Bifurcations in one-dimensional piecewise smooth continuous maps

Soumitro Banerjee

Department of Physical Sciences
Indian Institute of Science Education and Research-Kolkata

Outline

Introduction

The border collision normal form

The border collision bifurcations
- Transition of one orbit to another of the same type
- Border collision pair bifurcation
- Border crossing bifurcations

The parameter space

Introduction

$f(x;\mu)$ is a *piecewise smooth continuous map* if

1. $f(x;\mu)$ is continuous in $(x;\mu)$, and
2. $f(x;\mu)$ is smooth in $(x;\mu)$ on each of the regions R_A and R_B, but its derivative is discontinuous at x_b.

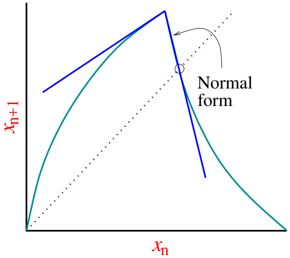
We say the map is piecewise monotonous, if $\partial f/\partial x$ can change sign only at the border x_b.

Let the map be given by

$$f(x;\mu) = \begin{cases} g(x;\mu) & \text{for } \; x \le x_b, \\ h(x;\mu) & \text{for } \; x \ge x_b \end{cases} \tag{1}$$

The border collision bifurcation

If the fixed point collides with the border with change in the parameter, there is a discontinuous change in the derivative $\partial f / \partial x$, and the resulting phenomenon is called *border collision bifurcation*.



Since the local structure of border collision bifurcations depends only on the local properties of the map in the neighborhood of the border, we study such bifurcations with the help of “normal form” — the piecewise affine approximation of f in the neighborhood of the border.

The border collision normal form

We make a parameter-dependent change of coordinate by letting $\overline{x} = x - x_b$. The border is now given by $\overline{x} = 0$, and the map is given by

$$f(\overline{x} + x_b; \mu) = F(\overline{x}; \mu).$$

We expand $F(x;\mu)$ to the first order about $x = \mu = 0$, and obtain

$$F(x;\mu) = \begin{cases} a\,x + \mu\,v_A + o(x;\mu) & \text{for} \quad x \leq 0, \\ b\,x + \mu\,v_B + o(x;\mu) & \text{for} \quad x \geq 0, \end{cases} \tag{2}$$

where

$$\begin{aligned} a &= \lim_{x\to 0^-} \frac{\partial}{\partial x} F(x;0), \\ b &= \lim_{x\to 0^+} \frac{\partial}{\partial x} F(x;0), \\ v_A &= \lim_{x\to 0^-} \frac{\partial}{\partial \mu} F(x;0), \\ v_B &= \lim_{x\to 0^+} \frac{\partial}{\partial \mu} F(x;0), \end{aligned}$$

The border collision normal form

The continuity of the map $F(x;\mu)$ for all μ requires that $v_A = v_B$, and we assume this value is non-zero. We can eliminate v_A and v_B from (2) by rescaling μ. Given $F(x;\mu)$, we compute a and b, and obtain the 1-D normal form

$$G_1(x;\mu) = \begin{cases} a\,x + \mu & \text{for } x \leq 0, \\ b\,x + \mu & \text{for } x \geq 0. \end{cases} \tag{3}$$

As μ is varied, the *local* bifurcation of the piecewise smooth map $F(x;\mu)$ is the same as that of the normal form G_1.

The border collision bifurcations

For the map G_1, there are two fixed points. The fixed point given by the left side of the map (let us call it L^*) is

$$x_L^* = \frac{\mu}{1-a}$$

and that given by the right side of the map is

$$x_R^* = \frac{\mu}{1-b}.$$

If L^* lies in the left hand side, i.e., if $x_L^* < 0$, this fixed point exists, otherwise it does not. Similarly, if R^* lies in the right hand side, this fixed point exists, otherwise it does not.

Transition of one orbit to another of the same type: Case 1: "Period-1 $\rightarrow$ period-1"

$$\text{If} \quad -1 < b \leq a < 1,$$

then there exists one stable fixed point for $\mu < 0$ and one stable fixed point for $\mu > 0$.

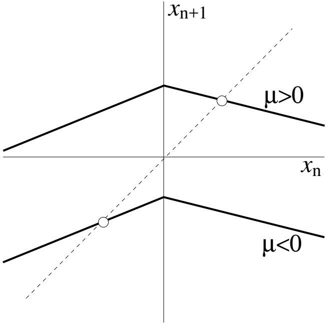


Figure: Illustration of case 1, $-1 < b \leq a < 1$

Case 2: "no attractor → no attractor"

$$\begin{array}{lll} \text{If} & (a) & a > 1, \ \ b > 1, \\ \text{or} & (b) & a < -1, \ \ b < -1, \end{array}$$

then there is one unstable fixed point for both positive and negative values of μ*. No attractors exist.*

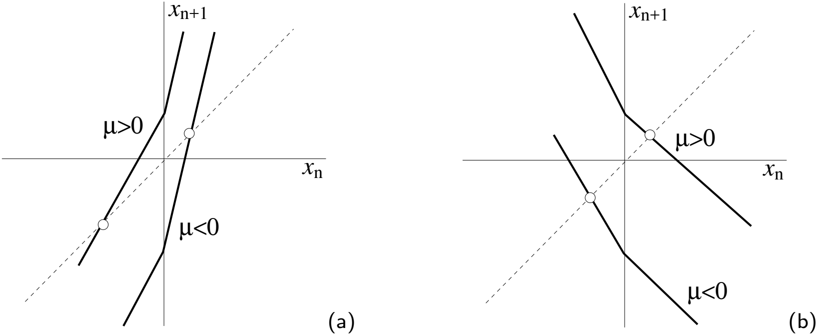


Figure: Illustration of case 2, (a) $a > 1, b > 1$ and (b) $a < -1, b < -1$

Border collision pair bifurcation

$$\text{If} \quad b < 1 < a,$$

then no fixed point exists for $\mu < 0$, but two fixed points exist, one on each side of $x = 0$, for $\mu > 0$. If μ is varied in the opposite direction, two fixed points merge and disappear.

Case 3: "No fixed point $\rightarrow$ period-1"

$$\text{If} \quad -1 < b < 1 < a,$$

then for $\mu < 0$ there is no fixed point (no attractor) while for $\mu \geq 0$ there is a fixed point attractor.

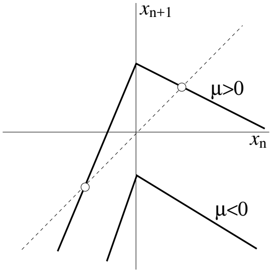


Figure: Illustration of Case 3

Case 4: "No fixed point $\rightarrow$ chaos".

$$\text{If} \quad a > 1 \quad \text{and} \quad \frac{a}{1-a} < b < -1,$$

then for $\mu < 0$ there is no fixed point (no attractor) while for $\mu \geq 0$ there is a chaotic attractor.

The chaotic attractor spans the interval $[\mu(b+1), \mu]$. The basin of this chaotic attractor is the interval between L^* and the point that maps to L^*, i.e., $[x_L^*, (x_L^* - \mu)/b]$.

Case 5: "No fixed point $\rightarrow$ no attractor".

$$\text{If} \quad a > 1 \quad \text{and} \quad b < \frac{a}{1-a},$$

then for $\mu < 0$ there is no fixed point (no attractor) while for $\mu \geq 0$ there is an unstable chaotic orbit (no attractor).

Border collision pair bifurcation

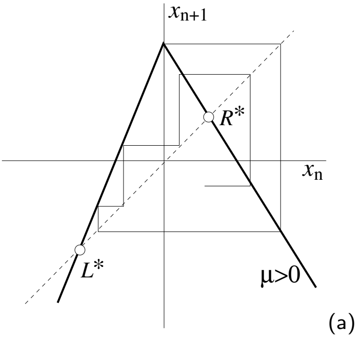


Figure: Illustration of the two cases of border collision pair bifurcation. (a) Case 4, and (b) Case 5.

Border crossing bifurcations

$$\text{If} \qquad -1 < a < 1 \qquad \text{and} \qquad b < -1,$$

a stable fixed point crosses the border as μ varies through zero, and becomes unstable. We call this a *border crossing bifurcation*.

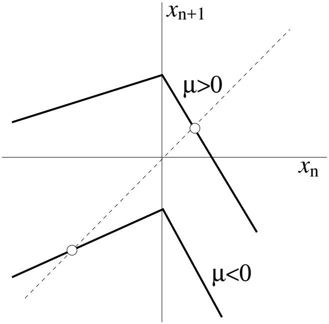


Figure: Illustration of border crossing bifurcation.

Border crossing bifurcations

To obtain the period-2 fixed point, we start from a point in L and apply the map to obtain

$$x_{n+1} = ax_n + \mu$$

and then apply the right side of the map to obtain

$$\begin{aligned} x_{n+2} &= b\,x_{n+1} + \mu \\ &= b(ax_n + \mu) + \mu \end{aligned}$$

Putting $x_{n+2} = x_n$, and solving, we get the point of the period-2 orbit that lies in L as

$$x_2^- = \frac{\mu(b+1)}{1-ab}$$

This point maps to

$$x_2^+ = \frac{\mu(a+1)}{1-ab}$$

Thus the period-2 orbit has two points:

$$x_2^- = \frac{\mu(b+1)}{1-ab} \quad \text{in } L \quad \text{and} \quad x_2^+ = \frac{\mu(a+1)}{1-ab} \quad \text{in } R$$

If $x_2^- < 0$ and $x_2^+ > 0$ then this fixed point exists, otherwise it does not. This has a few implications. For $b < -1$ and $-1 < a < 1$,

- If $ab < 0$ the period-2 fixed point exists for $\mu > 0$, and is stable
- If $ab > 0$ the period-2 fixed point exists for $\mu < 0$, and is unstable.

Notice that as $ab \to 1$ the two points of the orbit approach infinity. They exist only if $ab < 1$. Thus, this stability condition is identical with the condition of existence.

Case 6: "Period-1 $\to$ period-2"

$$\text{If} \quad -1 < a < 1, \quad b < -1, \quad \text{and} \quad -1 < ab < 1,$$

As μ is varied through 0, there is a bifurcation from a period-1 attractor to a period-2 attractor.

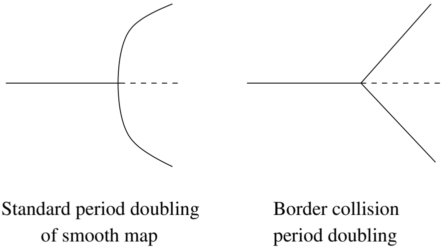


Figure: The difference between period doubling bifurcations in a smooth map and a nonsmooth map.

This orbit can become unstable in two ways: (a) when $ab > 1$ and (b) $ab < -1$.

Case 7: "Period-1 $\to$ no attractor"

$$\text{If} \quad -1 < a < 1, \quad b < -1 \quad \text{and} \quad ab > 1,$$

then there is a period-1 attractor for $\mu < 0$ and no attractor for $\mu > 0$. The unstable period-2 orbit occurs for $\mu < 0$. This means it is a subcritical period doubling bifurcation caused by border collision.

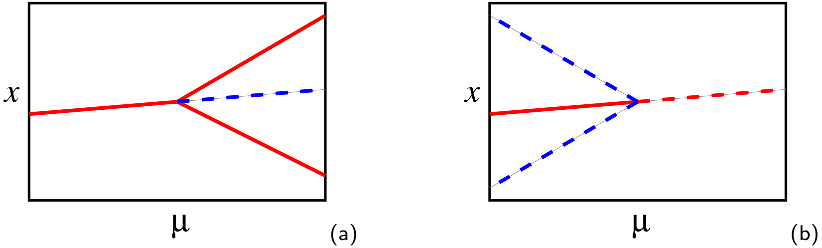


Figure: Typical bifurcation diagrams for (a) supercritical period doubling and (b) subcritical period doubling, in a nonsmooth map.

Period 3 and higher periodic and chaotic orbit

When the condition $ab < -1$ is satisfied, the period-2 orbit becomes unstable. But since a trapping region exists, a chaotic orbit appears. As a is further increased, beyond a critical parameter value, an LLR-type period-3 orbit appears.

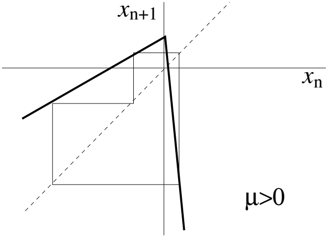


Figure: Occurrence of a period-3 orbit for $-1 < a < 1$, and $b < -1$.

The period-3 orbit remains stable so long as $|aab| < 1$, and can destabilize in two ways: when $aab > 1$ and when $aab < -1$. These give the range of occurrence of the period-3 orbit in the parameter space.

Case 8: "Period-1 $\to$ high periodic or chaotic attractor"

$$\text{If} \quad -1 < a < 1, \quad b < -1, \quad \text{and} \quad ab < -1$$

then a fixed point attractor can bifurcate into a periodic attractor with period greater than 2 or a chaotic attractor as μ is varied from less than 0 to greater than 0.

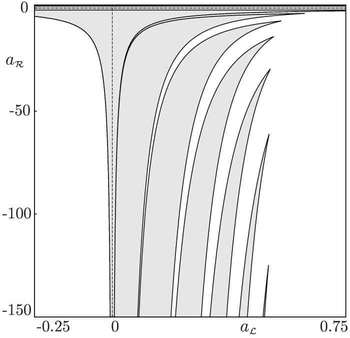


Figure: Schematic drawing of the parameter region $0 < a < 1$, $b < -1$ (Case 8) showing the type of attractor for $\mu > 0$. The shaded regions correspond to period-n attractors and white regions have chaotic attractors.

The parameter space

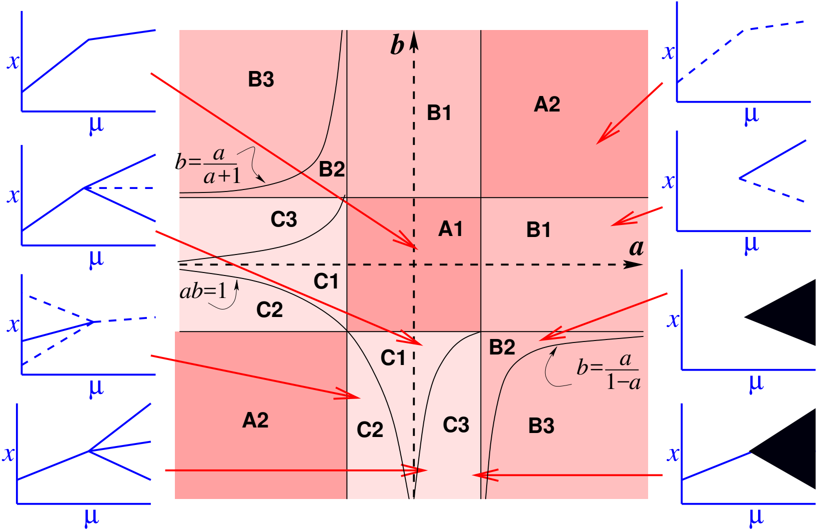


Figure: The partitioning of the parameter space into regions with the same qualitative bifurcation phenomena.

Thank You