On the Stability of periodic orbits in switching dynamical systems

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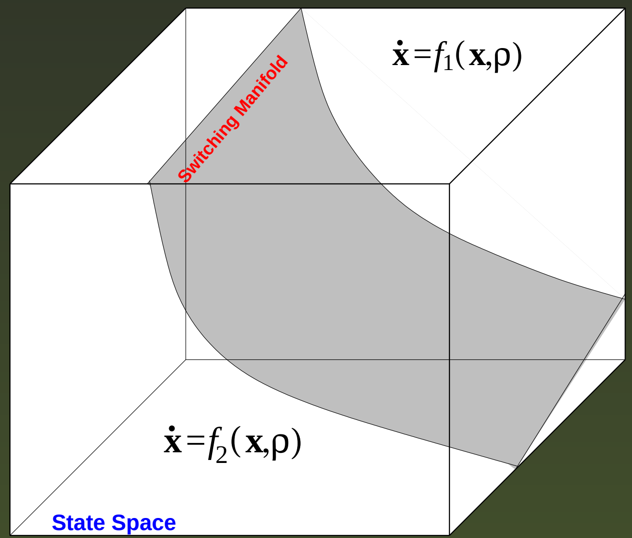
Class of systems

$$\dot{\mathbf{x}} = f(\mathbf{x}, \rho) = \begin{cases} f_1(\mathbf{x}, \rho) & \text{for} \quad \mathbf{x} \in R_1 \\ f_2(\mathbf{x}, \rho) & \text{for} \quad \mathbf{x} \in R_2 \\ \vdots & \\ f_n(\mathbf{x}, \rho) & \text{for} \quad \mathbf{x} \in R_n \end{cases}$$

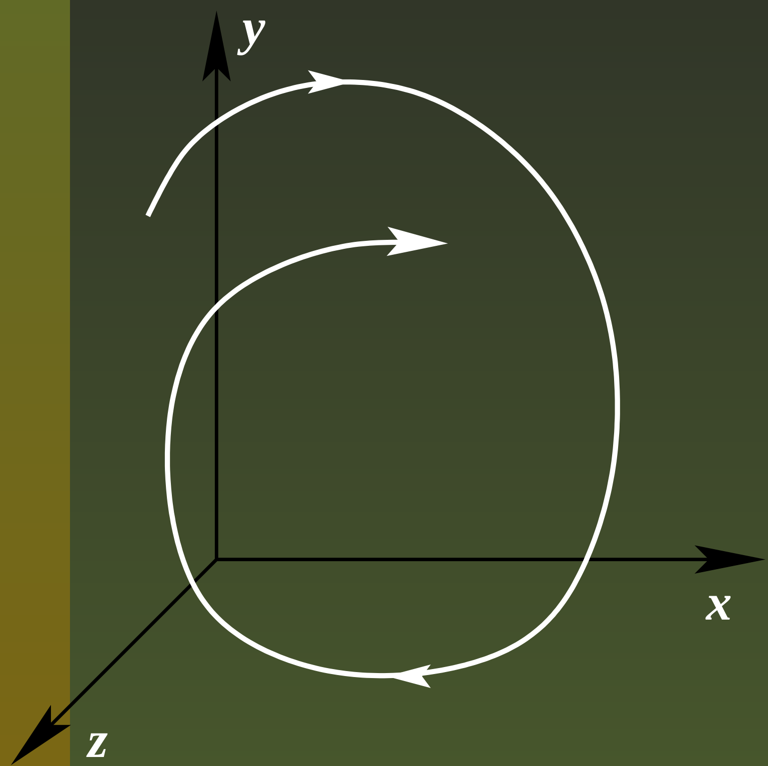


The regions R_1, R_2 etc. are divided by the discrete event conditions: $(n-1)$ dimensional surfaces given by algebraic equations of the form $\Gamma_n(\mathbf{x}) = 0$. These are the "switching manifolds."

Whose stability?

- If an equilibrium point is located in one of the partitions, the issue of its stability is trivial (in no way different from that of a smooth system). Eigenvalues of the Jacobian matrix at the equilibrium point has to be placed in the left half plane.
- In most hybrid systems, one is however concerned with *orbits* which visit more than one partition.
- The stability of a periodic orbit is distinctly different from the stability of an equilibrium point.

An orbit in state space



System equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t),$$

Initial condition:

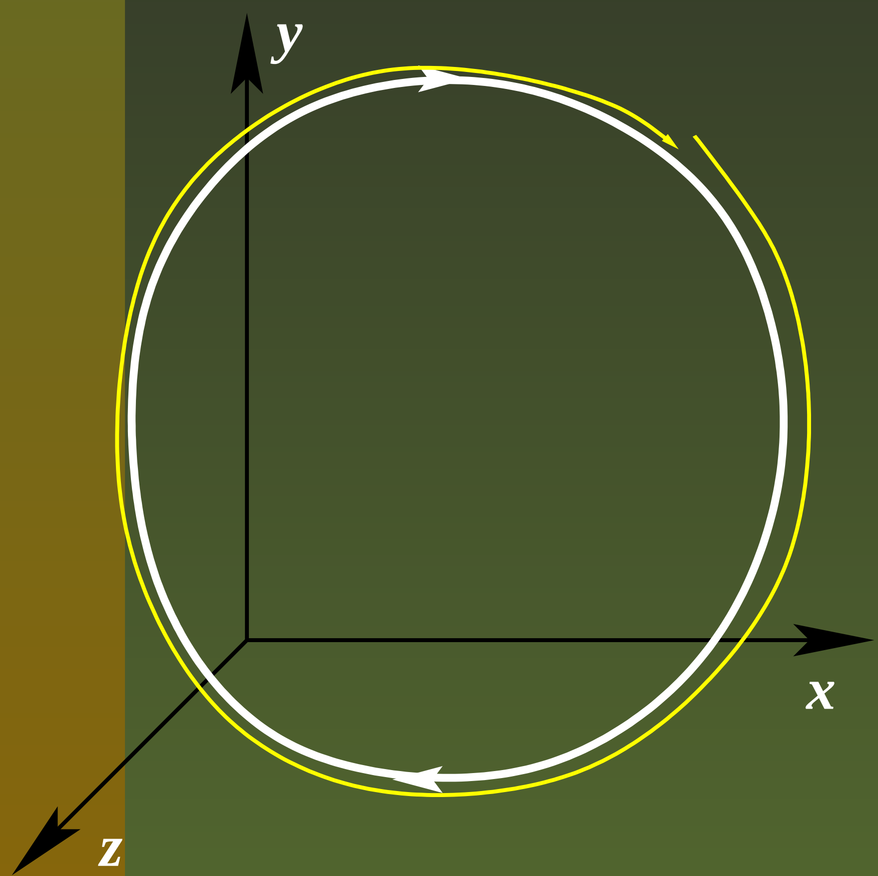
$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0.$$

The solution

$$\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{\varphi}(\tau, t_0, \boldsymbol{x}_0), \tau)\mathrm{d}\tau$$

Stability of orbits

Stability: If a small perturbation is added and the solution converges back to the orbit, then the orbit is stable.



How sensitive is the trajectory to perturbations in the initial condition?

State transition matrix

Suppose a system is governed by a set of differential equations

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t).$$

Starting from an initial condition $\boldsymbol{x}(0)$, it gives a solution $\boldsymbol{x}(t)$ after time t,

We may then relate the vectors $\boldsymbol{x}(t)$ and $\boldsymbol{x}(0)$ by a matrix $\boldsymbol{\Phi}$ such that

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0) \tag{1}$$

Since this matrix $\boldsymbol{\Phi}$ represents a transition from the state at time 0 to that at time t, it is called the *state transition matrix*.

State transition matrix

Consider a linear time-invariant homogenious equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}. \tag{2}$$

If the equation is 1D, i.e., $\dot{x} = ax$, then its solution is $x(t) = e^{at}\, x(0)$, and the exponential term can be expanded as

$$e^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots$$

State transition matrix

So, for $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, guess solution: $\boldsymbol{x}(t) = e^{\mathbf{A}t}\boldsymbol{x}(0)$, where the matrix $e^{\mathbf{A}t}$ is the power series :

$$e^{\mathbf{A}t} = 1 + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots$$

Substituting this guess solution into the equation $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, we get

$$\begin{aligned}\frac{d}{dt}e^{\mathbf{A}t}\boldsymbol{x}(0) &= \mathbf{A}\left[1 + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots\right]\boldsymbol{x}(0) \\ &= \mathbf{A}e^{\mathbf{A}t}\boldsymbol{x}(0)\end{aligned}$$

State transition matrix

Therefore, the solution is

$$\boldsymbol{x}(t) = e^{\mathbf{A}t}\,\boldsymbol{x}(0)$$

Thus, the exponential matrix $e^{\mathbf{A}t}$ is the state transition matrix $\boldsymbol{\Phi}(t)$ for the equation $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$

$$\begin{aligned} \boldsymbol{x}(t) &= e^{\mathbf{A}t}\,\boldsymbol{x}(0) \\ &= \boldsymbol{\Phi}(t)\,\boldsymbol{x}(0) \end{aligned}$$

State transition matrix

Now consider the nonhomogeneous state equation

$$\dot{\boldsymbol{x}} = \mathbf{A}\,\boldsymbol{x} + \mathbf{B}\,\mathbf{U}$$

where $\mathbf{U}$ is a vector of inputs, and $\mathbf{B}$ represents the coefficients.
Rearranging, we get

$$\dot{\boldsymbol{x}}(t) - \mathbf{A}\,\boldsymbol{x}(t) = \mathbf{B}\,\mathbf{U}(t)$$

Premultiplying both sides by $e^{-\mathbf{A}t}$, we get

$$e^{-\mathbf{A}t}\left[\dot{\boldsymbol{x}}(t) - \mathbf{A}\,\boldsymbol{x}(t)\right] = e^{-\mathbf{A}t}\,\mathbf{B}\,\mathbf{U}(t)$$

State transition matrix

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\boldsymbol{x}(t)\right] = e^{-\mathbf{A}t}\,\mathbf{B}\,\mathbf{U}(t)$$

By integrating this equation between 0 and t we get

$$e^{-\mathbf{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{U}(\tau)d\tau$$

or,

$$\boldsymbol{x}(t) = e^{\mathbf{A}t}\boldsymbol{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{U}(\tau)d\tau$$

State transition matrix

Now suppose we perturb the initial condition to $\boldsymbol{x}'(0)$. How does the perturbation $\delta\boldsymbol{x} = \boldsymbol{x}'(0) - \boldsymbol{x}(t)$ evolve? For the perturbed vector we get

$$\boldsymbol{x}'(t) = e^{\mathbf{A}t}\boldsymbol{x}'(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{U}(\tau)d\tau$$

Subtracting the two we get

$$\boldsymbol{x}'(t) - \boldsymbol{x}(t) = e^{\mathbf{A}t}\boldsymbol{x}'(0) - e^{\mathbf{A}t}\boldsymbol{x}(0)$$

or,

$$\delta\boldsymbol{x}(t) = e^{\mathbf{A}t}\delta\boldsymbol{x}(0) = \mathbf{\Phi}(t)\delta\boldsymbol{x}(0)$$

Trajectory sensitivity

Now consider a general system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. The trajectory evolves as

$$\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{\varphi}(\tau, t_0, \boldsymbol{x}_0), \tau)\mathrm{d}\tau$$

Differentiating w.r.t the initial condition,

$$\frac{\partial \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0} = \boldsymbol{I} + \int_{t_0}^{t} \mathbf{A}(\tau, \boldsymbol{x}) \frac{\partial \boldsymbol{\varphi}(\tau, t_0, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0} \mathrm{d}\tau \qquad (3)$$

$$\text{where} \quad \mathbf{A}(t, \boldsymbol{x}) = \left. \frac{\partial \boldsymbol{f}(\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0), t)}{\partial \boldsymbol{x}} \right|_{\boldsymbol{x}=\boldsymbol{x}(t)}$$

Trajectory sensitivity

By differentiating (3) with respect to time,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0}\right) = \mathbf{A}(t, \boldsymbol{x})\frac{\partial \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0} \qquad (4)$$

The solution of this matrix differential equation

$$\boldsymbol{\Phi} = \frac{\partial \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0}$$

is called the *sensitivity function*.

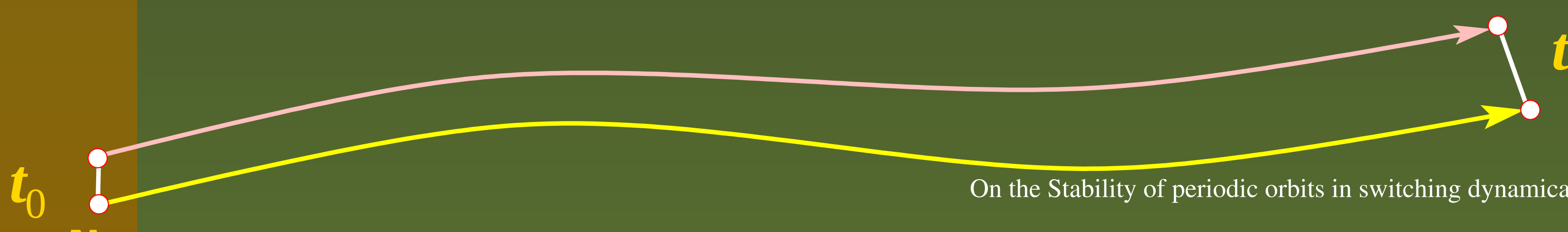
Trajectory sensitivity

The matrix differential equation

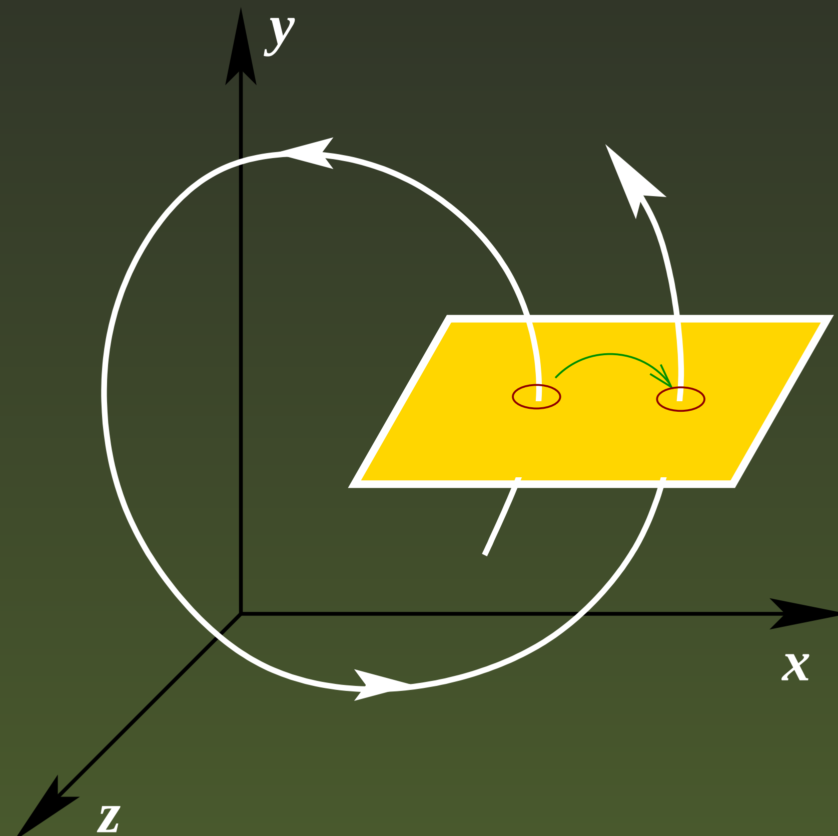
$$\dot{\mathbf{\Phi}} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}(t)} \mathbf{\Phi}$$

has to be solved with initial condition $\mathbf{\Phi}(0) = \boldsymbol{I}$. The sensitivity of the flow to initial condition can be obtained by the Taylor series expansion:

$$\delta\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0) = \frac{\partial \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0} \delta\boldsymbol{\varphi}(t_0, t_0, \boldsymbol{x}_0) + \text{H.O.T.} \quad (5)$$



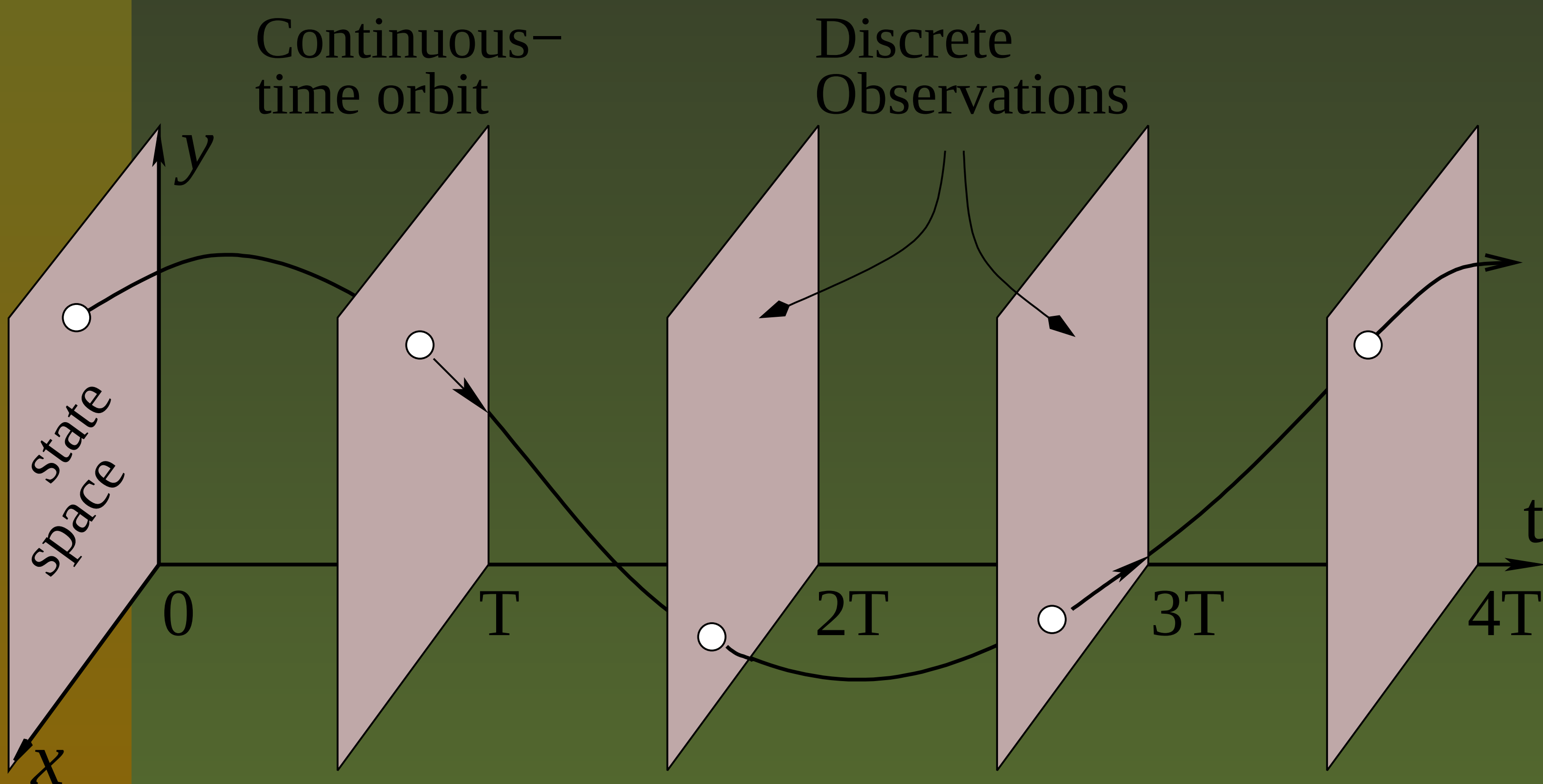
The Poincaré map approach



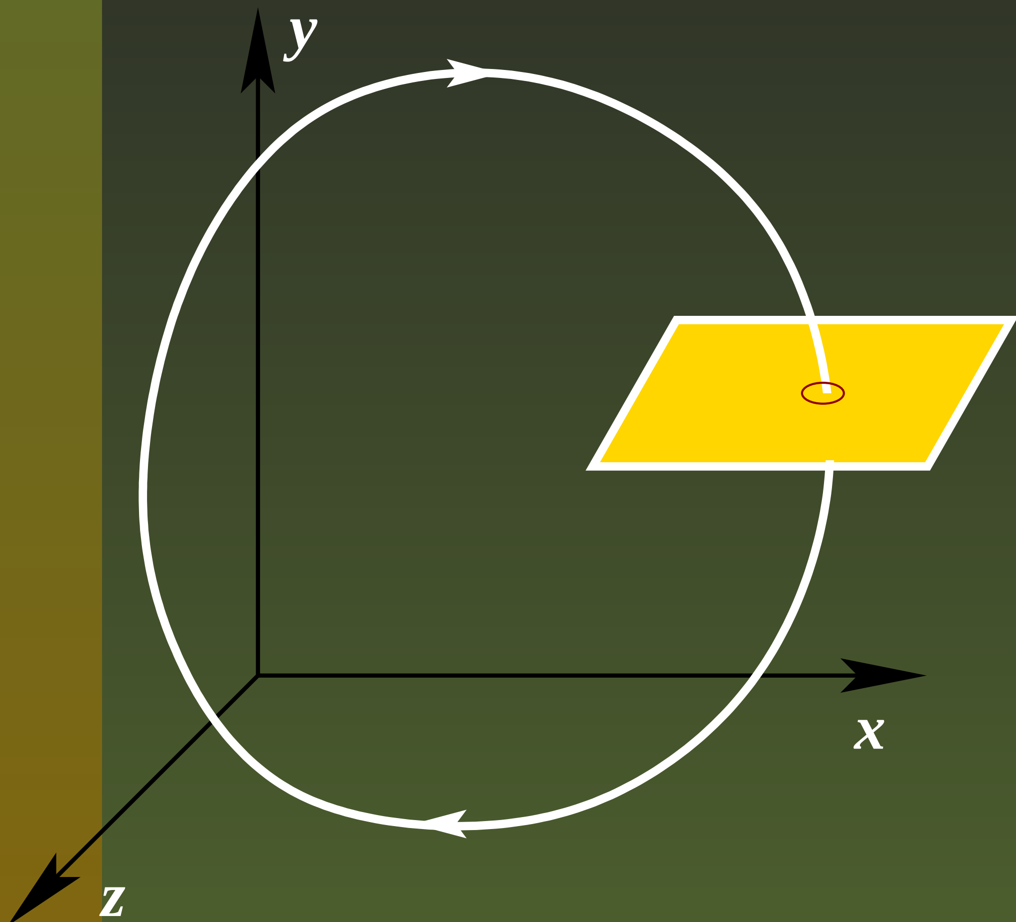
On the Poincaré section, $\boldsymbol{x}_{n+1} = f(\boldsymbol{x}_n)$.

Systems with external periodic input

The sampled data model is a nonlinear function that will map $\boldsymbol{x}(nT)$ to $\boldsymbol{x}((n+1)T)$. This gives $\boldsymbol{x}_{n+1} = f(\boldsymbol{x}_n)$



For a periodic orbit



Fixed point:
$\mathbf{x}_{n+1} = \mathbf{x}_n$
related to
periodic orbit.

But how to obtain the Poincaré map for a hybrid system?

Sampled data model

Consider a general non-autonomous piecewise linear system

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \mathbf{A}_i \boldsymbol{x}(t) + \mathbf{B}_i \boldsymbol{u}(t)$$

that goes through subsystems $i = 1, 2$, within a period T. First phase dT, second phase $(1-d)T, \quad 0 < d < 1.$

G. C. Verghese, M. E. Elbuluk, and J. G. Kassakian, “A general approach to sampled-data modeling for power electronic circuits,” IEEE Transactions on Power Electronics, vol. PE-1, no. 2, pp. 76-89, 1986.

The sampled data model

Through the period dT, the state evolves as

$$\begin{aligned}\boldsymbol{x}(dT) &= e^{\mathbf{A}_1 dT}\boldsymbol{x}(0) + \int_0^{dT} e^{\mathbf{A}_1(dT-\tau)}\mathbf{B}_1 u(\tau)\mathrm{d}\tau \\ &= \boldsymbol{\Phi}_1(dT,0)\boldsymbol{x}(0) + \mathbf{I}_1(d).\end{aligned}$$

After switching, through the period $(1-d)T$ the state evolves as

$$\begin{aligned}\boldsymbol{x}(T) &= e^{\mathbf{A}_2(T-dT)}\boldsymbol{x}(dT) + \int_{dT}^{T} e^{\mathbf{A}_2(T-\tau)}\mathbf{B}_2 u(\tau)\mathrm{d}\tau \\ &= \boldsymbol{\Phi}_2(T,dT)\boldsymbol{x}(dT) + \mathbf{I}_2(d).\end{aligned}$$

The sampled data model

Take final state before switching equal to initial state after switching,

$$\begin{aligned} \boldsymbol{x}(T) &= \boldsymbol{f}(\boldsymbol{x}_0, d) \\ &= \boldsymbol{\Phi}_2(T,dT)\{\boldsymbol{\Phi}_1(dT,0)\boldsymbol{x}(0)+\mathbf{I}_1(d)\}+\mathbf{I}_2(d) \end{aligned}$$

This is the sampled-data model of the system.

Obtaining the periodic orbit

For a periodic orbit, $\mathbf{x}(T) = \mathbf{x}(0)$, this gives

$$\boldsymbol{x}(0) = \left[\boldsymbol{I} - \boldsymbol{\Phi}_2(T, dT)\boldsymbol{\Phi}_1(dT, 0)\right]^{-1} \left[\boldsymbol{\Phi}_2(T, dT)\mathbf{I}_1(d) + \mathbf{I}_2(d)\right] \quad (6)$$

The switching events occur when the algebraic equation

$$h(\boldsymbol{x}(0), d) = 0 \quad (7)$$

is satisfied.

Substituting (6) into this equation, we have an equation involving only one unknown: the duty ratio d. This can be solved easily using any numerical routine. This procedure yields the location of the periodic orbit.

Stability of the periodic orbit

We need to obtain the linearization of the sampled data model. Differentiating this with respect to $\boldsymbol{x}(0)$ and using the chain rule we get

$$\frac{\partial \boldsymbol{x}(T)}{\partial \boldsymbol{x}(0)} = \frac{\partial \boldsymbol{f}(\boldsymbol{x}_0, d)}{\partial \boldsymbol{x}(0)} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_0, d)}{\partial d} \frac{\partial d}{\partial \boldsymbol{x}(0)} \tag{8}$$

Note that the duty ratio d is a function of $\boldsymbol{x}(0)$. By a similar procedure, the switching condition $h(\boldsymbol{x}(0), d) = 0$ yields

$$\frac{\partial h}{\partial \boldsymbol{x}(0)} + \frac{\partial h}{\partial d} \frac{\partial d}{\partial \boldsymbol{x}(0)} = 0.$$

Stability of the periodic orbit

By rearranging the last equation:

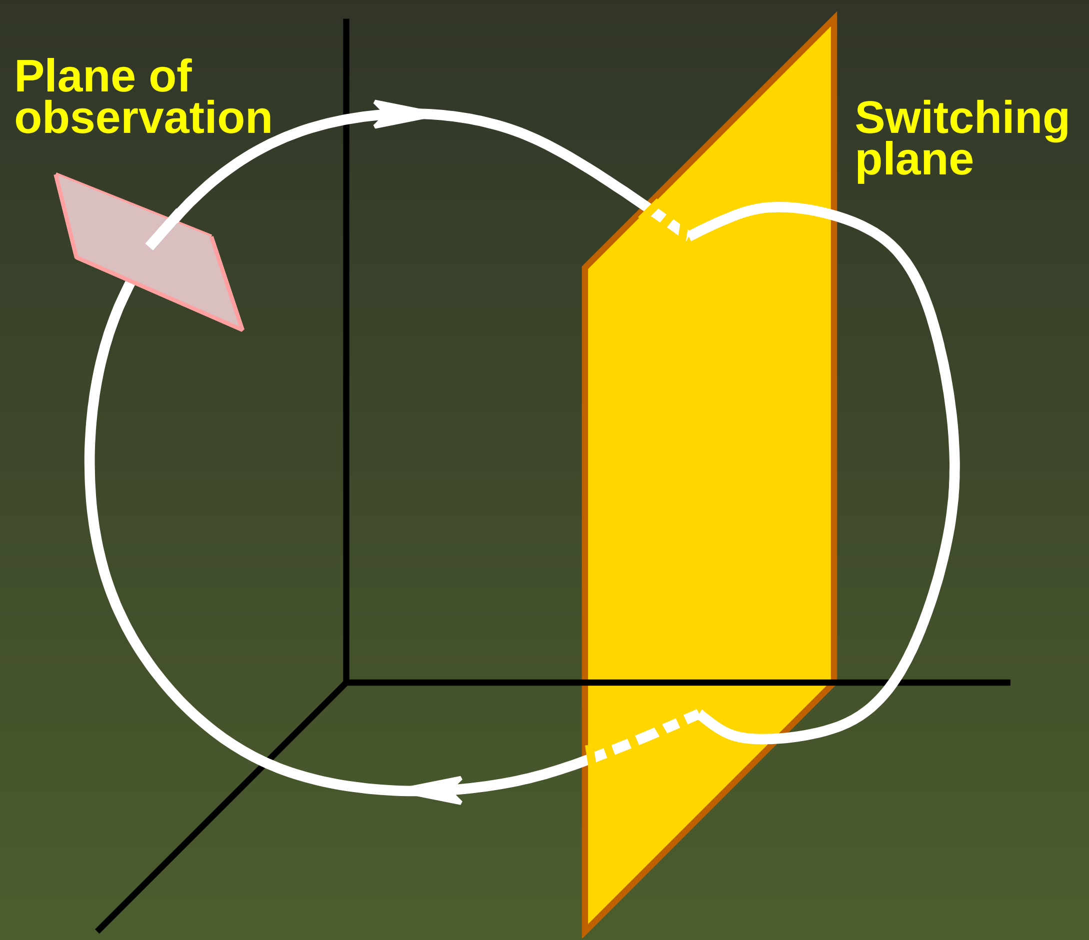
$$\frac{\partial d}{\partial \boldsymbol{x}(0)} = -\left(\frac{\partial h}{\partial d}\right)^{-1} \frac{\partial h}{\partial \boldsymbol{x}(0)}$$

Substituting this into (8), we get

$$\frac{\partial \boldsymbol{x}(T)}{\partial \boldsymbol{x}(0)} = \frac{\partial \boldsymbol{f}(\boldsymbol{x}_0, d)}{\partial \boldsymbol{x}(0)} - \frac{\partial \boldsymbol{f}(\boldsymbol{x}_0, d)}{\partial d}\left(\frac{\partial h}{\partial d}\right)^{-1} \frac{\partial h}{\partial \boldsymbol{x}(0)} \qquad (9)$$

This is the expression of the Jacobian matrix.

Orbit in a general hybrid system



The Floquet theory approach

Consider the periodic orbit $\boldsymbol{x}_p(t)$.

The perturbation: $\delta\boldsymbol{x}(t) = \boldsymbol{x}_{\bar{p}}(t) - \boldsymbol{x}_p(t)$.
We have seen that the perturbation $\delta\boldsymbol{x}(t)$ is modeled by a homogeneous linear time varying model:

$$\frac{\mathrm{d}\delta\boldsymbol{x}(t)}{\mathrm{d}t} = \left.\frac{\partial\boldsymbol{f}(\boldsymbol{x},t)}{\partial\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_p} \delta\boldsymbol{x}(t) = \mathbf{A}(t,\boldsymbol{x}_p)\delta\boldsymbol{x}(t)$$

Floquet, G., "Sur les équations differentielles linéaires à coefficients périodiques," *Ann. Sci. Ecole Norm. Sup.*, 1883, Ser. 2, **12**, page 47.

Floquet theory

Problem: $\mathbf{A}(t, \boldsymbol{x}_p)$ is a time varying matrix.

Floquet's conjecture was that the elements of the matrix

$$\mathbf{A} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_p}$$

vary in a periodic manner. Thus, if one makes observation of the behaviour of $\delta\boldsymbol{x}$ only at the discrete points in time $t = 0, T, 2T, 3T..$, one can obtain the matrix with constant elements.

From a geometrical point of view, this corresponds exactly to a Poincaré section.

Floquet theory

So the problem is to obtain an expression in the form

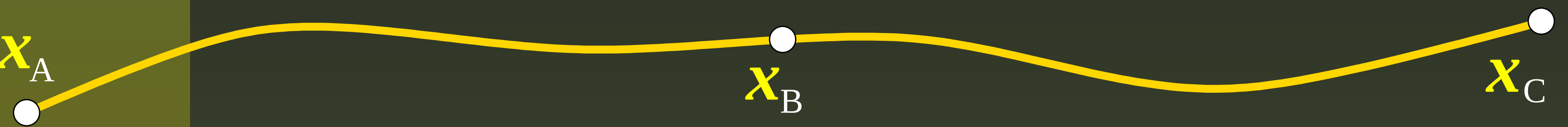
$$\delta \boldsymbol{x}(T) = \boldsymbol{\Phi}(T + t_0, t_0, \delta \boldsymbol{x}_0)\delta \boldsymbol{x}_0.$$

The matrix $\boldsymbol{\Phi}$ evaluated over a complete cycle, is called the *monodromy matrix*.

It is the state transition matrix over a complete cycle.

If the eigenvalues of the monodromy matrix, called the Floquet multipliers, are inside the unit circle, the system is stable.

Properties of state transition matrix

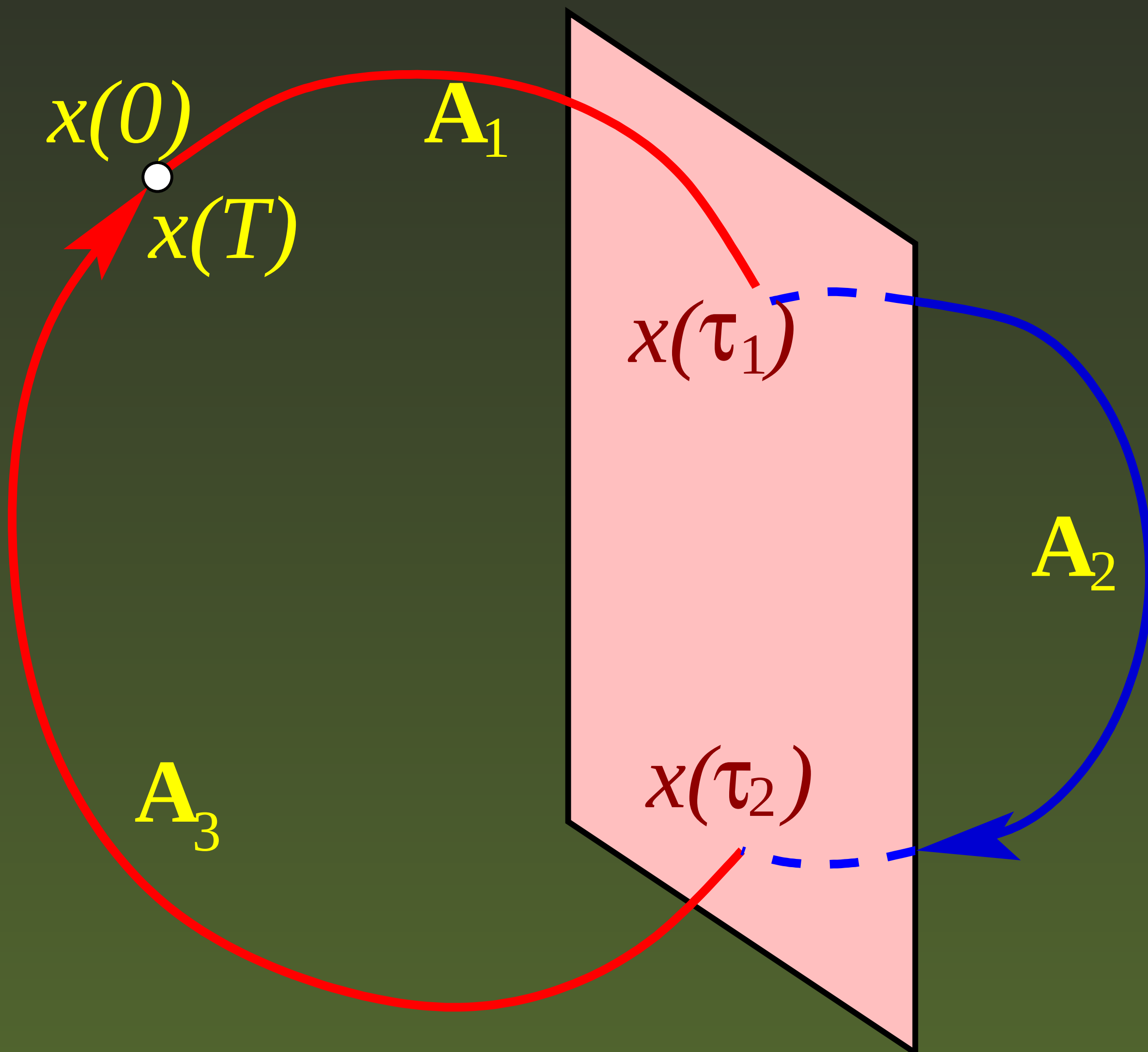


If $\quad \boldsymbol{x}_B = \boldsymbol{\Phi}_{AB}\,\boldsymbol{x}_A, \quad \boldsymbol{x}_C = \boldsymbol{\Phi}_{BC}\,\boldsymbol{x}_B$

then $\quad \delta\boldsymbol{x}_B = \boldsymbol{\Phi}_{AB}\,\delta\boldsymbol{x}_A, \quad \delta\boldsymbol{x}_C = \boldsymbol{\Phi}_{BC}\,\delta\boldsymbol{x}_B$

and $\quad \delta\boldsymbol{x}_C = \boldsymbol{\Phi}_{AC}\,\delta\boldsymbol{x}_A, \quad \boldsymbol{\Phi}_{AC} = \boldsymbol{\Phi}_{BC}\boldsymbol{\Phi}_{AB}$

Let us consider two partitions



Calculation of Jacobian by parts

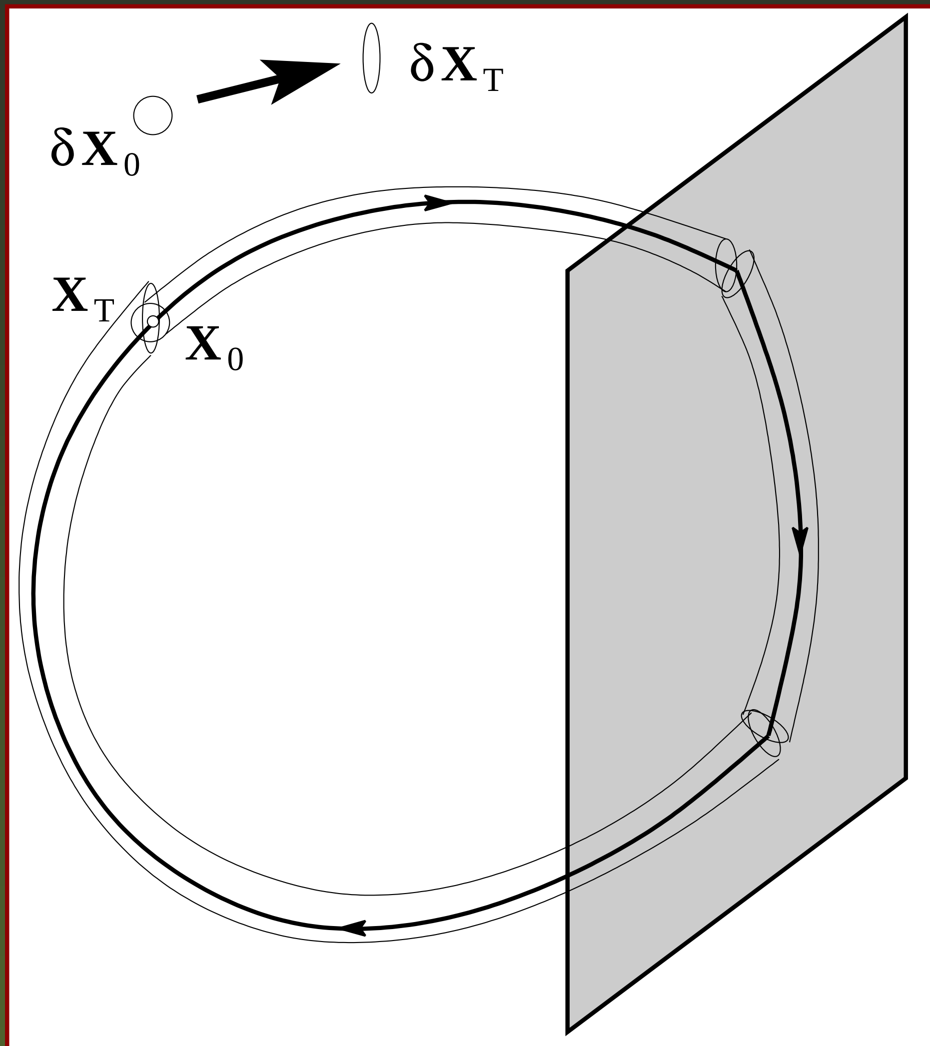
Suppose we are able to find the state transition matrices

$$\begin{aligned}\delta\boldsymbol{x}(\tau_1) &= \mathbf{A}_1\,\delta\boldsymbol{x}(0)\\ \delta\boldsymbol{x}(\tau_2) &= \mathbf{A}_2\,\delta\boldsymbol{x}(\tau_1)\\ \delta\boldsymbol{x}(T) &= \mathbf{A}_3\,\delta\boldsymbol{x}(\tau_2)\end{aligned}$$

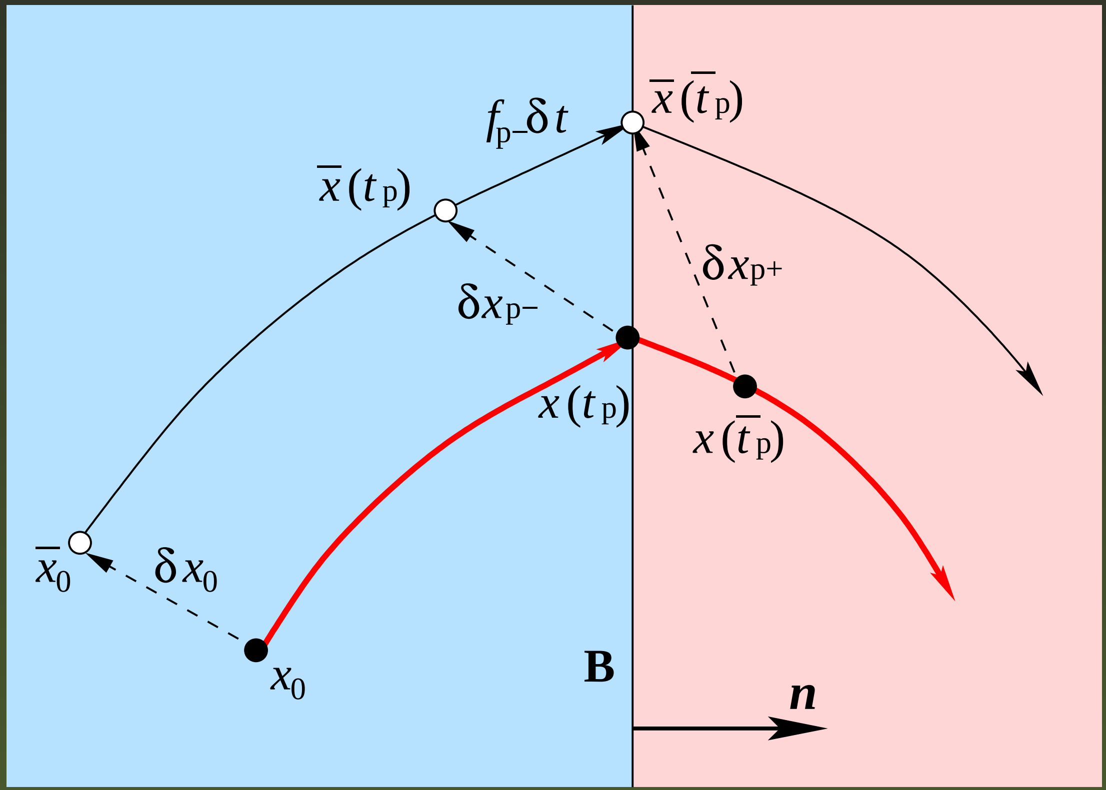
The product $\mathbf{A}_3 \cdot \mathbf{A}_2 \cdot \mathbf{A}_1$ does not give the state transition matrix over the whole cycle. One has to take into account how the perturbations change when they cross the border.

M. A. Aizerman and F. R. Gantmakher, “On the stability of periodic motions,” *Journal of Applied Mathematics and Mechanics* (translated from Russian), 1958, pages 1065-1078.

State transition matrix across border

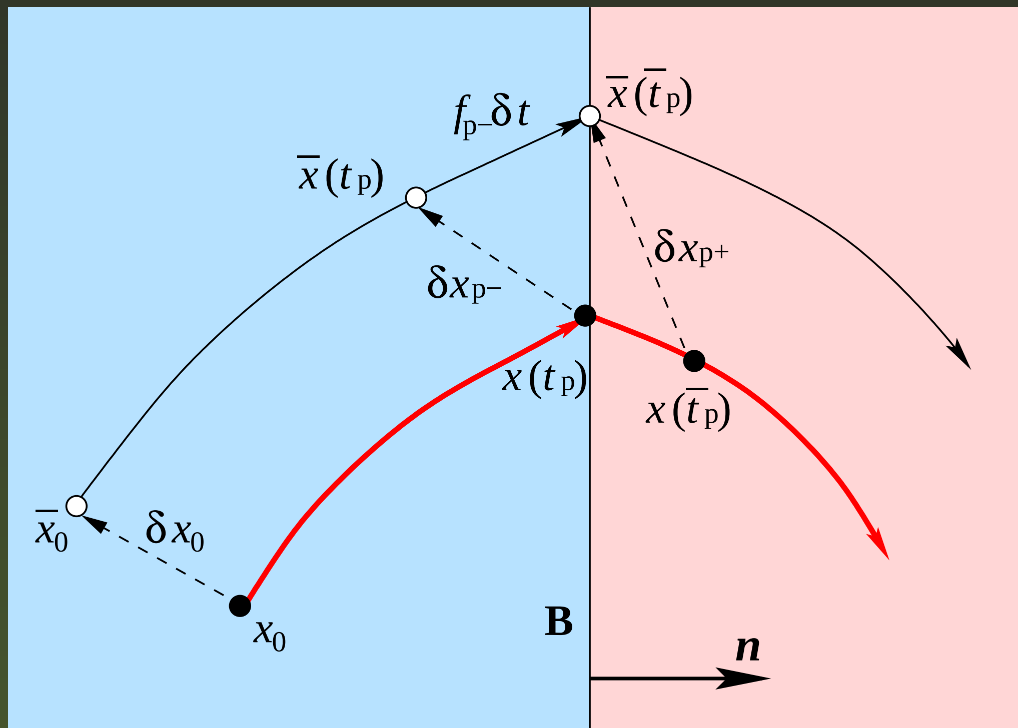


The saltation matrix



$$\bar{x}_0 = x_0 + \delta x_0$$
$$\delta x_{p-} = \bar{x}(t_p) - x(t_p)$$
$$\delta x_{p+} = \bar{x}(\bar{t}_p) - x(\bar{t}_p)$$

First order Taylor expansion



$$\begin{aligned} x(\bar{t}_p) &\approx x(t_p) + f_{p+}\delta t, \quad \text{where } f_{p+} := f_2(t_p, x(t_p)) \\ \bar{x}(\bar{t}_p) &\approx \bar{x}(t_p) + f_{p-}\delta t, \quad \text{where } f_{p-} := f_1(t_p, x(t_p)) \\ &\approx x(t_p) + \delta x_{p-} + f_{p-}\delta t \end{aligned}$$

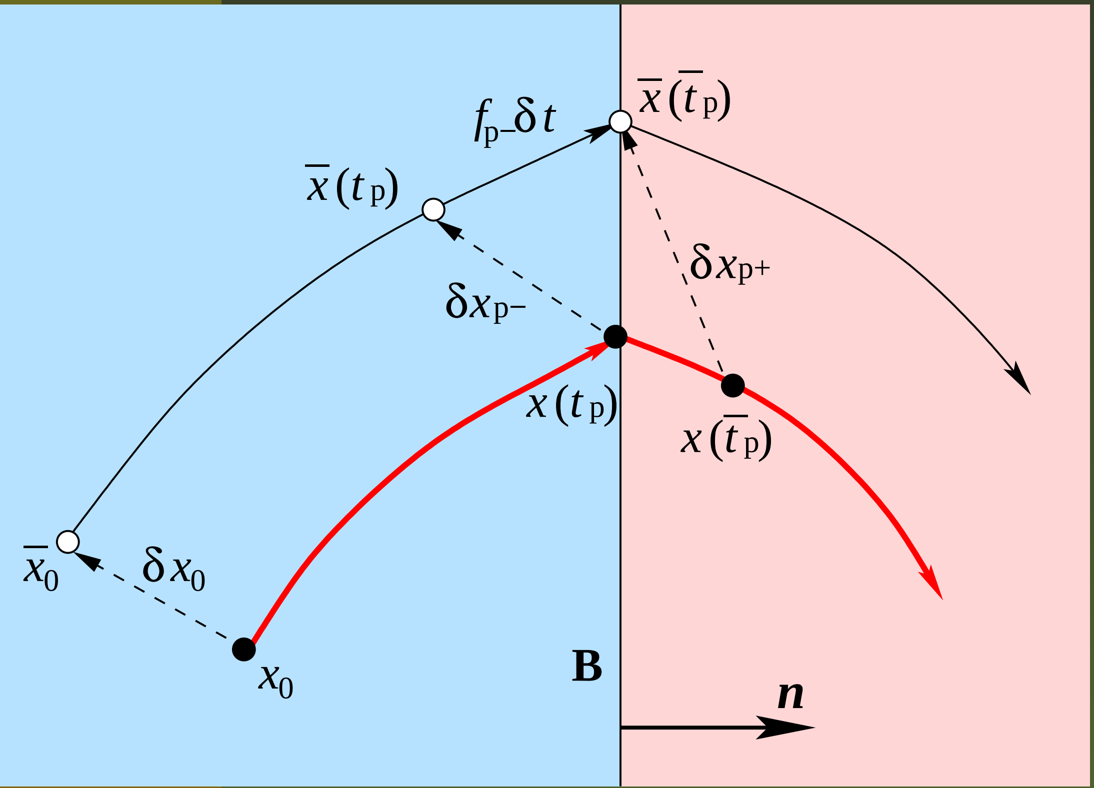
Substituting, we get

$$\begin{aligned}
\delta x_{p+} &= \bar{x}(\bar{t}_p) - x(\bar{t}_p) \\
&\approx x(t_p) + \delta x_{p-} + f_{p-}\delta t - x(t_p) - f_{p+}\delta t \\
&\approx \delta x_{p-} + f_{p-}\delta t - f_{p+}\delta t
\end{aligned}$$

Both the original solution and the perturbed solution satisfy the switching conditions: $\beta(x(t_p)) = 0$ and $\beta(\bar{x}(\bar{t}_p)) = 0$.

This gives

$$n^T f_{p-} \delta t = -n^T \delta x_{p-}$$



$$\Rightarrow \delta t = -\frac{n^T \delta x_{p-}}{n^T f_{p-}}$$

The saltation matrix

Substituting, we get

$$\delta x_{p+} \;=\; \delta x_{p-} + (f_{p+} - f_{p-})\,\frac{n^T \delta x_{p-}}{n^T f_{p-}}$$

Now, the saltation matrix relates how the perturbation before the crossing maps to the perturbation after the crossing, i.e.,

$$\delta x_{p+} = S\,\delta x_{p-}$$

Or,

$$S = I + \frac{(f_{p+} - f_{p-})\,n^T}{n^T f_{p-}}$$

The saltation matrix

Here we have assumed that the switching manifold does not vary with time. If it does, it can be shown that the resulting form of the saltation matrix is

$$S = I + \frac{(f_{p+} - f_{p-})\, n^T}{n^T f_{p-} + \frac{\partial h}{\partial t}\Big|_{t=t_p}}$$

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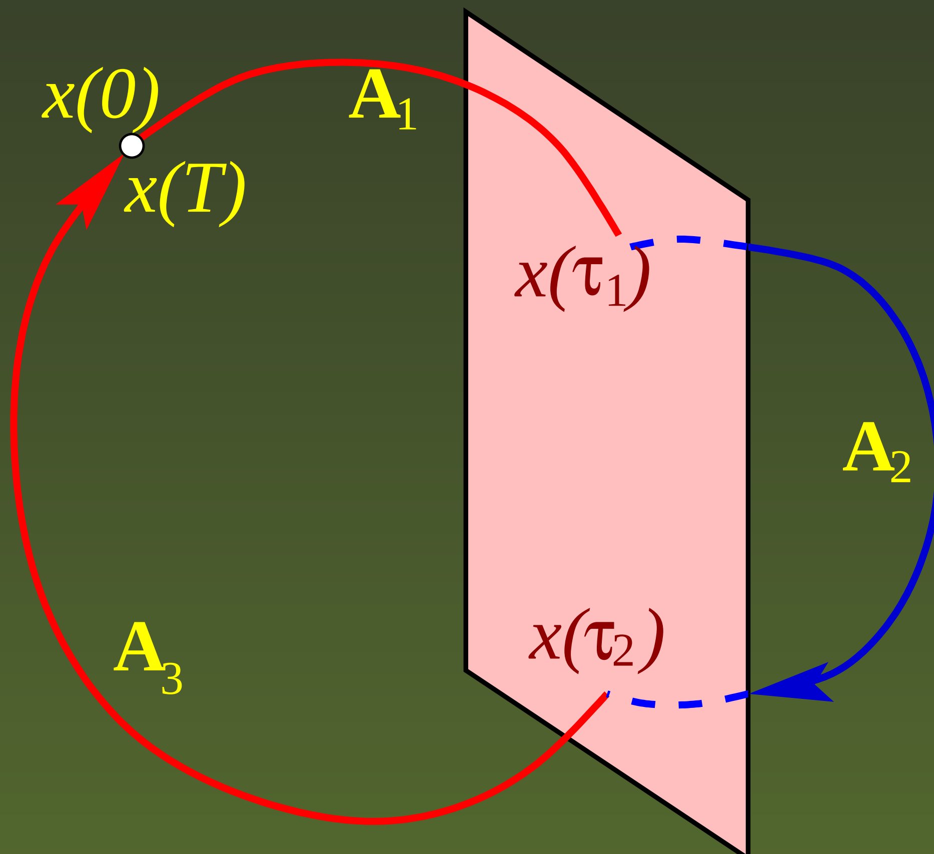
1. R. I. Leine and H. Nijmeijer, "Dynamics and Bifurcations in Non-Smooth Mechanical Systems," Springer Verlag, Berlin, 2004.
2. A. F. Filippov, "Differential equations with discontinuous righthand sides", Kluwer Academic Publishers, Dortrecht, 1988.

The monodromy matrix

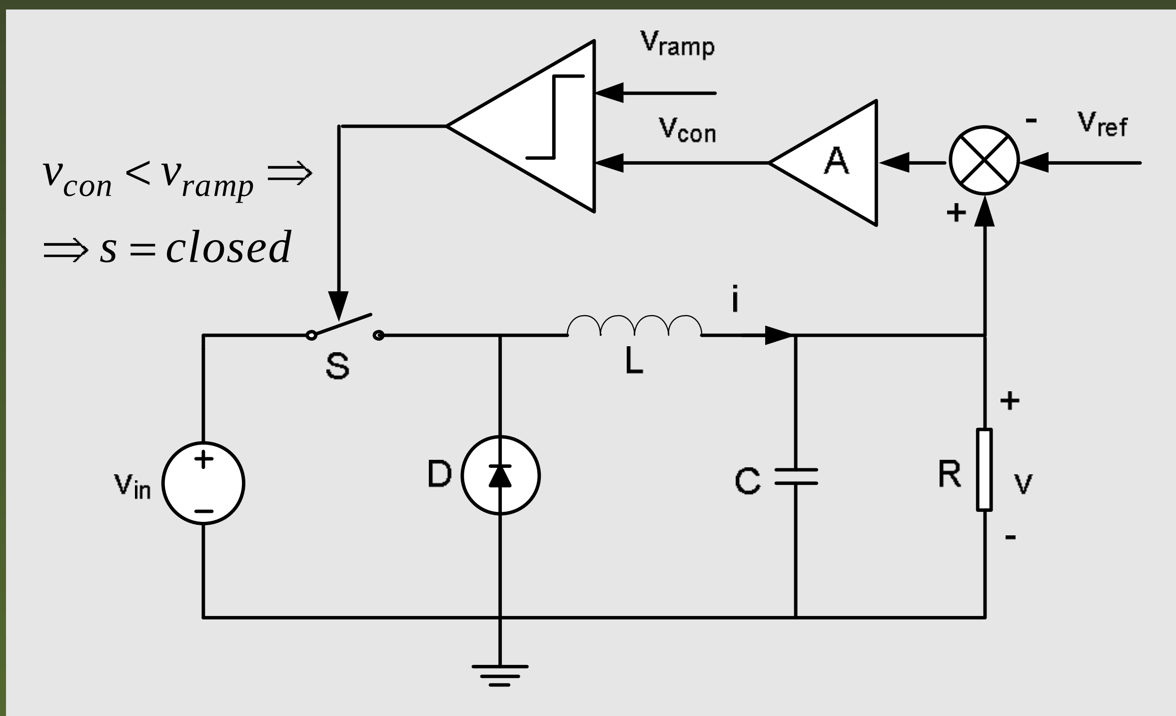
Thus the *monodromy matrix* is obtained as

$$\mathbf{A}_3 \cdot \mathbf{S}_2 \cdot \mathbf{A}_2 \cdot \mathbf{S}_1 \cdot \mathbf{A}_1$$

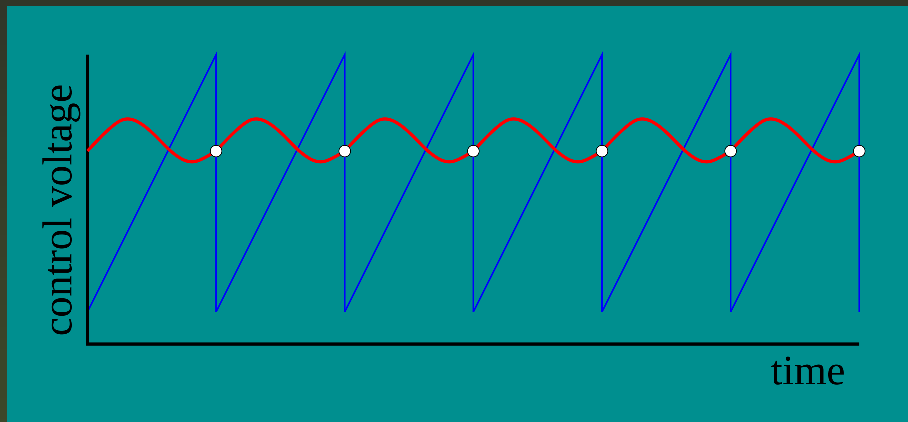


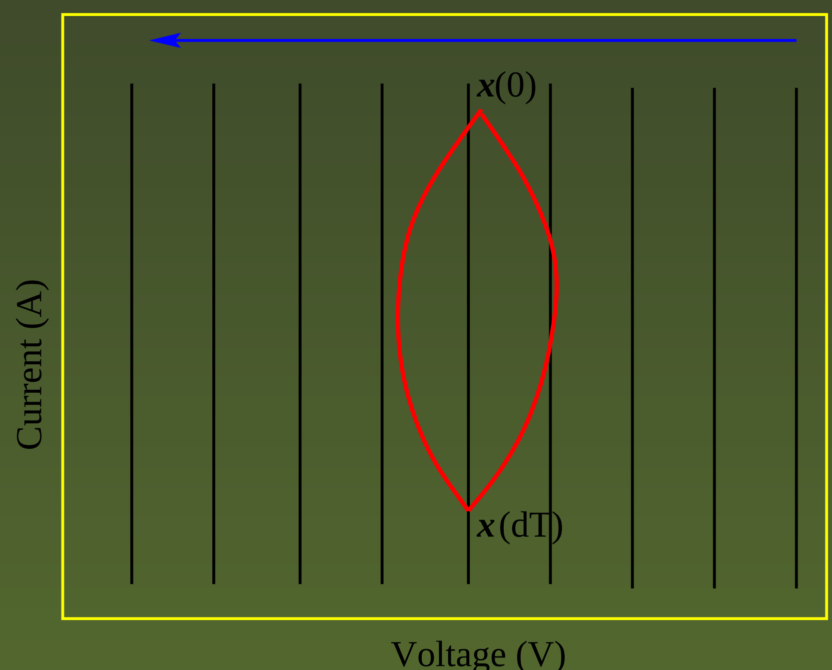
An example

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} = \begin{cases} \frac{v_{in} - v(t)}{L}, & \text{S is conducting} \\ -\frac{v(t)}{L}, & \text{S is blocking.} \end{cases}, \quad \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{i(t) - \frac{v(t)}{R}}{C}$$



The periodic orbit





The switching hypersurface

The switching hypersurface (h) is given by

$$\begin{aligned} h(\boldsymbol{x}(t), t) &= x_1(t) - V_{\text{ref}} - \frac{v_{\text{ramp}}(t)}{A} = 0, \\ v_{\text{ramp}}(t) &= V_L + (V_U - V_L)\left(\frac{t}{T} \mod 1\right) \end{aligned}$$

The normal to the hypersurface is:

$$\boldsymbol{n} = \nabla h(\boldsymbol{x}(t), t) = \begin{bmatrix} \dfrac{\partial h(\boldsymbol{x}(t), t)}{\partial x_1(t)} \\ \dfrac{\partial h(\boldsymbol{x}(t), t)}{\partial x_2(t)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

System representation

By defining $x_1(t) = v(t)$ and $x_2(t) = i(t)$, the system equations are

$$\dot{\boldsymbol{x}} = \begin{cases} \mathbf{A}_s\boldsymbol{x} + \mathbf{Bu}, & A(x_1(t) - V_{\text{ref}}) < v_{\text{ramp}}(t), \\ \mathbf{A}_s\boldsymbol{x}, & A(x_1(t) - V_{\text{ref}}) > v_{\text{ramp}}(t). \end{cases}$$

Where,

$$\mathbf{A}_s = \begin{bmatrix} -1/RC & 1/C \\ -1/L & 0 \end{bmatrix}, \quad \mathbf{Bu} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{\text{in}}$$

Calculation of the saltation matrix

When the state goes from the off state to the on state,

$$\boldsymbol{f}_{p-} = \lim_{t \uparrow t_\Sigma} \boldsymbol{f}_-(\boldsymbol{x}(t)) \;=\; \begin{bmatrix} x_2(t_\Sigma)/C - x_1(t_\Sigma)/RC \\ -x_1(t_\Sigma)/L \end{bmatrix},$$

$$\boldsymbol{f}_{p+} = \lim_{t \downarrow t_\Sigma} \boldsymbol{f}_+(\boldsymbol{x}(t)) \;=\; \begin{bmatrix} x_2(t_\Sigma)/C - x_1(t_\Sigma)/RC \\ (V_{\text{in}} - x_1(t_\Sigma))/L \end{bmatrix}.$$

where t_Σ is the switching instant.

Calculation of the saltation matrix

Thus

$$\boldsymbol{f}_{p+} - \boldsymbol{f}_{p-} = \begin{bmatrix} 0 \\ \frac{V_{\text{in}}}{L} \end{bmatrix},$$

$$\left(\boldsymbol{f}_{p+} - \boldsymbol{f}_{p-}\right)\boldsymbol{n}^T = \begin{bmatrix} 0 & 0 \\ \frac{V_{\text{in}}}{L} & 0 \end{bmatrix},$$

$$\boldsymbol{n}^T \boldsymbol{f}_{p-} = \frac{x_2(t_\Sigma)}{C} - \frac{x_1(t_\Sigma)}{RC}.$$

Calculation of the saltation matrix

$$\frac{\partial h(\boldsymbol{x}(t),t)}{\partial t} = \frac{\partial\left(x_1(t) - V_{\text{ref}} - \frac{TV_L + (V_U - V_L)t}{AT}\right)}{\partial t} = -\frac{V_U - V_L}{AT}.$$

Hence the saltation matrix is calculated as

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 \\ \dfrac{V_{\text{in}}/L}{\frac{x_2(t_\Sigma) - x_1(t_\Sigma)/R}{C} - \frac{V_U - V_L}{AT}} & 1 \end{bmatrix}$$

Calculation of stability

For a buck converter with the parameters

$$V_{\text{in}} = 24V,\ V_{\text{ref}} = 11.3V,\ L = 20m\text{H},\ R = 22\Omega,\ C = 47\mu\text{F},$$
$$A = 8.4,\ T = 1/2500\text{s},\ V_L = 3.8\text{V and } V_U = 8.2\text{V}$$

the switching instant was calculated to be $0.4993 \times T$.

The state at the switching instants are

$$\boldsymbol{x}(0) = \begin{bmatrix} 12.0222 \\ 0.6065 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}(d'T) = \begin{bmatrix} 12.0139 \\ 0.4861 \end{bmatrix}.$$

Calculation of stability

The saltation matrix is calculated as

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 \\ -0.4639 & 1 \end{bmatrix}$$

The state matrix is

$$\mathbf{A}_s = \begin{bmatrix} -1/RC & 1/C \\ -1/L & 0 \end{bmatrix} = \begin{bmatrix} 967.12 & 21276.6 \\ -50 & 0 \end{bmatrix}.$$

Calculation of stability

The state transition matrices for the two pieces of the orbit are:

1. Off period:

$$\boldsymbol{\Phi}(d'T, 0) = e^{\mathbf{A}_s d'T} = \begin{bmatrix} 0.8058 & 3.8366 \\ -0.0090 & 0.9802 \end{bmatrix}$$

2. On period:

$$\boldsymbol{\Phi}(T, d'T) = e^{\mathbf{A}_s dT} = \begin{bmatrix} 0.8052 & 3.8468 \\ -0.0090 & 0.9800 \end{bmatrix}.$$

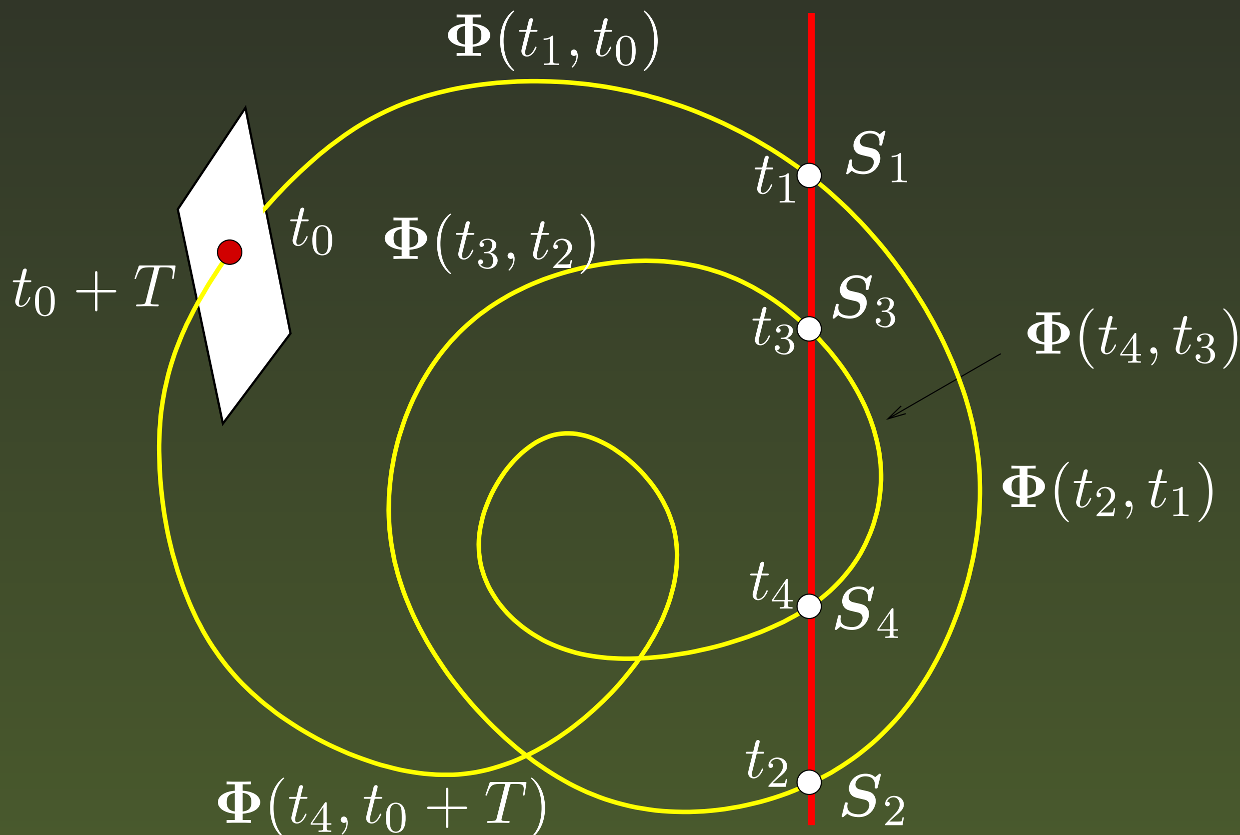
Calculation of stability

Hence the monodromy matrix is

$$\boldsymbol{\Phi}(T, 0, \boldsymbol{x}(0)) = \boldsymbol{\Phi}(T, d'T) \cdot \boldsymbol{S} \cdot \boldsymbol{\Phi}(d'T, 0)$$
$$= \begin{bmatrix} -0.8238 & 0.0131 \\ -0.3825 & -0.8184 \end{bmatrix}$$

The eigenvalues are $-0.8211 \pm 0.0708j$ implying that at the above parameter values the system is stable.

Arbitrary hybrid trajectory



$$\begin{aligned}\boldsymbol{\Phi}(t_0 + T, t_0) \quad = \quad & \boldsymbol{\Phi}(t_4, t_0 + T) \times \boldsymbol{S}_4 \times \boldsymbol{\Phi}(t_4, t_3) \times \boldsymbol{S}_3 \times \\ & \boldsymbol{\Phi}(t_3, t_2) \times \boldsymbol{S}_2 \times \boldsymbol{\Phi}(t_2, t_1) \times \boldsymbol{S}_1 \times \boldsymbol{\Phi}(t_1, t_0)\end{aligned}$$

Thank You