Dynamics of piecewise smooth maps
Lecture I: Introduction

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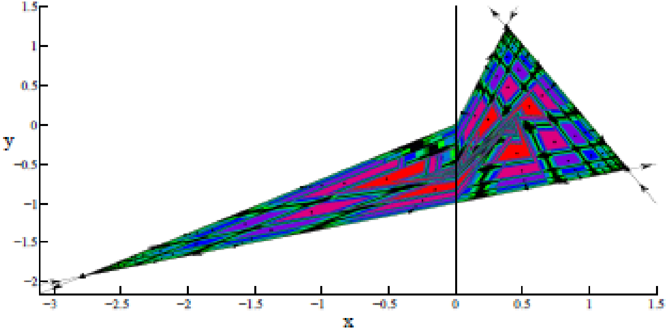
Summer School, ICTS Bangalore, June 2018

Piecewise smooth maps

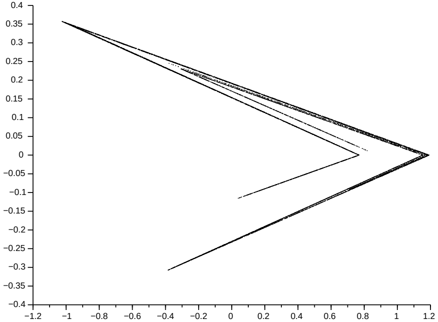
- Map is smooth on open regions of phase space with transitions across boundaries from one map to another.
- Piecewise smooth and continuous across the boundary or discontinuous across the boundary.
- Less is more: try to avoid zoology (i.e. lists) of different behaviours.
- Possible bifurcation sequences can be very complicated compared with smooth maps.

Levels of complexity: periodic orbits



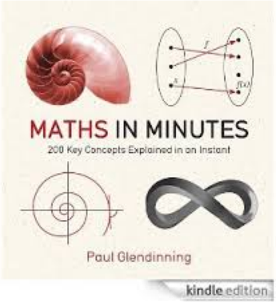
Infinitely many stable periodic orbits (which also occurs in typical families of smooth maps near homiclinic tangencies). From Simpson (2014).

Levels of complexity: robust chaos



Stable chaotic solutions persist under appropriate perturbation (unlike typical smooth cases).

Introductions: Paul Glendinning



- Professor at University of Manchester, UK, and Scientific Director, ICMS, Edinburgh.
- I work on many areas of bifurcation theory: homoclinic bifurcations in smooth systems, quasi-periodically forced systems and strange non-chaotic attractors, piecewise smooth maps and flows...
- ... and applications (cardiac arrhythmias, ecological models, neuro-models, paleo-climate,...)
- Also write popular maths.

Aim of the course

Transitions and bifurcation structure can be very complicated. I'll try to concentrate more on techniques and how these can be applied.

I hope this will enable you to analyze your own examples and/or see how theory can be developed.

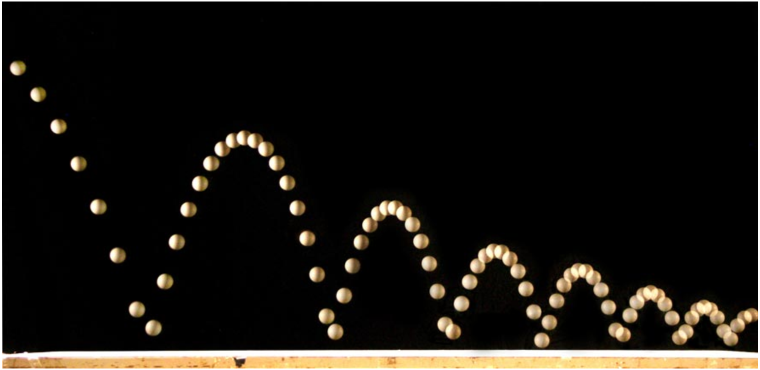
Necessarily sketchy, don't expect full details.

Plan of the lectures

I. Introduction and scope.
II. Piecewise Smooth Maps of the Interval..
III. Piecewise Smooth Maps of the Plane.
IV. Higher dimensions.

Why study non-smooth? Bouncing Ball

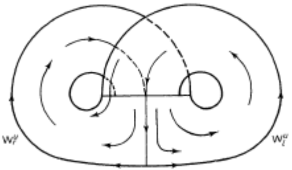
Impacts (mechanics) provide a natural source of non-smooth behaviour.



Reset initial conditions after each bounce. (This is from Andrew Davidhazy: real photos of ping pong ball!)

Why study non-smooth? Lorenz semi-flow

Smooth dynamics can lead to piecewise smooth maps (e.g. near saddle).



(a)

(b)

From Guckenheimer and Williams (1979): note how early smooth dynamics encountered piecewise smooth models.

Why study non-smooth? 'Simpler' models of complex behaviour

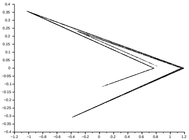
Either for applications or a test-bed for theory.

- Integrate and fire neurons (Izikevitch)
- The heart (Glass)
- Hénon to Lozi (Misieurewicz) More next slide.
- One-dimensional unimodal (Collet)
- Existence of invariant measures (Young)

Example: Lozi and Hénon

Lozi map (non-smooth) 1978

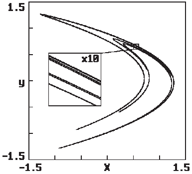
$$\begin{aligned} x_{n+1} &= 1 - a|x_n| + by_n \\ y_{n+1} &= x_n \end{aligned}$$



• first non-trivial 2D example with proved chaotic attractor (Misieurewicz 1980)

Hénon map (smooth) 1976

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + by_n \\ y_{n+1} &= x_n \end{aligned}$$



• still no explicit proof of chaotic attractor at the 'standard' parameters (but know there are positive measure of parameters with

Smooth dynamics: bifurcation theory

- Stable/centre manifold theorems classification by dimensions of stable and unstable manifolds, two generic bifurcations, plus a bit of complication, but independent of dimension.
- Homoclinic orbits dependent on leading eigenvalues not ambiant dimension. Shilnikov classification: three for maps, three for stationary points of flows. Though infinite modulus of stability.
- Horseshoes Yorke: ubiquity of period-doubling cascades.

Yes, you can make life more complicated – three Ss: symmetry, solenoids, stochasticity and sircle maps... – but you have to try!

Nonsmooth dynamics

Filippov classification of sliding singularities for differential equations *in the plane*

- 3 of 6 types have infinite numbers of topological classes;
- boundary equilibrium bifurcations (BEBs): 12 different codimension one types.

Hard to keep track (e.g. Hogan et al 2015: the case of the missing BEBS).

But:

- often have much easier expansion conditions (no 'infinite' contraction where $f'(x) = 0$ for example);
- piecewise linear examples can allow much more explicit calculation.

Introduction to Maps

By definition a PWS system is smooth in regions, so any dynamics that does not interact with a boundary can be described using smooth theory.

This includes the existence and stability of fixed points and periodic orbits in smooth regions and their bifurcations (though there are new bifurcations involving the boundary).
A fixed point of a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a solution of

$$x = f(x) \quad \text{so } x_{n+1} = f(x_n) = x_n.$$

Stability

By considering small perturbations, a fixed point is stable (or more accurately, linearly stable) if all the eigenvalues of the Jacobian matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

evaluated at the fixed point lie inside the unit circle (i.e. have modulus less than one).

Periodic Points

A point is periodic of period p if

$$x = f^p(x)$$

where $f^p(x) = f(f^{p-1}(x))$, i.e. it denotes the p^{th} iterate of f,

$$f^p = f \circ f \circ \cdots \circ f \quad (p \text{ times}),$$

and not the p^{th} power of $f(x)$ which we will denote by $[f(x)]^p$ or similar. If x is a point of period p then the periodic orbit containing x is

$$\{x, f(x), \ldots, f^{p-1}(x)\}$$

and if all the points are distinct then it is sometimes worth emphasising that p is the minimal possible period of the orbit (though usually this is left unstated). Note that if x has period p then it also has period mp for all $m > 1$.

Stability

Since a periodic point can be viewed as a fixed point of f^p, the linear stability of a periodic orbit is determined by the eigenvalues of the Jacobian matrix

$$Df^p(x) = Df(f^{p-1}(x))Df(f^{p-2}(x))\dots Df(x).$$

This is a big difference with continuous time systems, where periodic orbits need to be considered using different methods.

Bifurcations occur is an eigenvalue passes through the unit circle, so there are three generic cases: a simple eigenvalue of $+1$, a simple eigenvalue of $+1$, or a pair of simple eigenvalues $e^{\pm i\theta}$, $\theta \neq m\pi$, $m \in \mathbb{Z}$.

Bifurcations

By the centre manifold theorem, typical bifurcations are independent of the phase space dimension. An eigenvalue passing through $+1$ gives the standard saddlenode bifurcation (or transcritical or pitchfork if some partial derivatives vanish). The new cases are:

- Period-doubling, eval of -1 A fixed point changes stability at the bifurcation value and an orbit of period two is created. If this period two orbit is stable it is called a supercritical period-doubling bifurcation.
- Neimark-Sacker (Hopf), complex evals $e^{\pm 2\pi i\theta}$ The fixed point changes stability and an invariant curve bifurcates on which there can be other attractors (e.g. periodic orbits) near resonances when θ is rational.

Monotonic maps

Very useful but easy result with applications in PWS systems:

Lemma 1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous map. If f is increasing then every bounded orbit is either a fixed point or tends to a fixed point. If f is decreasing then every bounded orbit is either a fixed point or a point of period two or tends to a fixed point or a point of period two.*

Proof starts with a simple trichotomy. Suppose that f is increasing, i.e. $x < y$ implies that $f(x) \leq f(y)$. Take $x \in \mathbb{R}$. Then either

$$f(x) = x, \quad \text{or} \quad f(x) > x, \quad \text{or} \quad f(x) < x.$$

Also recall that an increasing sequence bounded above tends to a limit.
Now use the blackboard!

Describing chaos

Throughout rest of the lecture f will be a continuous map $f : \mathbb{R} \to \mathbb{R}$.
For intervals J and K say J f-covers K if $K \subseteq f(J)$.

f-covers provide a language to understand the existence of periodic orbits etc via combinatorics (graphs).

Lemma 3. *If a closed interval J f-covers itself then J contains a fixed point of f.*

The proof is just an exercise with the Intermediate Value Theorem.

Markov Graphs

Let $J_1, \ldots, J_m$ be closed intervals with disjoint interiors. A Markov graph of f is a directed graph with vertices $1, \ldots, m$ and a directed edge from i to j iff J_i f-covers J_j. The transition matrix associated with this graph is the $m \times m$ matrix T with

$$T_{ij} = \begin{cases} 1 & \text{if } J_i \ f\text{–covers } J_j \\ 0 & \text{otherwise.} \end{cases}$$

Paths

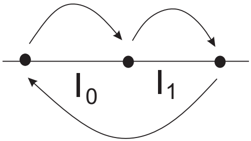
A path in a directed graph is an ordered sequence of vertices $a_0 a_1 \dots a_k$ such that there is a directed edge from a_i to a_{i+1} for each $i = 0, \dots, k-1$. The length of the path is the number of edges traversed (i.e. k in the example). Note that if there is a path from a_0 to a_k of length k if and only if $T^k_{a_0 a_k} \neq 0$.

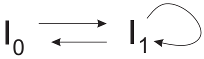
It is not hard to show (by induction on the length of a path) that if there is a path of length k from $a_0 \dots a_k$ in the Markov graph then there exists a closed interval $L \subset J_{a_0}$ such that $f^k(L) = J_{a_k}$ and $f^r(L) \subseteq J_{a_r}$. In particular a closed path implies the exience of a periodic orbit.

This is the basic tool for proving classic theorems such as Sharkovskii's Theorem. It also provides a motivation for the definition of a one-dimensional horseshoe.

Application: Period Three implies all periods (Li and Yorke, 1973)

If f has an orbit of period three with points $x_1 < x_2 < x_3$ then either $f(x_1) = x_2$, $f(x_2) = x_3$ and $f(x_3) = x_1$ or the orbit has this order after $x \to -x$. Assume this order and let $I_0 = [x_1, x_2]$ and $I_1 = [x_2, x_3]$.





Thus I_0 f-covers I_1 and I_1 f-covers I_0 and I_1 giving the Markov graph shown. read off the periodic orbits: I_1, $I_1 I_2$, three by assumption (using the end-points) and $I_1^{n-1} I_0$ for any $n \geq 4$.

Horseshoes

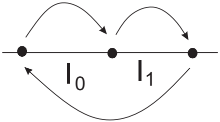
f has a horseshoe if there exist closed intervals J_0 and J_1 with disjoint interiors such that J_0 f-covers both J_0 and J_1 and J_1 f-covers both J_0 and J_1.

So if f has a horseshoe then for any sequence of 0s and 1s $a_0 a_1 \ldots$ there exists $x \in J_{a_0}$ such that $f^r(x) \in J_{a_r}$ for all $r > 0$.

This is sometimes described as f having dynamics equivalent to a full shift on two symbols. We will say that f is chaotic if f^n has a horseshoe for some $n \geq 1$.

Application: Period Three implies chaos (Li and Yorke, 1973)

Each of I_0 and I_1 f^2-cover both I_0 and I_1 so f^2 has a horsehoe.



And for PWS systems?

Note that these results only need f to be continuous on the intervals J_k; what happens between these intervals is immaterial. This means that the methods are often applicable in PWS systems.

This will be developed in later lecture.