Bifurcations in 2D Piecewise Smooth maps

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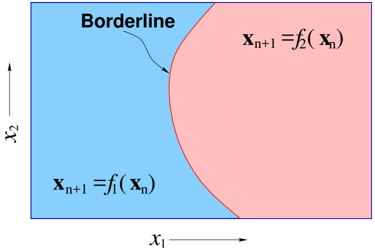
The system under consideration

For simplicity, a PWS map is considered that involves only two regions of smooth behavior:

$$f(x,y,\mu) = \begin{cases} f_1(x,y,\mu), & (x,y) \in R_A \\ f_2(x,y,\mu), & (x,y) \in R_B \end{cases} \tag{1}$$



Since the system is two-dimensional, the border is a curve separating the two regions of smooth behavior and is denoted by Γ_μ.

The normal form

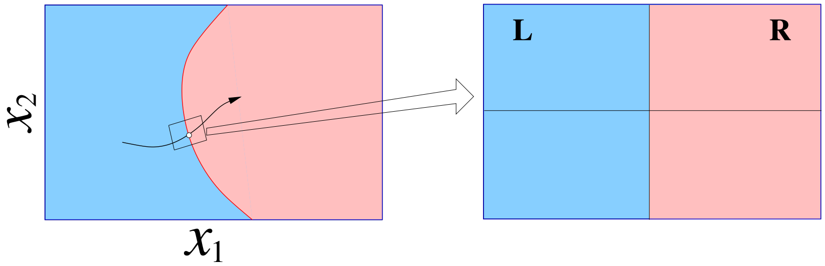


FIGURE: Schematic representation of obtaining the piecewise linear normal form map from the piecewise smooth map.

The normal form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = G_2(x_k, y_k, \mu) = \begin{cases} \underbrace{\begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}}_{\mathbf{J}_L} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \leq 0 \\ \underbrace{\begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}}_{\mathbf{J}_R} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \geq 0 \end{cases}$$

where τ_L is the trace and δ_L is the determinant of the Jacobian matrix $\mathbf{J}_L$ of the system at a fixed point in R_A and close to the border and τ_R is the trace and δ_R is the determinant of the Jacobian matrix $\mathbf{J}_R$ of the system evaluated at a fixed point in R_B near the border. Coordinate transformation and scaling have been done such that the border collision occurs at $\mu = 0$.

Banerjee, Ranjan, and Grebogi, IEEE Trans. CAS-I, **47**, 2000.

A check

Exercise: Show that any system of the form

$$\begin{pmatrix} \bar{x}_{k+1} \\ \bar{y}_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \bar{x}_k \\ \bar{y}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu \qquad (2)$$

(provided **A** is either non-diagonal, or diagonal with distinct eigenvalues, i.e., $\mathbf{A} \neq \lambda \mathbf{I}$) can be transformed to the 2-D normal form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tau & 1 \\ -\delta & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu \qquad (3)$$

using the transformation

$$x_k = T\bar{x}_k \quad \text{and} \quad T = \begin{pmatrix} 1 & \frac{a_4}{a_3} \\ 0 & -\frac{\delta}{a_3} \end{pmatrix}$$

where $\tau :=$trace$(A) = a_1 + a_4$ and $\delta :=$det$(A) = a_1 a_4 - a_2 a_3$, and $a_3 \neq 0$.

Properties of the 2D normal form map

The fixed points are: $L^* := (x_L^*, y_L^*) \in R_A$ and $R^* := (x_R^*, y_R^*) \in R_B$:

$$(x_L^*, y_L^*) = \left(\frac{\mu}{1 - \tau_L + \delta_L}, \frac{-\mu\delta_L}{1 - \tau_L + \delta_L} \right), \quad (4)$$

$$(x_R^*, y_R^*) = \left(\frac{\mu}{1 - \tau_R + \delta_R}, \frac{-\mu\delta_R}{1 - \tau_R + \delta_R} \right). \quad (5)$$

For admissible L^*, one needs $\frac{\mu}{1-\tau_L+\delta_L} \leq 0$, otherwise L^* is in R_B and is denoted as $\bar{L}^*$, a *virtual fixed point*.
Similarly, for admissible R^*, one needs $\frac{\mu}{1-\tau_R+\delta_R} \geq 0$.
Eigenvalues of the Jacobian matrix

$$\lambda_{1,2} = \frac{1}{2} \left(\tau \pm \sqrt{\tau^2 - 4\delta} \right)$$

In this discussion, we assume that the system is dissipative, i.e., $|\delta_L| < 1$ and $|\delta_R| < 1$

Feigin's classification

$$\sigma_L^+ := \text{number of real eigenvalues of } \mathbf{J}_L \; > +1$$
$$= \begin{cases} 1 & \text{if} \quad \tau_L > (1+\delta_L) \\ 0 & \text{if} \quad \tau_L < (1+\delta_L) \end{cases}$$

$$\sigma_L^- := \text{number of real eigenvalues of } \mathbf{J}_L \; < -1$$
$$= \begin{cases} 1 & \text{if} \quad \tau_L < -(1+\delta_L) \\ 0 & \text{if} \quad \tau_L > -(1+\delta_L) \end{cases}$$

$$\sigma_R^+ := \text{number of real eigenvalues of } \mathbf{J}_R \; > +1$$
$$= \begin{cases} 1 & \text{if} \quad \tau_R > (1+\delta_R) \\ 0 & \text{if} \quad \tau_R < (1+\delta_R) \end{cases}$$

$$\sigma_R^- := \text{number of real eigenvalues of } \mathbf{J}_R \; < -1$$
$$= \begin{cases} 1 & \text{if} \quad \tau_R < -(1+\delta_R) \\ 0 & \text{if} \quad \tau_R > -(1+\delta_R) \end{cases}$$

Feigin's classification

$$\sigma_{LR}^{+} := \text{number of real eigenvalues of } \mathbf{J}_L \mathbf{J}_R \; > +1$$
$$= \begin{cases} 1 & \text{if} \quad \tau_L \tau_R > (1+\delta_L)(1+\delta_R) \\ 0 & \text{if} \quad \tau_L \tau_R < (1+\delta_L)(1+\delta_R) \end{cases}$$

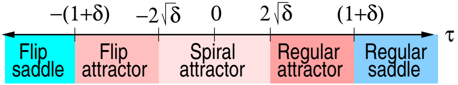
$$\sigma_{LR}^{-} := \text{number of real eigenvalues of } \mathbf{J}_L \mathbf{J}_R \; < -1$$
$$= \begin{cases} 1 & \text{if} \quad \tau_R \tau_L < -(1-\delta_R)(1-\delta_L) \\ 0 & \text{if} \quad \tau_R \tau_L > -(1-\delta_R)(1-\delta_L) \end{cases}$$

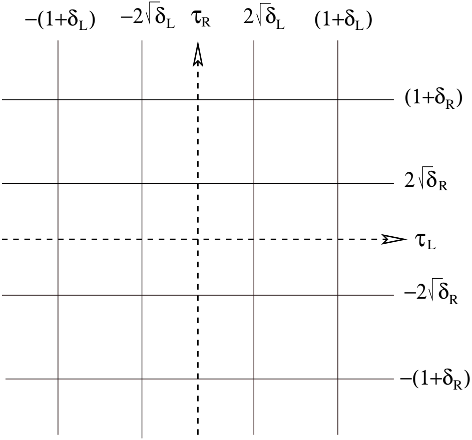
Feigin's classification

TABLE: The possible types of fixed points of the normal form map.

Type	eigenvalues	condition	identifiers
For positive determinant			
Regular attractor	real, $0<\lambda_1,\lambda_2<1$	$2\sqrt{\delta}<\tau<(1+\delta)$	$\sigma^+=0,\ \sigma^-=0$
Regular saddle	real, $0<\lambda_1<1,\ \lambda_2>1$	$\tau>(1+\delta)$	$\sigma^+=1,\ \sigma^-=0$
Flip attractor	real, $0>\lambda_1>-1, 0>\lambda_2>-1$	$-2\sqrt{\delta}>\tau>-(1+\delta)$	$\sigma^+=0,\ \sigma^-=0$
Flip saddle	real, $0<\lambda_1<1,\ \lambda_2<-1$	$\tau<-(1+\delta)$	$\sigma^+=0,\ \sigma^-=1$
Spiral attractor	complex, $\lvert\lambda_1\rvert,\lvert\lambda_2\rvert<1$		
(a) Clockwise spiral		$0<\tau<2\sqrt{\delta}$	$\sigma^+=0,\ \sigma^-=0$
(b) Counter-clockwise spiral		$-2\sqrt{\delta}<\tau<0$	$\sigma^+=0,\ \sigma^-=0$
For negative determinant			
Flip attractor	$0>\lambda_1>-1,\ 1>\lambda_2>0$	$-(1+\delta)<\tau<(1+\delta)$	$\sigma^+=0,\ \sigma^-=0$
Flip saddle	$\lambda_1>1, -1<\lambda_2<0$	$\tau>1+\delta$	$\sigma^+=1,\ \sigma^-=0$
Flip saddle	$0<\lambda_1<1, \lambda_2<-1$	$\tau<-(1+\delta)$	$\sigma^+=0,\ \sigma^-=1$

Primary partitioning of parameter space





Feigin's classification

In the work of Feigin, three basic classes of border collision bifurcations were shown to occur under the following conditions:

1. If $\sigma_L^+ + \sigma_R^+$ is even, then there is a smooth transition of one orbit to another at a border collision.
2. If $\sigma_L^+ + \sigma_R^+$ is odd, then two orbits merge and disappear at the border.
3. If $\sigma_L^- + \sigma_R^-$ is odd, then a period-2 orbit exists after border collision.

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di Bernardo, Feigin, Hogan, and Homer, Chaos, Solitons & Fractals, **10**, 1999.

Properties of the 2D normal form map

Proposition

If $\sigma_L^- + \sigma_R^-$ is odd, or equivalently if

$$\tau_L > -(1+\delta_L) \quad \text{and} \quad \tau_R < -(1+\delta_R) \tag{6}$$

$$\text{or} \quad \tau_L < -(1+\delta_L) \quad \text{and} \quad \tau_R > -(1+\delta_R) \tag{7}$$

a period-2 orbit exists in one side of the border collision event. If $\tau_L\tau_R < (1+\delta_L)(1+\delta_R)$, i.e., if $\sigma_{LR}^+ = 0$, under condition (6), the period-2 fixed point exists only for $\mu > 0$ and if $\sigma_{LR}^+ = 1$, the period-2 fixed point exists only for $\mu < 0$. The situation for (7) is symmetrical against change of sign of μ.

Proof

The dynamics of period-2 orbit is governed by the second return map with one point in R_A and the other point in R_B. For the period-2 fixed point in R_A, the x-component is given by

$$x_{2L}^* = \frac{-\mu(1+\tau_R+\delta_R)}{\tau_L\tau_R-(1+\delta_L)(1+\delta_R)} \tag{8}$$

The period-2 orbit exists if this quantity is a negative number. Likewise, the period-2 fixed point in R_B has x-component

$$x_{2R}^* = \frac{-\mu(1+\tau_L+\delta_L)}{\tau_L\tau_R-(1+\delta_L)(1+\delta_R)} \tag{9}$$

For the period-2 orbit to exist, this quantity must be a positive number. It is straightforward to obtain the conditions of existence of the period-2 orbits from these. □

Properties of the 2D normal form map

Proposition

If a period-2 orbit exists, it is stable if

$$\tau_R \tau_L < (1+\delta_R)(1+\delta_L), \quad (10)$$

$$\tau_R \tau_L > -(1-\delta_R)(1-\delta_L). \quad (11)$$

Inequality (10) is equivalent to $\sigma_{LR}^{+} = 0$ *and (11) is equivalent to* $\sigma_{LR}^{-} = 0$.

Proof

The Jacobian of the second return map is given by

$$J_{LR} = J_L J_R = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} = \begin{pmatrix} \tau_L\tau_R - \delta_R & \tau_L \\ -\delta_L\tau_R & -\delta_L \end{pmatrix} \quad (12)$$

Let δ_{LR} :=det$(J_{LR}) = \delta_R\delta_L$ and τ_{LR} :=trace$(J_{LR}) = \tau_L\tau_R - \delta_R - \delta_L$. Applying the conditions for stability $|\delta_{LR}| < 1$ and $-(1+\delta_{LR}) < \tau_{LR} < (1+\delta_{LR})$, we find that the period-2 orbit will be stable if and only if

$$\begin{aligned} -1 &< \delta_R\delta_L < 1 \\ \tau_R\tau_L &< (1+\delta_R)(1+\delta_L) \\ \tau_R\tau_L &> -(1-\delta_R)(1-\delta_L) \end{aligned}$$

□

Properties of the 2D normal form map

Proposition

The unstable manifolds associated with saddle fixed points fold at every intersection with the x-axis, and the images of every fold point is a fold point. The stable manifolds fold at every intersection with the y-axis and the pre-image of every fold point is a fold point.

Argument

Under the action of G_2, the line $x = 0$ maps to the line $y = 0$. As the map has different functional forms at the two sides of y axis, the slopes will be different at the two sides of the x axis.
Under the action of G_2^{-1}, points on the x-axis map to points on the y-axis, and hence the stable manifold must have different slopes in the two sides of the y-axis. □

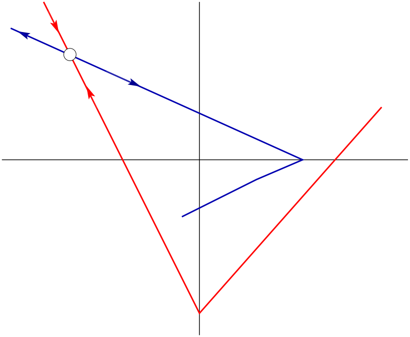
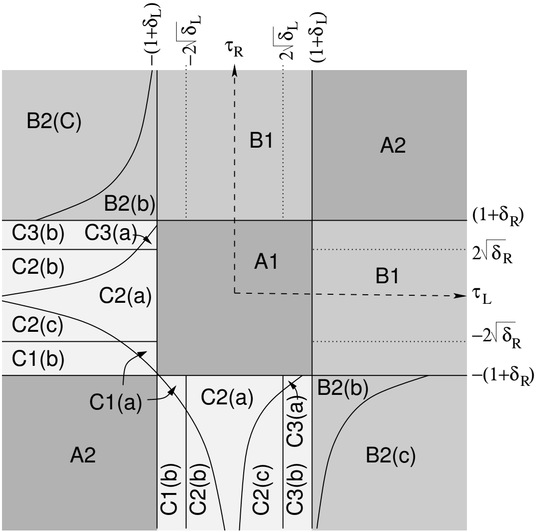


FIGURE: Folding of the stable and unstable sets.

Partitioning depending on the observed border collision bifurcations



Scenario A. Persistent fixed point

Proposition

If $\sigma_L^- + \sigma_R^-$ is an even number and $\sigma_L^+ + \sigma_R^+$ is also an even number, then a fixed point persists as μ is varied through zero.

Proof: The above conditions are satisfied if

$$(\sigma_L^- = 0 \text{ and } \sigma_R^- = 0) \quad \text{or} \quad (\sigma_L^- = 1 \text{ and } \sigma_R^- = 1)$$
$$(\sigma_L^+ = 0 \text{ and } \sigma_R^+ = 0) \quad \text{or} \quad (\sigma_L^+ = 1 \text{ and } \sigma_R^+ = 1)$$

These conditions are equivalent to
$\tau_L > 1 + \delta_L$ and $\tau_R > 1 + \delta_R$,
or $\tau_L < -(1 + \delta_L)$ and $\tau_R < -(1 + \delta_R)$,
or $-(1 + \delta_L) < \tau_L < (1 + \delta_L)$ and $-(1 + \delta_R) < \tau_R < (1 + \delta_R)$.

It is easy to check that for the above conditions, L^* exists for $\mu < 0$ while $\bar{R}^*$ is a virtual fixed point, and for $\mu > 0$ R^* exists and $\bar{L}^*$ is a virtual fixed point.

Scenario A1: (Persistence of Stable Fixed Point)

$$\begin{aligned} \text{If} \quad & -(1+\delta_L) < \tau_L < (1+\delta_L) \\ \text{and} \quad & -(1+\delta_R) < \tau_R < (1+\delta_R), \end{aligned} \tag{13}$$

then a stable fixed point persists as the bifurcation parameter μ is increased through zero $(A \to B)$.

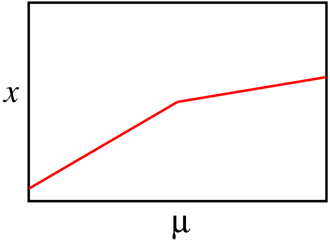


FIGURE: Schematic bifurcation diagram of scenario A1

Multiple attractor bifurcation

Multiple attractors can be born simultaneously at a border collision.

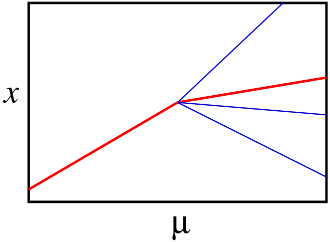
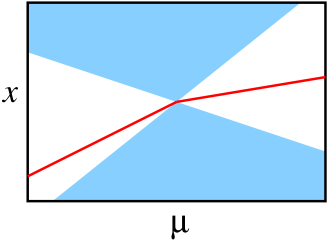


FIGURE: Schematic bifurcation diagram showing multiple attractor bifurcation.

Fundamental source of uncertainty.

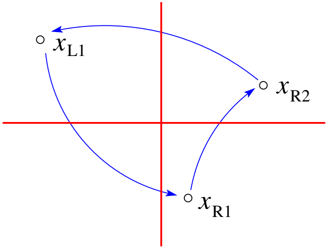
Dangerous border collision bifurcation

The fixed point remains stable for $\mu < 0$ and $\mu > 0$, but the basin of attraction shrinks to measure zero at $\mu = 0$.



Hassouneh, Abed, and Nusse, Phys. Rev. Lett., **92**, 070201, 2004

Dangerous border collision bifurcation



LRR orbit: $(x_{L1}, y_{L1}) \mapsto (x_{R1}, y_{R1}) \mapsto (x_{R2}, y_{R2})$. The conditions of existence $x_{L1} < 0$, $x_{R1} > 0$, and $x_{R2} > 0$. From the other two conditions we get the inequalities

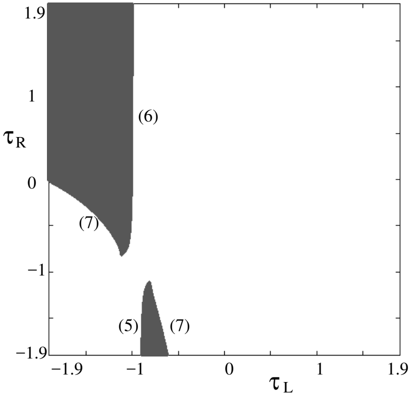
$$\frac{(1 + \tau_L - \delta_R + \tau_R\tau_L + \delta_L\delta_R + \delta_L\tau_R)\,\mu}{1 + {\delta_R}^2\delta_L + \tau_L\delta_R + \delta_L\tau_R + \delta_R\tau_R - \tau_L{\tau_R}^2} > 0, \qquad (14)$$

$$\frac{(1 + \tau_R - \delta_L + \tau_R\tau_L + \delta_L\delta_R + \tau_L\delta_R)\,\mu}{1 + {\delta_R}^2\delta_L + \tau_L\delta_R + \delta_L\tau_R + \delta_R\tau_R - \tau_L{\tau_R}^2} > 0. \qquad (15)$$

Dangerous border collision bifurcation

Contours of the existence region will be formed by

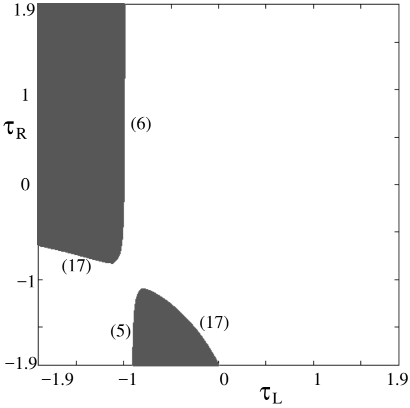
$$\begin{aligned} 1+\tau_L-\delta_R+\tau_R\tau_L+\delta_L\delta_R+\delta_L\tau_R &= 0,\\ 1+\tau_R-\delta_L+\tau_R\tau_L+\delta_L\delta_R+\tau_L\delta_R &= 0,\\ 1+{\delta_R}^2\delta_L+\tau_L\delta_R+\delta_L\tau_R+\delta_R\tau_R-\tau_L{\tau_R}^2 &= 0. \end{aligned}$$



Dangerous border collision bifurcation

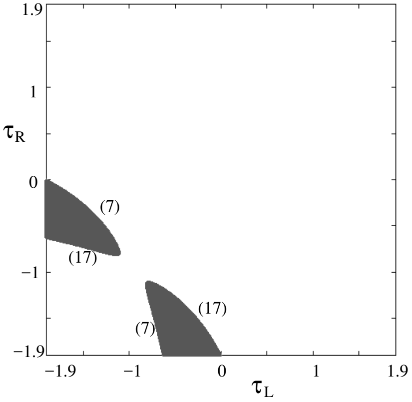
Similarly, the region of existence of the LLR orbit is delimited by

$$\begin{aligned}1+\tau_R-\delta_L+\tau_L\tau_R+\delta_R\delta_L+\delta_R\tau_L &= 0,\\ 1+\tau_L-\delta_R+\tau_L\tau_R+\delta_R\delta_L+\tau_R\delta_L &= 0,\\ 1+{\delta_L}^2\delta_R+\tau_R\delta_L+\delta_R\tau_L+\delta_L\tau_L-\tau_R{\tau_L}^2 &= 0.\end{aligned}$$



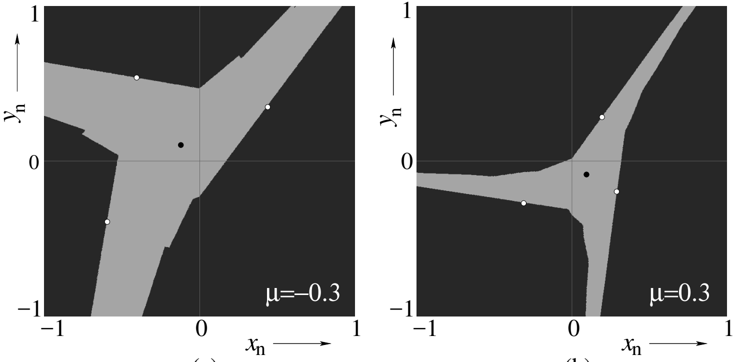
Dangerous border collision bifurcation

Subtraction yields the regions in which the LLR orbit exists (along with the fixed point L^*), but the complementary LRR orbit does not.



Dangerous border collision bifurcation

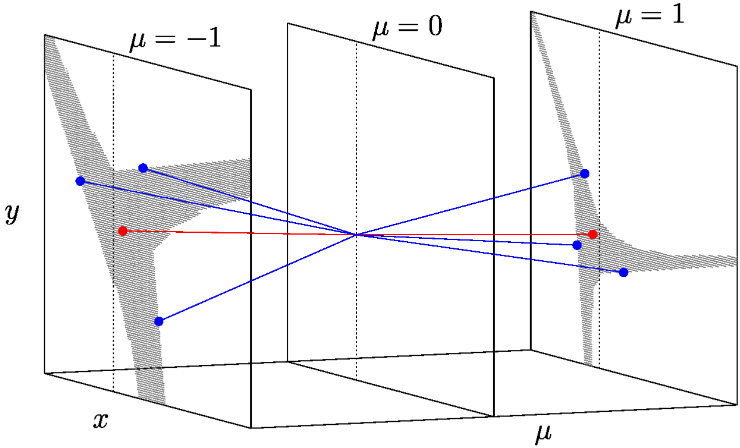
The orbits and their basins of attraction for $\mu < 0$ and $\mu > 0$.



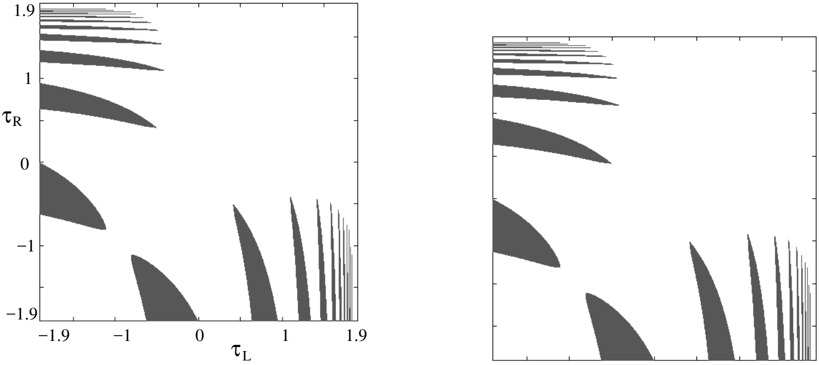
(a) (b)

The mechanism of dangerous border collision bifurcation

All orbits converge on the origin at $\mu = 0$



Dangerous border collision bifurcation



(a) (b)

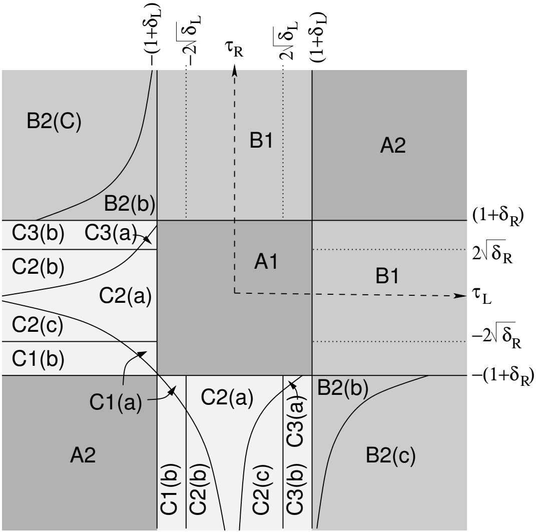
(a) The regions obtained analytically in Ganguli and Banerjee, Phys. Rev. E, 2005. (b) Regions numerically obtained by Hassouneh, Abed, Nusse, PRL, 2004.

Scenario A2: (Persistence of Unstable Fixed Point)

In this scenario, an unstable fixed point persists as μ is varied through zero ($a \to b$). There are no attractors for both $\mu < 0$ and $\mu > 0$. All initial conditions diverge to infinity. This occurs if

$$\begin{aligned} &\tau_L < -(1+\delta_L) \quad \text{and} \quad \tau_R < -(1+\delta_R) \\ \text{or} \quad &\tau_L > (1+\delta_L) \quad \text{and} \quad \tau_R > (1+\delta_R). \end{aligned}$$

Partitioning of the parameter space



Scenario B. Border collision pair bifurcation

Proposition

$$\textit{If} \quad \tau_L > 1 + \delta_L \quad \textit{and} \quad \tau_R < 1 + \delta_R, \tag{16}$$

the map has no fixed point for negative values of μ, and two fixed points for positive values of μ.

$$\textit{If} \quad \tau_R > 1 + \delta_R \quad \textit{and} \quad \tau_L < 1 + \delta_L, \tag{17}$$

then two fixed points exist for negative values of μ, and no fixed point exists for positive values of μ. Inequalities (16) and (17) are equivalent to the condition that

$$\sigma_L^+ + \sigma_R^+ \quad \textit{is odd.}$$

Proof:

The fixed points are: $L^* := (x_L^*, y_L^*) \in R_A$ and $R^* := (x_R^*, y_R^*) \in R_B$:

$$(x_L^*, y_L^*) = \left(\frac{\mu}{1-\tau_L+\delta_L}, \frac{-\mu\delta_L}{1-\tau_L+\delta_L} \right), \tag{18}$$

$$(x_R^*, y_R^*) = \left(\frac{\mu}{1-\tau_R+\delta_R}, \frac{-\mu\delta_R}{1-\tau_R+\delta_R} \right). \tag{19}$$

We see that if $\tau_L > 1+\delta_L$ and $\tau_R < 1+\delta_R$,, when $\mu < 0$ that the fixed point of R_A is located in R_B and that of R_B is located in R_A. Therefore the no fixed point exists in the system. For $\mu > 0$, L^* exists in R_A and R^* exists in R_B. The situation for $\tau_R > 1+\delta_R$ and $\tau_L < 1+\delta_L$ can be proved in a similar manner. □

For $\mu > 0$ L^* is a regular saddle, therefore unstable, and R^* can be stable or unstable.

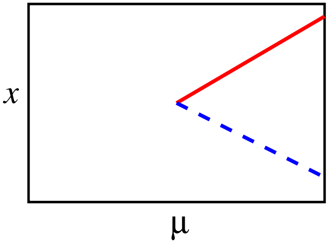
Scenario B1: $\emptyset \to a, B$ (Merging and Annihilation of Stable and Unstable Fixed Points)

$$\text{If } \tau_L > 1 + \delta_L \text{ and } -(1 + \delta_R) < \tau_R < 1 + \delta_R, \tag{20}$$

then there is a bifurcation from no fixed point to two period-1 fixed points. Condition (20) is analogous to

$$\sigma_L^+ + \sigma_R^+ \text{ is odd, and } \sigma_L^- + \sigma_R^- \text{ is even.}$$

This is similar to saddle-node bifurcation (or tangent bifurcation) in smooth maps.



Scenario B2: $\emptyset \to a, b$ (Merging and Annihilation of Two Unstable Fixed Points)

$$\text{If} \quad \tau_L > (1+\delta_L) \quad \text{and} \quad \tau_R < -(1+\delta_R) \tag{21}$$

or, analogously

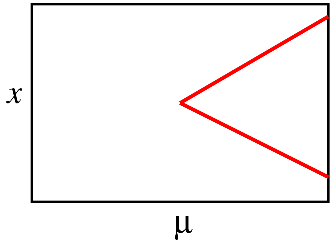
$$\text{If } \sigma_L^+ + \sigma_R^+ \text{ is odd, and } \sigma_L^- + \sigma_R^- \text{ is odd,}$$

then there is no fixed point for $\mu < 0$, while there are two *unstable fixed points* for $\mu > 0$.
The parameter space can be further subdivided into three regions.

Scenario B2(a): $\emptyset \rightarrow a, b$ (Merging and Annihilation of Two Unstable Fixed Points)

$$\text{If} \qquad \tau_L \tau_R > -(1-\delta_L)(1-\delta_R) \tag{22}$$

then by Proposition 1 and 2 a stable period-2 orbit exists, and therefore there is a bifurcation from a no fixed point to two unstable fixed points plus a period-2 attractor. The above condition is equivalent to $\sigma_{LR}^{-} = 0$.



This condition does not occur if the determinants are positive, but does occur in the case of negative determinants.

Scenario B2(b): $\emptyset \to a, b$, CHAOS (Birth/disappearance of a stable chaotic attractor)

Proposition

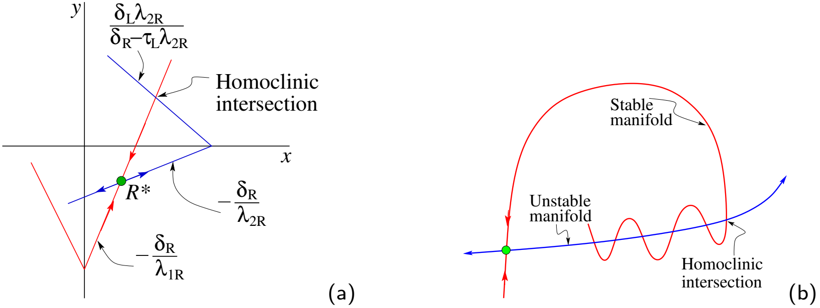
If the system has a flip-saddle type fixed point, and if the stable manifold and the unstable manifold of the fixed point undergo a transverse homoclinic intersection, then a unique chaotic orbit exists. This orbit may be stable or unstable.

Robust chaos: There exists a range of parameters where

- A chaotic orbit occurs
- There is no periodic window
- No coexisting attractor exists.

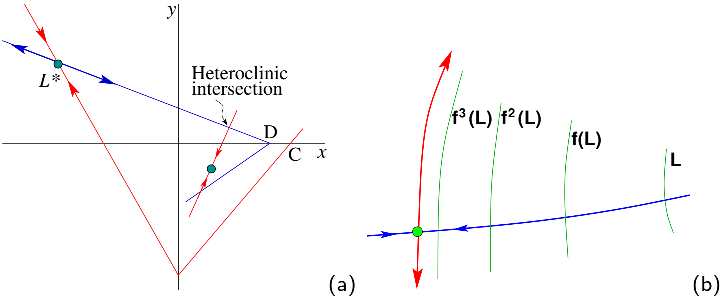
Mechanism of creation of robust chaos:

$$\tau_L > 1 + \delta_L, \quad \tau_R < 1 + \delta_R, \quad \mu > 0,$$



A homoclinic intersection must exist $\Longrightarrow$ Chaos.

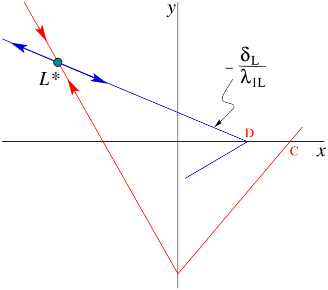
Mechanism of creation of robust chaos:



A transverse heteroclinic intersection must exist.

Since the unstable manifold of L^* intersects with the stable manifold of R^*, by the Lambda Lemma, the two unstable manifolds of the two fixed points must come arbitrarily close to each other. And since all attractors must lie of the unstable manifold, the above result implies that the attractor must be unique.

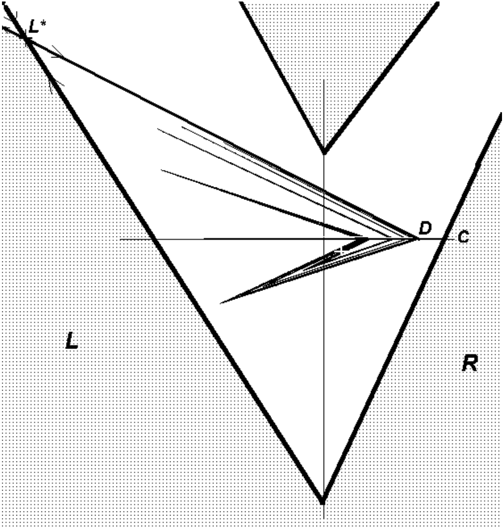
Stability of chaotic orbit:



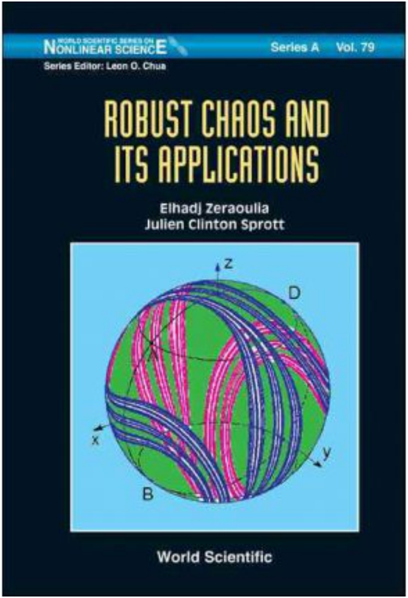
The basin of attraction of the chaotic orbit is bounded by the stable manifold of L^*. So long as this line does not intersect the attractor, it will be stable. So long as the point D is to the left of point C, the chaotic orbit will be stable. This leads to the condition for stability:

$$\delta_L \tau_R \lambda_{1L} - \delta_R \lambda_{1L} \lambda_{2L} + \delta_R \lambda_{2L} - \delta_L \tau_R + \tau_L \delta_L - \delta_L^2 - \lambda_{2L} \delta_L > 0$$

Robust Chaos



A book on Robust Chaos



Scenario B2(c): $\emptyset \to a, b,$ chaos

If the condition

$$\delta_L\tau_R\lambda_{1L} - \delta_R\lambda_{1L}\lambda_{2L} + \delta_R\lambda_{2L} - \delta_L\tau_R + \tau_L\delta_L - \delta_L^2 - \lambda_{2L}\delta_L > 0$$

is not satisfied, then there is a bifurcation from no fixed point to two unstable fixed points plus an *unstable chaotic orbit* as μ is increased through zero.

Scenario C. Border crossing bifurcation:

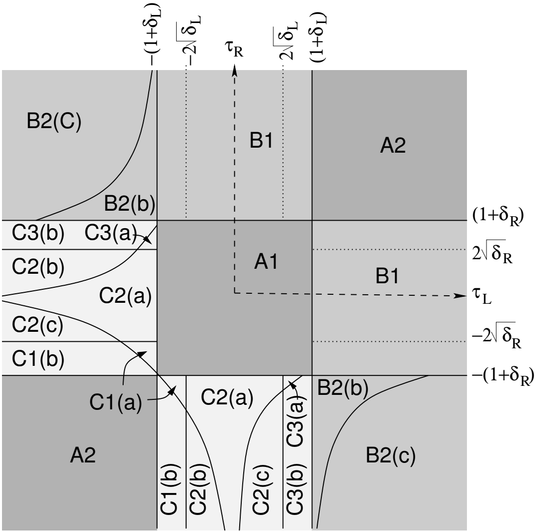
In this class of bifurcation, the fixed point crosses the border as μ is varied through zero, and becomes unstable. In terms of the Feigin classification, this happens when there is a smooth transition from one orbit to another (i.e., $\sigma_L^+ + \sigma_R^+$ is even) *and* a period-2 orbit exists (i.e., $\sigma_L^- + \sigma_R^-$ is odd). In terms of the traces and determinants, this situation occurs when

$$-(1+\delta_L) < \tau_L < (1+\delta_L) \tag{23}$$

and

$$\tau_R < -(1+\delta_R) \tag{24}$$

Partitioning of the parameter space



Scenario C1: Flip attractor changes to flip saddle

There are two possible outcomes:

SCENARIO C1(A): $A \to b, AB$ (supercritical border collision period-doubling)

$$\text{If} \quad \tau_R \tau_L \; < \; (1+\delta_R)(1+\delta_L), \tag{25}$$

i.e., if $\sigma_{LR}^{+} = 0$, then there is a bifurcation from a stable fixed point to an unstable fixed point plus a stable period-2 orbit.

SCENARIO C1(B): $A, ab \to b$ (subcritical border collision period-doubling)

$$\text{If} \quad \tau_R \tau_L \; > \; (1+\delta_R)(1+\delta_L), \tag{26}$$

i.e., if $\sigma_{LR}^{+} = 1$, then for $\mu < 0$ an unstable period-2 orbit coexists with the stable fixed point. For $\mu > 0$ the period-2 orbit disappears and fixed point is unstable, thus there is no stable orbit.

Scenario C1: Flip attractor changes to flip saddle

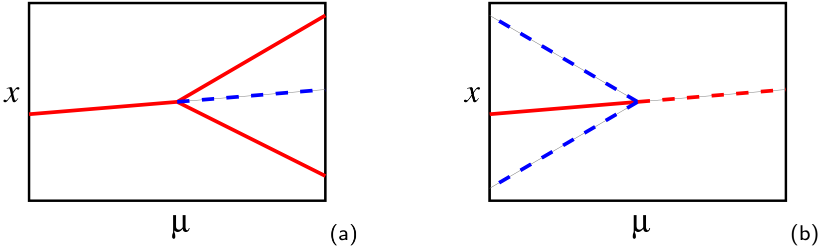


FIGURE: Schematic bifurcation diagrams for (a) supercritical period doubling and (b) subcritical period doubling, at border collision.

Scenario C2: Spiral attractor changes to flip saddle

This happens when

$$-2\sqrt{\delta_L} < \tau_L < 2\sqrt{\delta_L} \quad \text{and} \quad \tau_R < -(1+\delta_R)$$

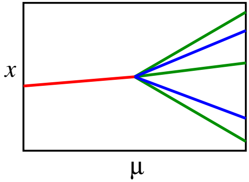
Then there are three possibilities.

Scenario C2(a): $A, \text{HPO} \to b, AB, \text{HPO}$

$$\text{If} \quad \tau_R\tau_L < (1+\delta_R)(1+\delta_L) \quad (27)$$

$$\text{and} \quad \tau_R\tau_L > -(1-\delta_R)(1-\delta_L) \quad (28)$$

i.e., if $\sigma_{LR}^{+} = 0$ and $\sigma_{LR}^{-} = 0$, then there is a supercritical border collision period doubling. If the conditions of occurrence of HPOs are satisfied, coexisting high-period orbits may occur at both sides of the bifurcation.

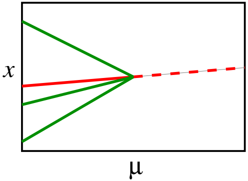


Scenario C2: Spiral attractor changes to flip saddle

SCENARIO C2(B): $A, ab, \text{HPO} \to b$

$$\text{If} \quad \tau_R \tau_L \; > \; (1+\delta_R)(1+\delta_L) \tag{29}$$

i.e., if $\sigma^{+}_{LR} = 1$, then there is a subcritical border collision period doubling. High period orbits may coexist with the stable fixed point for $\mu < 0$, and there is no attractor for $\mu > 0$.



Scenario C2: Spiral attractor changes to flip saddle

SCENARIO C2(C): $A, \text{HPO} \rightarrow b, ab, \text{HPO}$ or CHAOS

$$\text{If} \quad \tau_R \tau_L < -(1-\delta_R)(1-\delta_L) \tag{30}$$

i.e., if $\sigma^-_{LR} = 1$, then there is a bifurcation from a stable fixed point to a high-period orbit or chaos. Coexisting orbits can occur on both sides of μ.

Scenario C3: Regular attractor changes to flip saddle

This happens when

$$2\sqrt{\delta_L} < \tau_L < 1 + \delta_L \tag{31}$$

$$\text{and} \qquad \tau_R < -(1 + \delta_R) \tag{32}$$

In this case there can be two outcomes:

SCENARIO C3(A): $A \to b, AB$

$$\text{If} \quad \tau_R \tau_L \; > \; -(1 - \delta_R)(1 - \delta_L) \tag{33}$$

i.e., if $\sigma_{LR}^{-} = 0$, then there is a supercritical border collision period doubling as μ is varied across zero.

SCENARIO C3(B): $A \to b, ab$, CHAOS

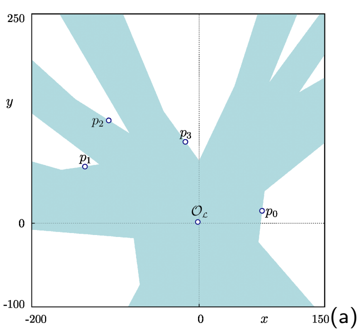
$$\text{If} \quad \tau_R \tau_L \; < \; -(1 - \delta_R)(1 - \delta_L) \tag{34}$$

i.e., if $\sigma_{LR}^{-} = 1$, then there is a transition from a stable period-1 orbit to a stable chaotic orbit.

Coexisting HPOs may occur in both these cases.

Scenario C3

Dangerous border collision bifurcation may occur in this region also, where the orbits involved may not be period-1. Any periodic or chaotic orbit may occur for $\mu < 0$ and $\mu > 0$ with the property that the basin of attraction shinks to zero size at $\mu = 0$.



The case of negative determinant

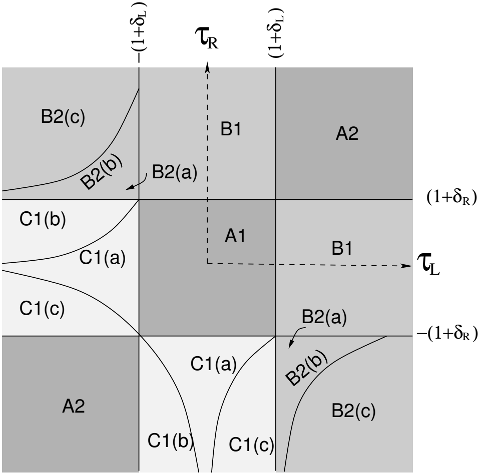


FIGURE: The partitioning of the parameter space when both the determinants are negative.

Thank You