbf{x}_n$ or $\mathbf{x}_{n-\ell}$ or $\mathbf{x}_{n-\ell+1}$.



Border collision

Suppose that it is $\mathbf{x}_1$ intersecting the switching surface. Then the bifurcation will involve two periodic orbits: one with code s and the other with code ${}^0s$, defined to be s with the initial symbol 1 replaced by 0. In other words, the 'partner' orbit has is a $(n, m, \ell+1)$-orbit. Similarly, if the border collision point is $\mathbf{x}_{n-\ell}$ then it crosses at the border collision creating another code with one of the 1s in s replaced by a zero – the partner is again a $(n, m, \ell+1)$-orbit. It turns out (see Figure) that these are generalized saddle-node orbits, so there is a lobe in which a (n, m, ℓ)-orbit coexists with a $(n, m, \ell+1)$-orbit.
Bifurcations involving $\mathbf{x}_n$ or $\mathbf{x}_{n-\ell+1}$ are similar, except each of these involves the existence of a $(n, m, \ell-1)$-orbit.

Summary of structure

The regions (lobes) are thus defined by orbits whose ℓ description differs by one. At the shrinking point (for the piecewise affine BCNF) there is a degenerate invariant circle. This beautiful structure does not persist for typical nonlinear perturbations of the BCNF: the codimension two pinching point has a natural unfolding, see Simpson and Meiss (2010) for details.

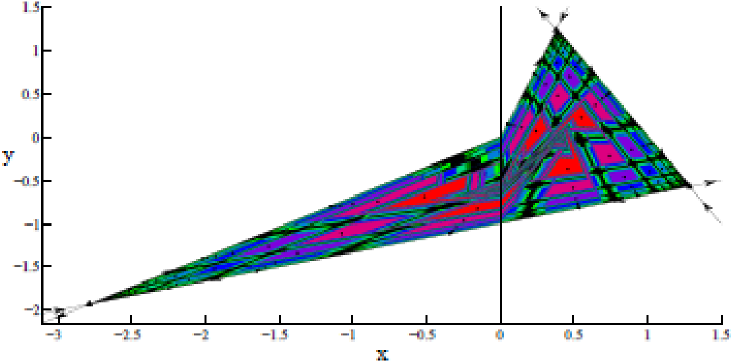
Infinitely many sinks

The previous section might give the impression that periodic orbits exist in splendid isolation. However, it has been recognised for many years that complicated regions of multistability exist in the border collision normal form (e.g. Gardini, 1992). More recently Simpson 2014) has shown that there are parameter values for the BCNF at which there are infinitely many stable periodic orbits.

This is (probably?) analogous to the Newhouse phenomenon for smooth maps.

Picture

We will not go into the details here: figure shows numerically computed basins of attraction at an approximation of the critical parameter.



Summary

- some progress in understranding structures;
- issues about coexistence in general
- what can be deduced about periodic orbits with nonlinearity;
- (less is more) what is it actually useful to say in general????
- what about other types of attractor?