Dynamics of piecewise smooth maps
Lecture 3: Piecewise Smooth Maps of the Interval.

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First example: Tent maps

Want to go through an example where we can show structure of the non-wandering set using the the locally eventually onto (LEO) property. Recall:

Non-wandering: a point x is wandering if there exists an open neighbourhood U of x such that $f^n(U) \cap U = \emptyset$ for all $n \geq 0$. Otherwise non-wandering. The non-wandering set of a map f, $\Omega(f)$ is the set of non-wandering points.

A map $f: I \to I$ is LEO if for all open $U \subset I$ there exists $n \geq 0$ such that $f^n|U$ is a homeopmorphism and $c\ell(f^n(U)) = I$.

This is a strong form of expansion. Clearly LEO implies that $\Omega(f) = I$.

Weakly PWS

Tent maps: Milnor-Thurston: every chaotic unimodal map f is semi-conjugate to a symmetric tent map T_s with $|slope| = s \in (1, 2]$ and $h(f) = \log s$.

If $T_s^n(\frac{1}{2}) \neq \frac{1}{2}$, $n = 1, 2, \ldots$ then semiconjugacy only collapses homtervals;

if $T_s^n(\frac{1}{2}) = \frac{1}{2}$ then may collapse an interval J (and preimages) s.t. $f^n|J$ unimodal.

Starting point of decomposition for chaotic maps (non-chaotic is easy). Proof uses kneading series as semi-conjugacy. Note periodicity implies Markov partition.

Extensions

Extends to multi-modal and piecewise monotonic (e.g. Glendinning and Hall (1996) for two increasing branches, Glendinning (Adams Prize, 1992) for two monotonic branches), but messy. Two natural parameters, one full family known to me (Glendinning and Sidorov, 2015).

Many others worked on this in the 1980s: Alseda, Gambaudo, Llibre, Misieurewicz, Tresser....

Tent maps: $\sqrt{2} < s \leq 2$

Even tent maps tricky – not full, so how do they behave. Neat way of seeing (van Strien 1981): expansion argument.

$$T_s(x) = \begin{cases} sx & \text{if } x \in [0, \frac{1}{2}] \\ s(1-x) & \text{if } x \in [c, 1] \end{cases}$$

so if $s > 1$ attractor in $I = [T_s^2(\frac{1}{2}), T_s(\frac{1}{2})]$.

Take any $U \subseteq I$ with length $|U|$. If $\frac{1}{2} \notin U$ then $|T(U)| = s|U|$ so expanding in finite interval I so eventually $\frac{1}{2} \in T^k(U) = V_0 \cup V_1$ (left and right).

Expansion

$$|V_0| = \alpha |T^k(U)|, \quad |V_1| = (1-\alpha)|T^k(U)|, \quad \alpha \in (0,1)$$

so

$$|T(V_0)| = s\alpha |T^k(U)|, \quad |T(V_1)| = s(1-\alpha)|T^k(U)|$$

and if $\frac{1}{2}$ not in one of these
$\max\{|T^2(V_0)|, |T^2(V_1)|\} = s^2 \max\{\alpha, 1-\alpha\}|T^k(U)| \geq \frac{1}{2}s^2|T^k(U)|$
so if $s^2 > 2$ then still expanding.

Expansion

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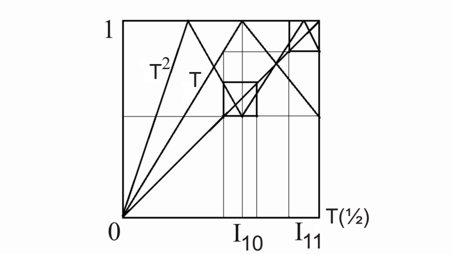
so if $s^2 > 2$ then still expanding.

So eventually map over $\frac{1}{2}$ with $T^n(U)$ and $\frac{1}{2} \in T^{n+1}(U)$ so $T^{n+2}(U) = I$.

TRANSITIVE (dense periodic orbits, dense orbit etc etc).

$\sqrt{2} < s^2 \leq 2$: Renormalize

Look at T_s^2: slope s^2 and so rescaled version of T_{s^2} which satisfies previous conditions for transitivity.



So non-wandering set is $\{0\} \cup P_0 \cup I_1$ where P_0 is period 2^0 and I_1 is union of 2^1 intervals $I_{10} \cup I_{11}$ disjoint unless $s^2 = 2$ in which case intersect on P_0.

$\sqrt{2} < s^{2^2} \leq 2$

Induction: The map T^2 restricted to the interval I_{10} or I_{11} has slope $u = s^2$ and

$$\sqrt{2} < u^2 \leq 2$$

and since T^2 is just a rescaled version of T on each of these intervals, by the result of the previous slide the non-wandering set in I_{10} is $P_{10} \cup I_{210}$ where P_{10} is period 2^0 for T^2 and I_{210} is a union of two $= 2^1$ intervals. Similarly
the non-wandering set in I_{11} is $P_{11} \cup I_{211}$ where P_{11} is period 2^0 for T^2 and I_{211} is a union of two $= 2^1$ intervals.

Since $T(I_{10}) = I_{11}$ and $T(I_{11}) = I_{10}$ this translates (with a bit of work) to $\Omega(T)$ having an orbit of period two, P_1 and a union of four intervals I_2 in $I_{10} \cup I_{11}$ and hence

$$\Omega(T) = \{0\} \cup P_0 \cup P_1 \cup I_2.$$

$\sqrt{2} < s^{2^n} \leq 2$: keep going

By induction

$$\Omega(T) = \{0\} \cup \left(\cup_0^{n-1} P_k\right) \cup I_n$$

where P_k is period 2^k and I_n is union of 2^n intervals disjoint unless $s^{2^n} = 2$ in which case intersect on P_{n-1}.

And so...

Idea here was to used induced maps (renormalization) to understand structure of dynamics.

This is a very powerful idea in smooth and nonsmooth dynamics. Often able to understand lots of structure using this method.

e.g. (Glendinning 1992,2014) there are uncountably many routes to chaos in piecewise monotonic maps with a single discontinuity.

Transitive maps

One of the fundamental ideas of complexity for dynamics (should have given this yesterday)

Suppose $f\colon I \to I$ is a PWS map of the interval I. f is transitive if for every open interval $J \subseteq I$ there exists $N < \infty$ such that

$$I = \mathrm{c}\ell \cup_{k=0}^{N} f^k(U).$$

Weaker than LEO, but still implies that $\Omega(f) = I$. Some people call this locally onto and say f is transitive if for all U and V there exists n such that

$$f^n(U) \cap V \neq \emptyset.$$

(Danger of too many definitions!)

Renormalization

Renormalization has a fundamental role to play in the description of dynamics. Here we will prove a restricted version of a more general result we return to in Lecture V.

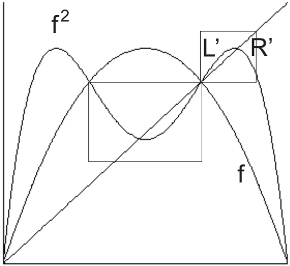
Suppose $f\colon [0,1] \to [0,1]$ is a PWS map and there exists $c \in (0,1)$ such that f is monotonic and continuous on $(0,c)$ and on $(c,1)$. f is renormalizable if there exist positive integers n_0 and n_1 with $n_0 + n_1 > 2$ and non-trivial intervals $J_0 = (x_1, c)$ and $J_1 = (c, x_2)$ such that $f^{n_k}|J_k$ is continuous and monotonic, $k = 0, 1$ and

$$f^{n_k}(J_k) \subseteq J_0 \cup \{c\} \cup J_1, \quad k = 0, 1.$$

In some sense, apart from stable periodic orbits, renormalization is the only obstruction to transitivity in PWS maps with two monotonic branches.

The classic case

The classic smooth case is period-doubling. At the accumulation of period-doubling f can be renormalized with f^2 infinitely often:



$$R' \to RL, \quad L' \to RR$$

So what?

Labelling orbits by R if they are in $x > c$ and L if $x < c$ then this makes it possible to write down the sequence of $f(c)$ (TA=DA).

Exercise

If r_n is the number of Rs in the first 2^n terms and ℓ_n is the number of Ls (so $r_n + \ell_n = 2^n$), find a recursion formula for (r_n, ℓ_n) in terms of (r_{n-1}, ℓ_{n-1}) and solve for r_n and ℓ_n.

Decomposition

Theorem 19 *Suppose* $f: [0,1] \to [0,1]$ *is a PWS map with two monotonic branches separated by* $c \in (0,1)$. *If there exists* $s > 1$ *such that* $|f'(x)| \geq a$ *for all* $x \in (0,1) \backslash \{c\}$ *then either* f *is transitive or* f *is renormalizable. If* f *is renormalizable on an interval* J *containing* c *then* $\Omega(f) = T \cup R$ *where* T *is described by a Markov graph and* R *is the nonwandering set of the induced map on* J *and its iterates under* f.

This means we have one set that we understand (Markov graph) and a second which can be approached in the same way (the induced map is again PWS with two monotonic branches and (at most) a single discontinuity. This sets up an inductive framework almost precisely the same as for the tent map, but with a greater number of possible induced maps.

Sketch Proof part 1

Without loss of generality assume that $[0, 1]$ is the smallest interval mapped into itself by f. Note that f has no stable periodic orbits.

Take any open interval U. By the expansion arguments used for the tent map there exists $n_0 \geq 0$ such that $c \in f^{n_0}(U) = U_0$ and follow both branches to their next intersection with c, i.e. let $U_0 = V_0 \cup \{c\} \cup V_1$ in the standard way and choose the smallest m_k, $k = 0, 1$ such that $c \in f^{m_k}(V_k)$, $k = 0, 1$ (these exist by the expansion argument). If $f^{m_k}(V_k) \subseteq U_0$ then f is renormalizable. Otherwise set

$$U_1 = f^{m_0}(V_0) \cup f^{m_1}(V_1) \cup U_0$$

and note $U_0 \subset U_1$.

Proof part 2

Now repeat the argument using U_1 and note that the equivalent of the return times for U_1 are less than or equal to the return times m_k for U_0. Either f is renormalizable or there exists U_2 with $U_1 \subseteq U_2$ which is a union of iterates of subsets of U.

Either there exists $m < \infty$ such that $U_m = (0,1)$ and so U satisfies the transitivity condition (but not necessarily all U satisfy the condition) or $U_n \to U_\infty$ as $n \to \infty$ and by continuity appropriate iterates of f map U_∞ into itself.

Proof part 3

Hence once again either f is renormalizable or $U_\infty = (0,1)$. But since the return times are decreasing, they also tend to a limit, m_k^∞, $k = 0, 1$, and these are reached in finite steps. Thus if $U_n \neq (0,1)$ for all $n > N_0$ (transitivity again) the minimality of $(0,1)$ implies that $U_n \cup f(U_n) = (0,1)$ for large enough n; the transitivity condition again.

Thus for each U either U satisfies the transitivity condition or f is renormalizable. Hence either f is renormalizable or f is not renormalizable and every open U satisfies the transitivity condition and hence f is transitive.

Proof part 4

If f is renormalizable let J_k be the intervals as in the definition and choose the maximal intervals satisfying the renormalization criterion.
Let

$$K = J_0 \cup \left(\cup_1^{n_0} f^r(J_0)\right) \cup \{c\} \cup J_1 \cup \left(\cup_1^{n_1} f^r(J_1)\right)$$

and let $L = I \backslash K$. Then L is a (possibly empty) finite union of closed intervals and since the sets K are mapped to themselves if $f(L_i) \cap L_j \neq \emptyset$ then $L_j \subseteq f(L_i)$, i.e. L_i f-covers L_j and so the dynamics in L can be described by a Markov graph. Setting $T = \Omega(f) \cap L$ and $R = \Omega(f) \cap c\ell(K)$ produces the stated descomposition of the non-wandering set.

END of proof.

Application: an example

Consider the map

$$x_{n+1} = \begin{cases} 1 - 2x_n & \text{if} x_n > 0 \\ b + x_n & \text{if } x_n < 0 \end{cases}$$

with

$$b = \frac{1}{5}.$$

What can you say about the dynamics?

Application: hint

If my algebra is correct, the orbits of 0 approached from both above and below are periodic, and therefore these points can be used to construct a Markov partition.

The detailed dynamics is left as an exercise.