Introduction to Dynamical Systems

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Ecology as dynamical system

The number of individuals in all biological populations fluctuate. $\Rightarrow$ dynamical system.

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Can be represented in discrete time:

$$x_{n+1} = f(x_n)$$

Or in continuous time:

$$\begin{aligned} \frac{dx}{dt} &= f_1(x, y) \\ \frac{dy}{dt} &= f_2(x, y) \end{aligned}$$

Example: Discrete-time

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- First approximation: $x_{n+1} = \mu x_n$
- When the population is small, the above is a good model.
- But when it is large, overpopulation and scarcity of food slows down the growth.
- So a reasonable model of a single population will be

$$x_{n+1} = \mu x_n(1 - x_n)$$

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- In absence of prey, predator population declines as $\dot{y} = -cy$. When prey are present, the predator population increases at a rate proportional to the predator/prey encounters, or dxy.
- Thus the whole model:

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= -cy + dxy \end{aligned}$$

ODE: $\dot{x} = f(x)$ and $\dot{x} = f(x, t)$

- If the system equations do not have any externally applied time-varying input, the system is said to be *autonomous* e.g., the Lorenz system

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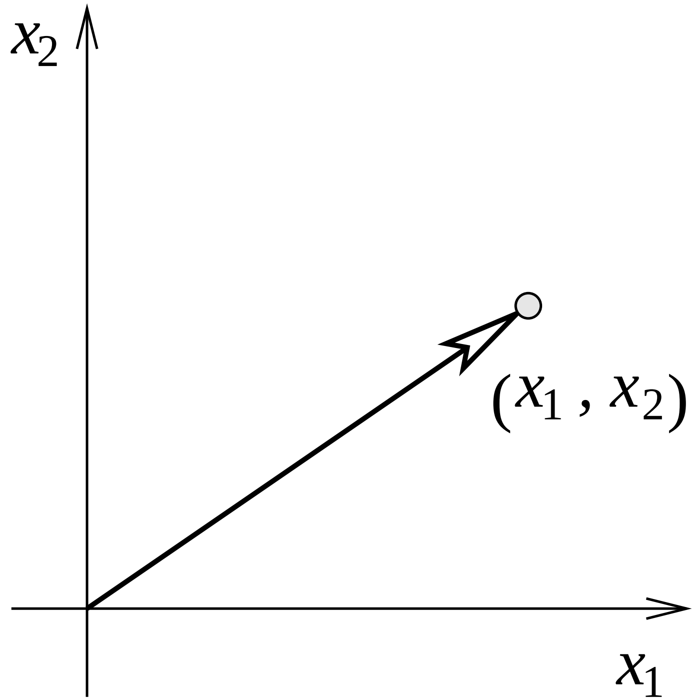
- Systems with external inputs or forcing functions are called *non-autonomous*, e.g., a periodically forced pendulum:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g \sin x + F \cos \omega t. \end{aligned}$$

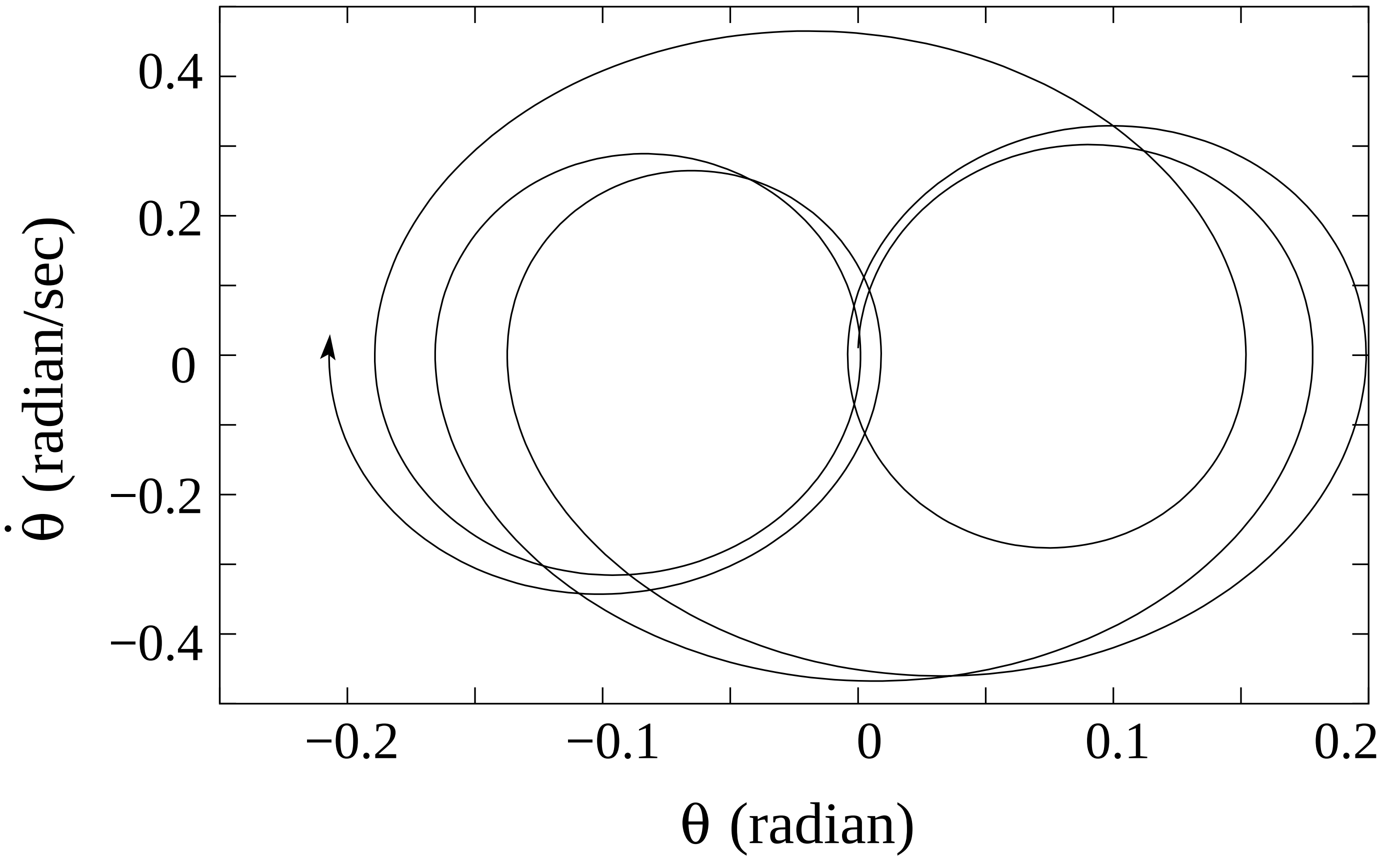
The phase space

Geometrically, the dynamics can be visualized by constructing a space with the variables as the coordinates.



The trajectory

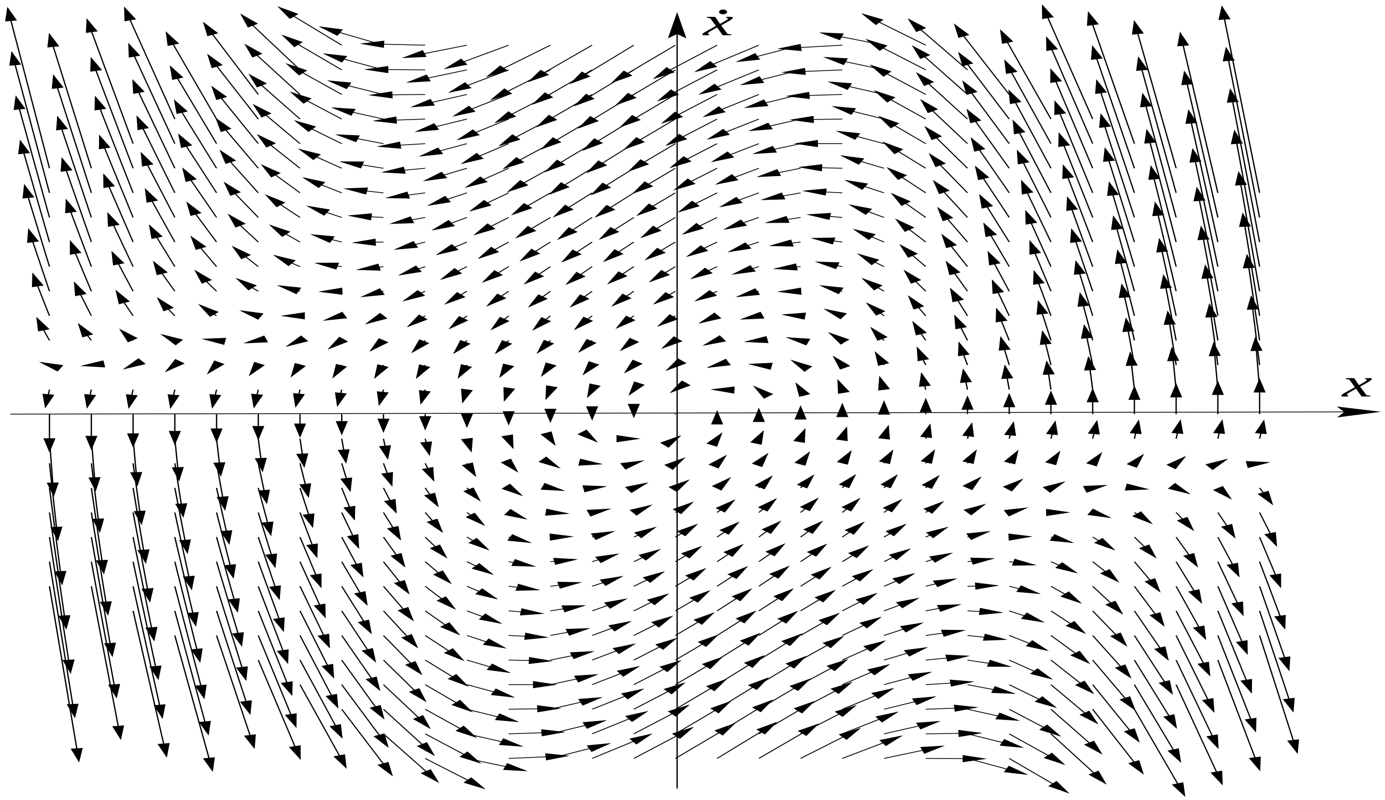
Example:



The trajectory of the pendulum with oscillating support for 10 seconds starting from a standstill position.

The vector field

The differential equations define a vector at every point in the phase space. The solution follows the "flow" of the vectors.



Vector field for $\ddot{x} - (1 - x^2)\dot{x} + x = 0.$

Equilibrium points

- The points where $\dot{\mathbf{x}} = f(\mathbf{x}) = 0$, are called the *equilibrium points*.

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- The behaviour in the neighbourhood of an equilibrium point (x^*, y^*) is studied by local linearization:

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix},$$

where $\delta x = x - x^*$, $\delta y = y - y^*$.

$\Rightarrow$ Jacobian matrix

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- Find the equilibrium point: $(0, 0)$.
- Obtain the Jacobian: $\begin{pmatrix} 0 & 1 \\ -2xy - 1 & 1 - x^2 \end{pmatrix}$.
- Substituting $x = 0$ and $y = 0$ we get

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}.$$

Solution of linear ODEs

- Consider the dynamics in the neighbourhood of an equilibrium point:

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- If the system is 2D, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution of linear ODEs

If we can find linearly independent solutions $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ and $\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}$, then a general solution starting from any given initial condition is

$$\begin{aligned} x(t) &= c_1\, x_1(t) + c_2\, x_2(t) \\ y(t) &= c_1\, y_1(t) + c_2\, y_2(t). \end{aligned}$$

These linearly independent solutions are obtained from the eigenvalues and eigenvectors of the matrix **A**.

Eigenvalues and eigenvectors

Eigenvalues and eigenvectors

In $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the matrix $\mathbf{A}$ operates on the vector $\mathbf{x}$ to give the vector $\dot{\mathbf{x}}$. There exist some special directions such that the resulting vector remains in the same direction, i.e.,

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where λ is a number, called the eigenvalue. Any vector along these special directions are called eigenvectors.

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where λ is a number, called the eigenvalue. Any vector along these special directions are called eigenvectors.

Thus, if we place an initial condition on an eigenvector, the $\dot{\mathbf{x}}$ vector will be in the same direction. Thus we can use the 1D solution $x(t) = x_0\, e^{at}$ as one possible solution.

Calculation of eigenvalues

Let the linearized equations close to an equilibrium point be

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalue equation:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0.$$

This is true if $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

Calculation of eigenvalues

Thus

$$
\begin{aligned}
&\left| \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \\
&(-2-\lambda)(-2-\lambda) - 1 = 0 \\
&\lambda^2 + 4\lambda + 3 = 0.
\end{aligned}
$$

Solutions of this characteristic equation (the eigenvalues) are -1 and -3.

Calculation of eigenvectors

Let us take the eigenvalue -1, and calculate its corresponding eigenvector:

$$\mathbf{A}\mathbf{x} = (-1)\mathbf{x} \quad \Rightarrow \quad (\mathbf{A} + \mathbf{I})\mathbf{x} = 0$$

$$\begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Both the rows lead to the same equation

$$x - y = 0.$$

This is the eigenvector associated with $\lambda_1 = -1$. Similarly, the eigenvector associted with $\lambda_2 = -3$ is found to be $x + y = 0$.

Back to solution of ODEs

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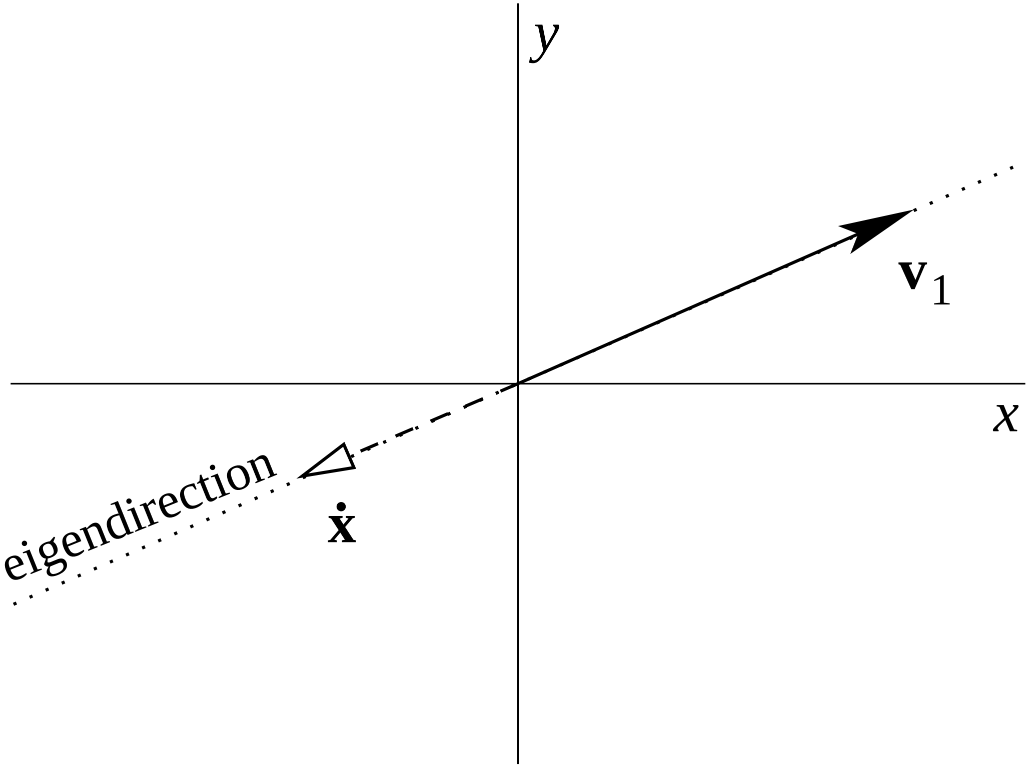
Let λ_1 and λ_2 be the eigenvalues and $\mathbf{v}_1$ and $\mathbf{v}_2$ be the corresponding eigenvectors.



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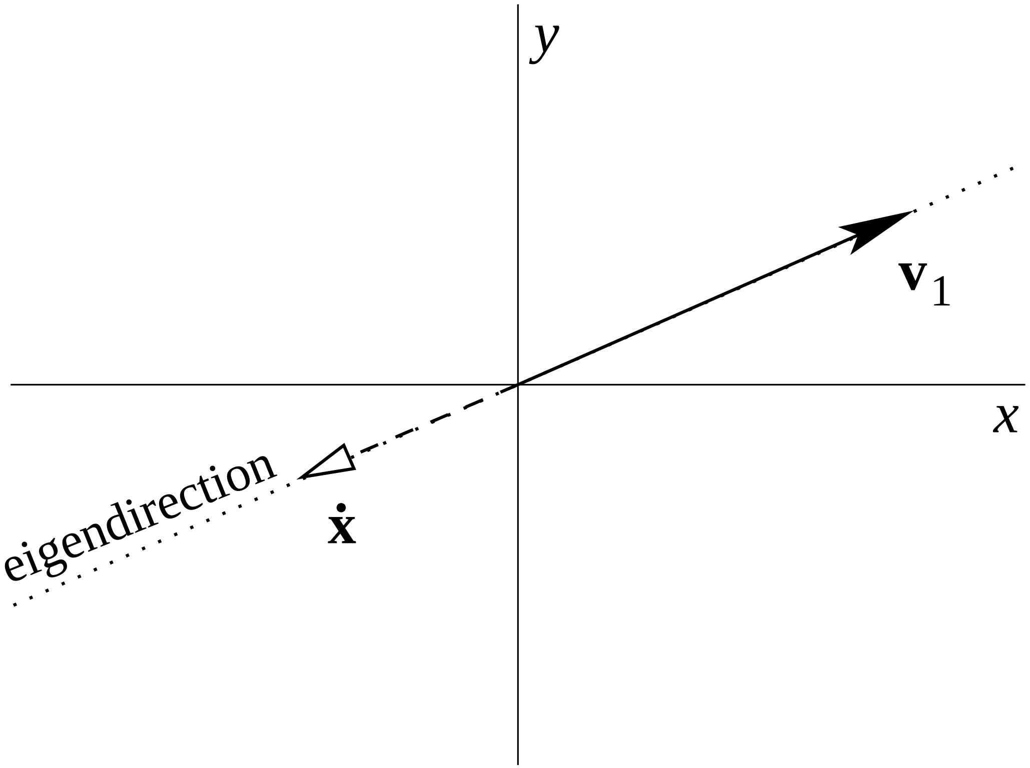


If we place an initial condition on $\mathbf{v}_1$, we have $\dot{\mathbf{x}} = \mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$. The solution is $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$.

Solution of ODEs

Similarly, for any initial condition placed along $\mathbf{v}_2$ we have another solution

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2.$$

Therefore the general solution can be constructed as

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Example

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$. For λ_1, the eigenvector is given by $2x = -y$. To choose any point on this eigenvector, set $x = 1$. This gives $y = -2$. Thus $\mathbf{v}_1 = [1 \quad -2]^T$. For this initial condition the solution is

$$\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Example

Similarly, for $\lambda_2 = -3$, the eigenvector is $x = -3y$. To choose a point on this eigenvector, take $x = 3$. This gives $y = -1$. Thus the second eigenvector becomes $\mathbf{v}_2 = [3 \quad -1]^T$, and the solution along this eigenvector is

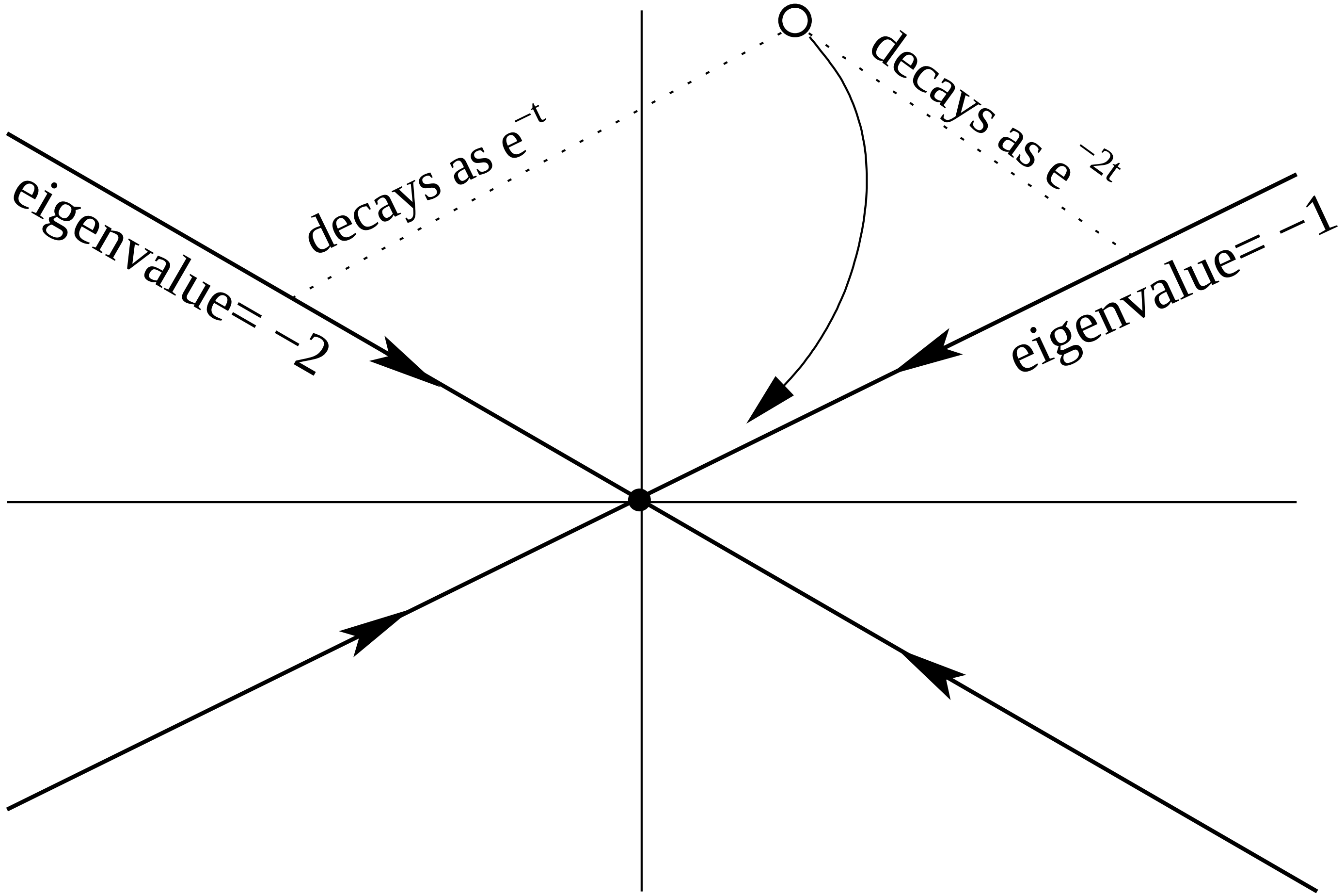
$$\mathbf{x}_2(t) = e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Hence the general solution of the system of differential equations is

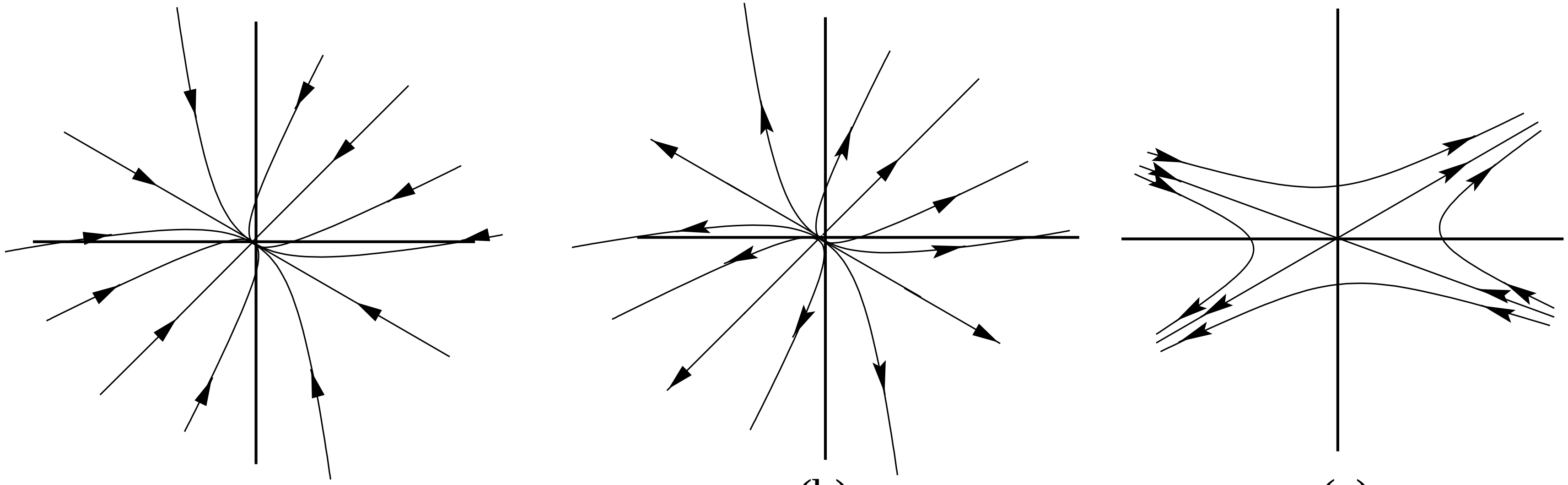
$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Local vector fields

The dynamics in the system with eigenvalues -1 and -2:



Local vector fields



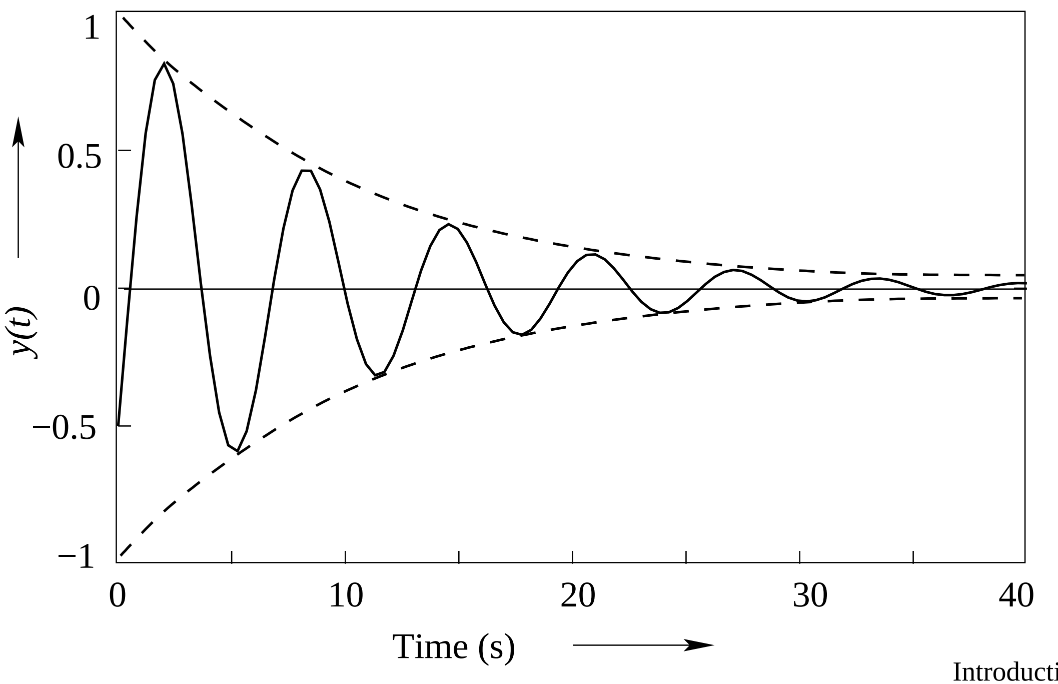
(a) (b) (c)

Vector fields of linear systems with real eigenvalues, (a) both eigenvalues negative, (b) both eigenvalues positive, and (c) one eigenvalue negative and the other positive.

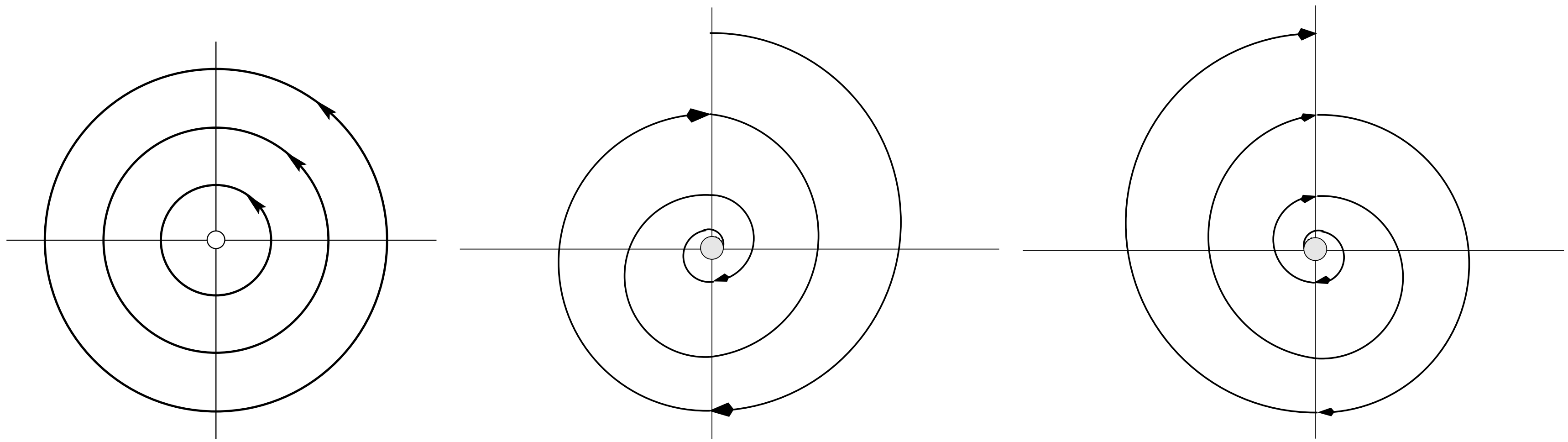
Complex eigenvalues

If the eigenvalues are $\lambda = \sigma \pm i\omega$, we get spiralling solutions (combination of exponentials and sinusoids):

$$\mathbf{x}(t) = c_1 e^{\sigma t} \begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix} + c_2 e^{\sigma t} \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}.$$

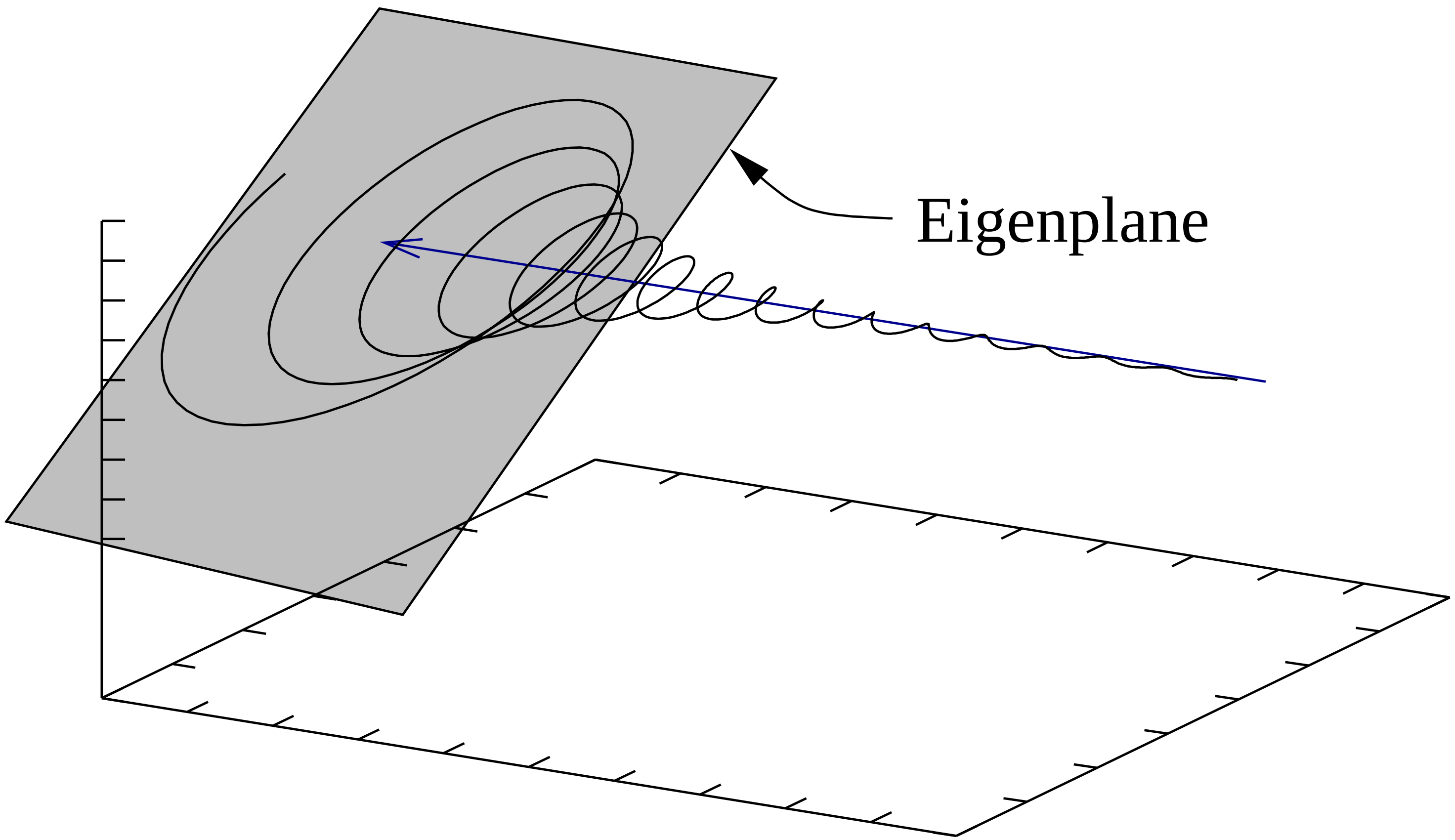


Complex eigenvalues



The structure of the vector field in the state space for (a) imaginary eigenvalues, (b) complex eigenvalues with negative real part, and (c) complex eigenvalues with positive real part.

3D systems



The typical orbit for a system with one negative real eigenvalue and two complex conjugate eigenvalues with positive real part.

On to nonlinear systems

Take the simple pendulum

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\sin x.\end{aligned}$$

- Multiple equilibrium points: $(0,0)$, $(-\pi,0)$, $(\pi,0)$, $(-2\pi,0)$, $(2\pi,0)$, etc.

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- Multiple equilibrium points: $(0,0)$, $(-\pi,0)$, $(\pi,0)$, $(-2\pi,0)$, $(2\pi,0)$, etc.
- At $(0,0)$, local linearization yields

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

whose eigenvalues are purely imaginary, $\pm i$.

The simple pendulum

At $(-\pi, 0)$ and $(\pi, 0)$, local linearization yields

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

whose eigenvalues are $+1$ and -1. For the eigenvalue $\lambda_1 = +1$, the eigendirection is $x = y$. For $\lambda_2 = -1$, the eigendirection is $x = -y$.

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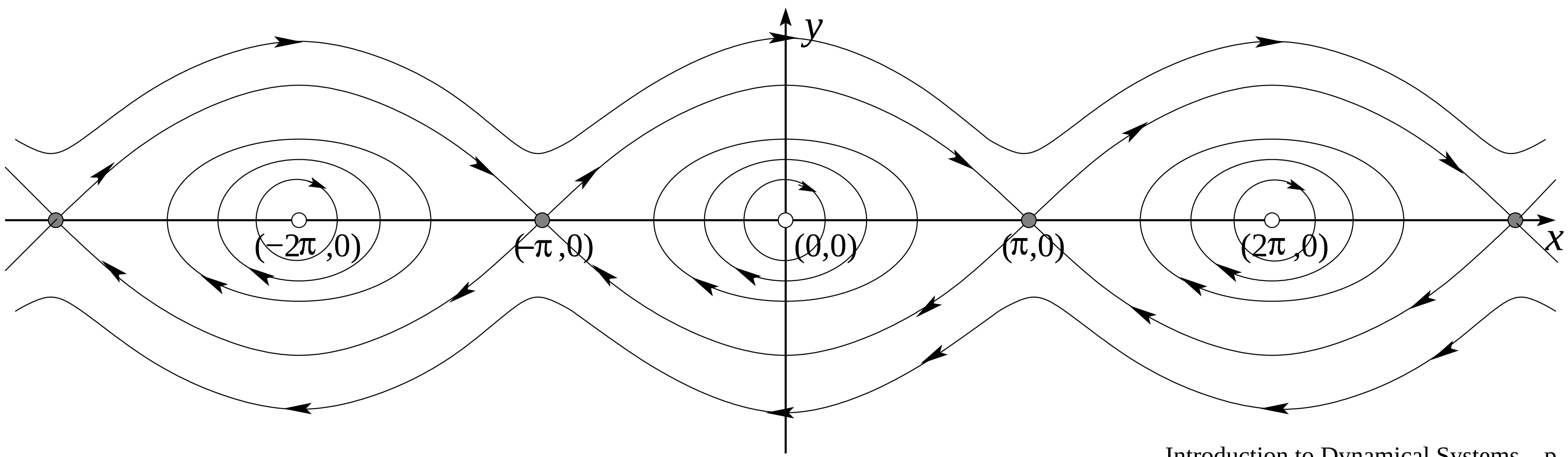

Attractors in nonlinear systems

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Consider the van der Pol system:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

In first order form:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu(1 - x^2)y - x. \end{aligned}$$

Equilibrium point: (0,0)

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Equilibrium point: (0,0)
Local linearization:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2\mu xy - 1 & \mu - \mu x^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Attractors in nonlinear systems

Evaluated at $(0, 0)$, it yields the Jacobian matrix

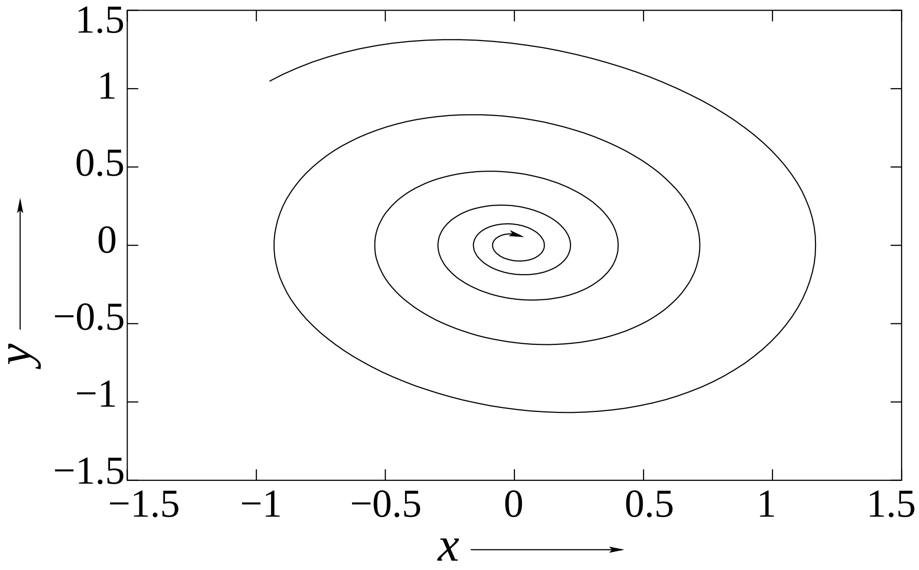
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}.$$

Its eigenvalues are

$$\lambda_{1,2} = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}.$$

Conclusion:
1. For $\mu < 2$ the eigenvalues are complex conjugate.
2. An incoming spiraling vector field should change into an outgoing spiraling vector field as μ is varied through zero.

Attractors in nonlinear systems



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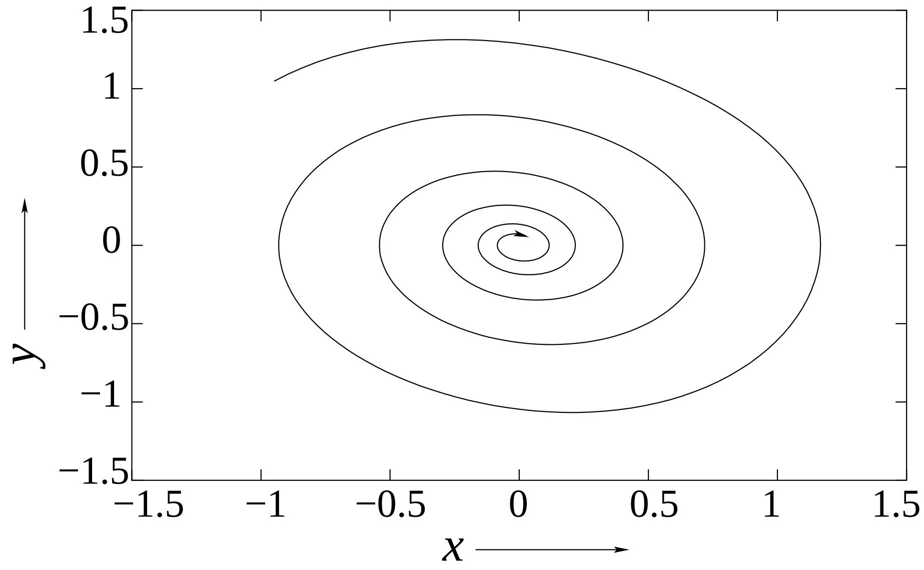
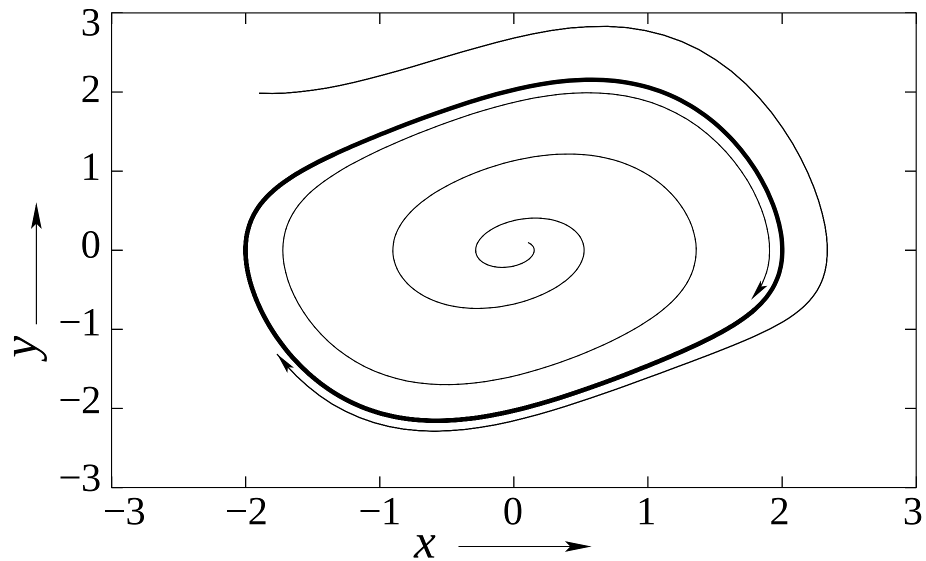
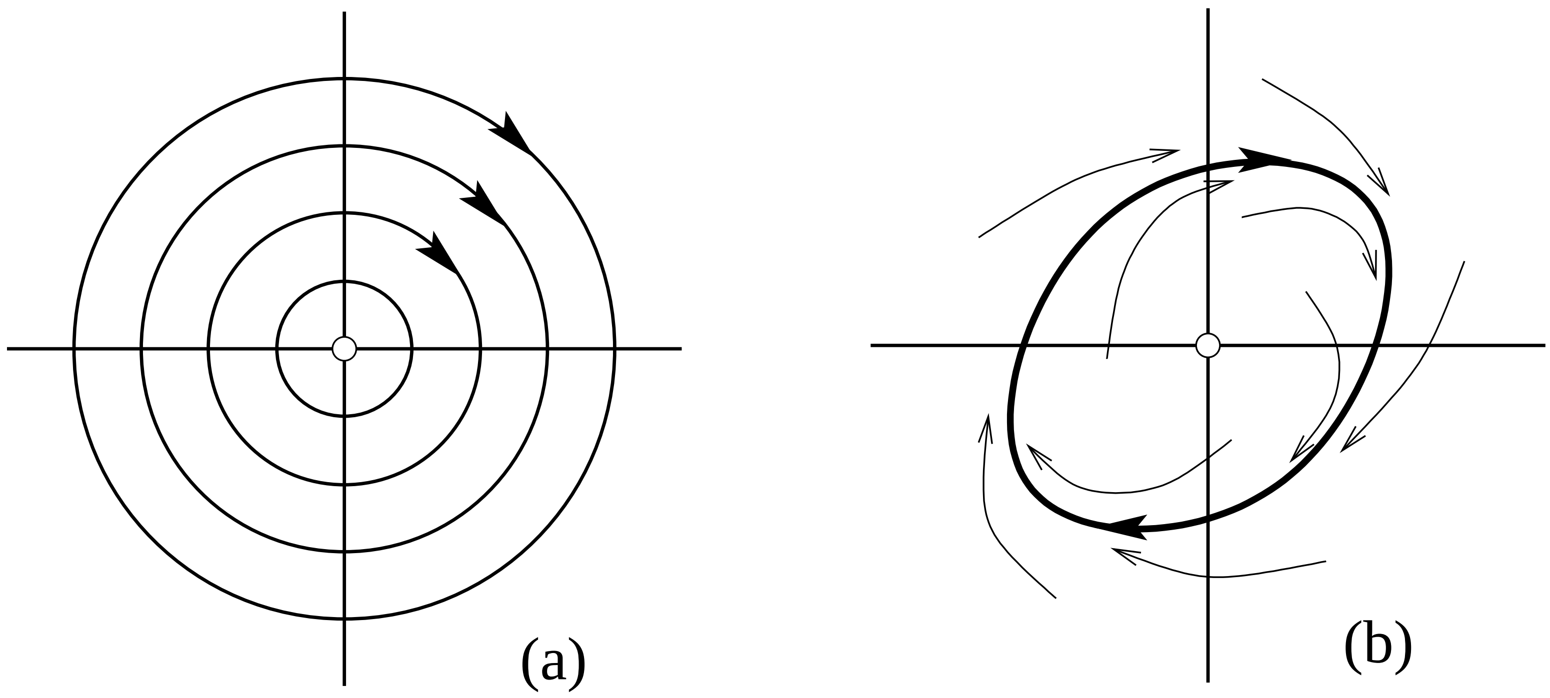




(a) for $\mu < 0$, and (b) for $\mu > 0$.

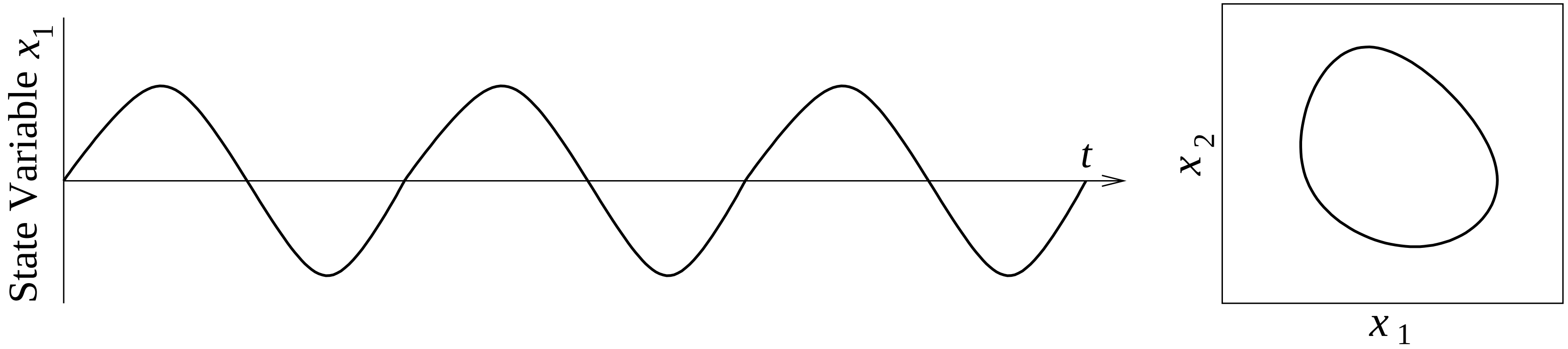
Limit cycle



(a) Periodic orbits in a linear system with imaginary eigenvalues, and (b) limit cycle in a nonlinear system.

Limit cycles

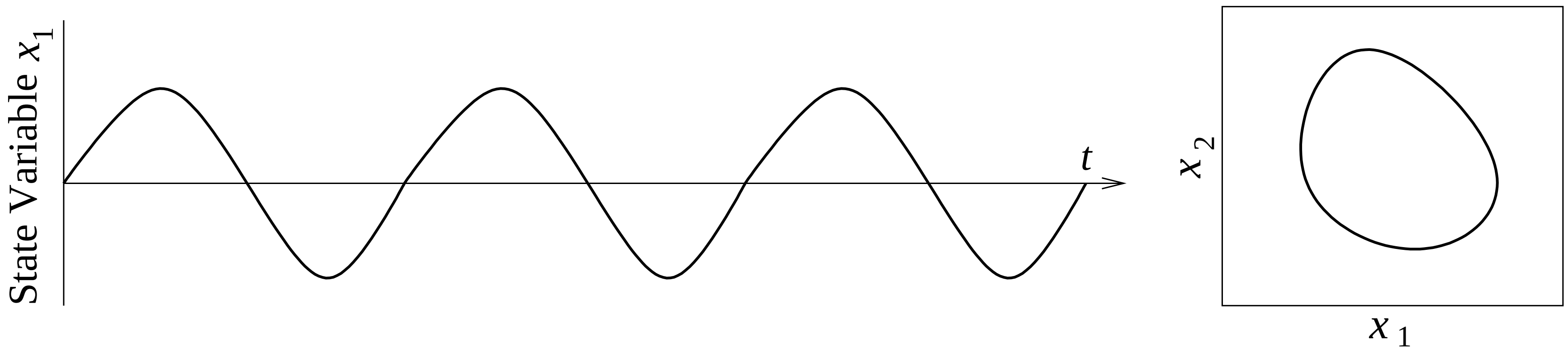
Limit cycles



(a) (b)

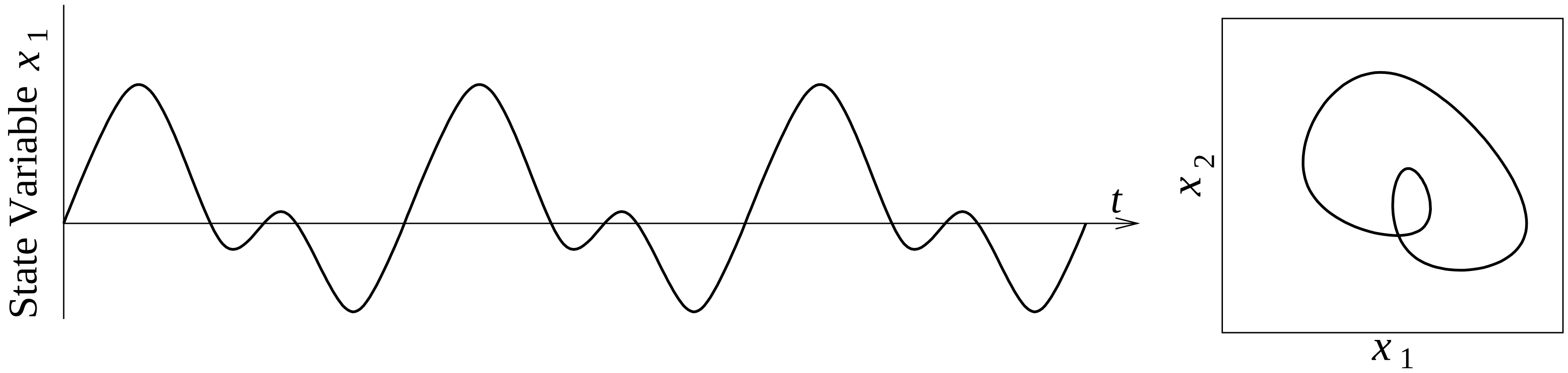
Period-1 limit cycle.

Limit cycles



(a) (b)

Period-1 limit cycle.



(a) (b)

Period-2 limit cycle. Not possible in a 2D system, but possible in 3 or higher dimensions.

The Lorenz system

The Lorenz system

$$
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The Lorenz system

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Equilibrium points are

$$\begin{aligned}
A &= (0,0,0), \\
B &= \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right), \\
C &= \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right).
\end{aligned}$$

The Lorenz system

The local behavior around these equilibrium points will be given by the Jacobian matrix

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ -z+r & -1 & -x \\ y & x & -b \end{bmatrix}.$$

The Lorenz system

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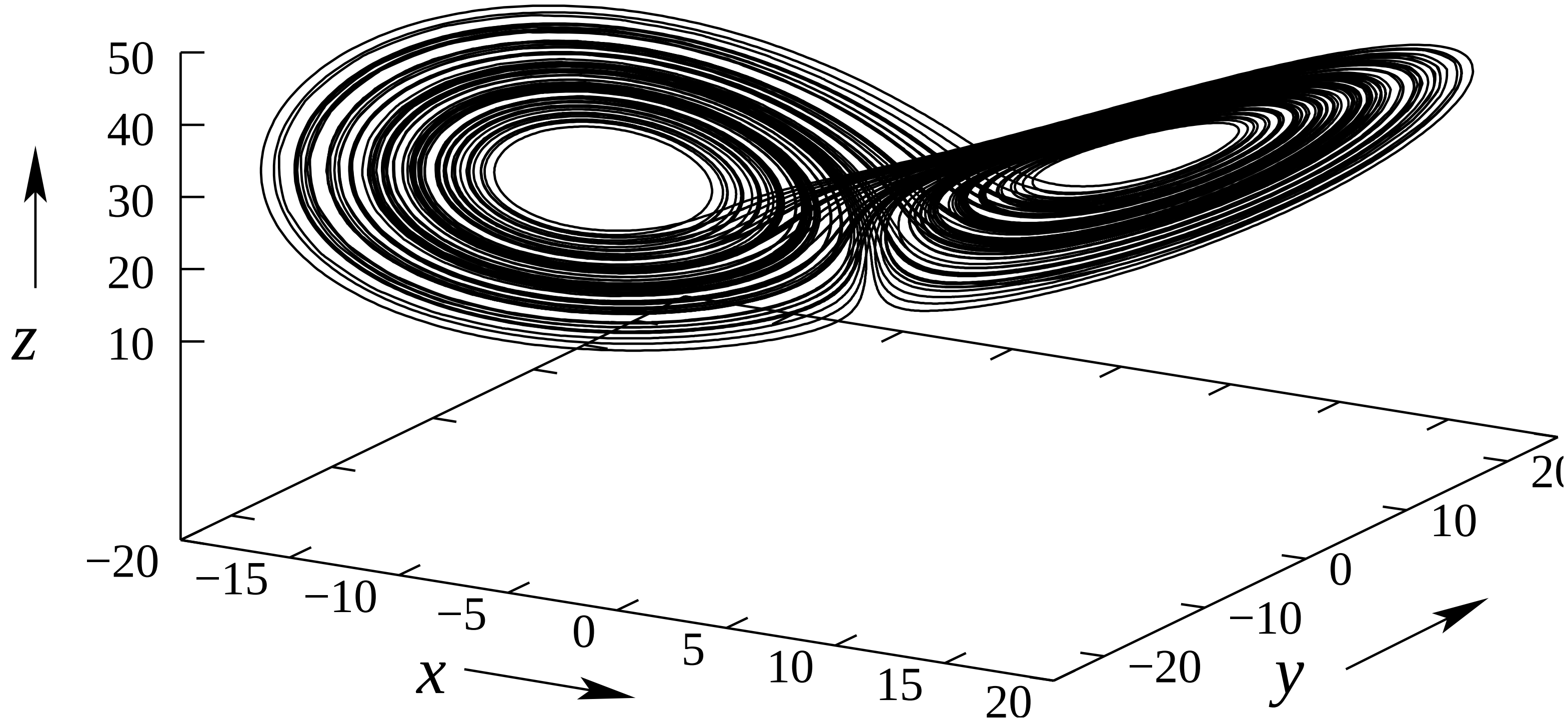
$$\begin{bmatrix} -\sigma & \sigma & 0 \\ -z+r & -1 & -x \\ y & x & -b \end{bmatrix}.$$

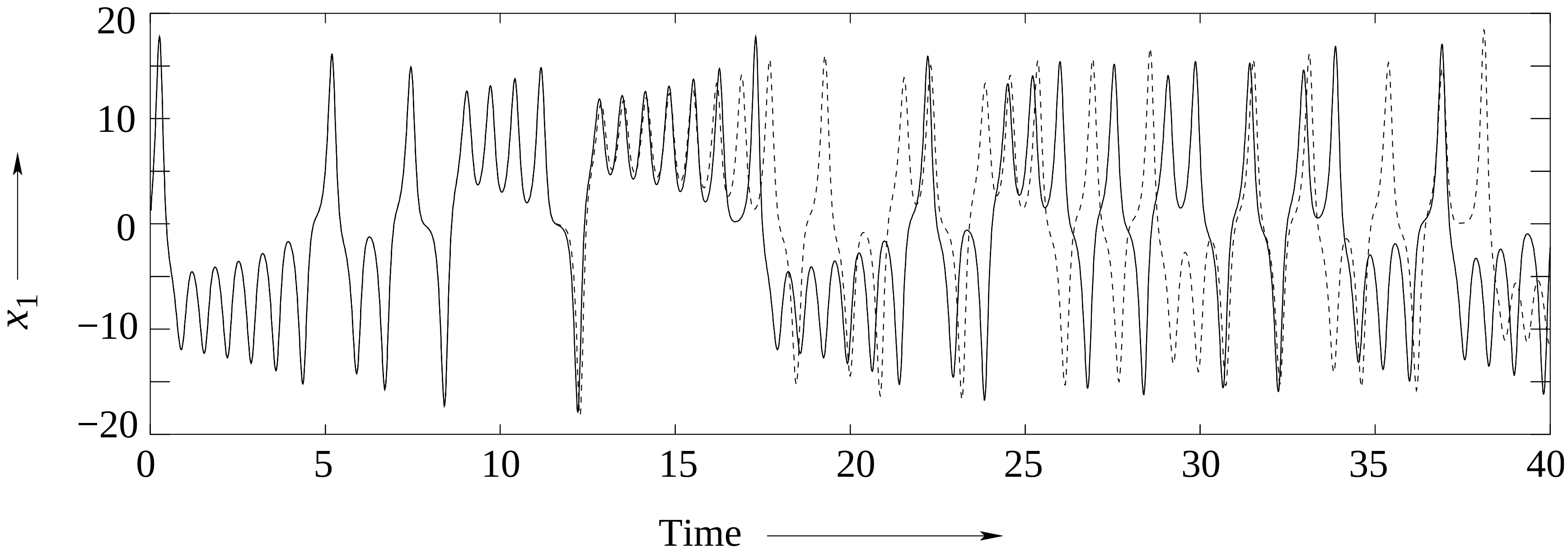
1. For $r < 1$, the equilibrium points B and C do not exist, and the point A is stable. For $r > 1$, the point A at the origin becomes unstable, and the equilibria B and C come into existence.
2. For $1 < r < 24.74$, these two equilibria have complex conjugate eigenvalues with real part negative. For $r > 24.74$, the real part becomes positive.

Chaos



The trajectory of the Lorenz system for $\sigma = 8/3$, $b = 10$, and $r = 28$.

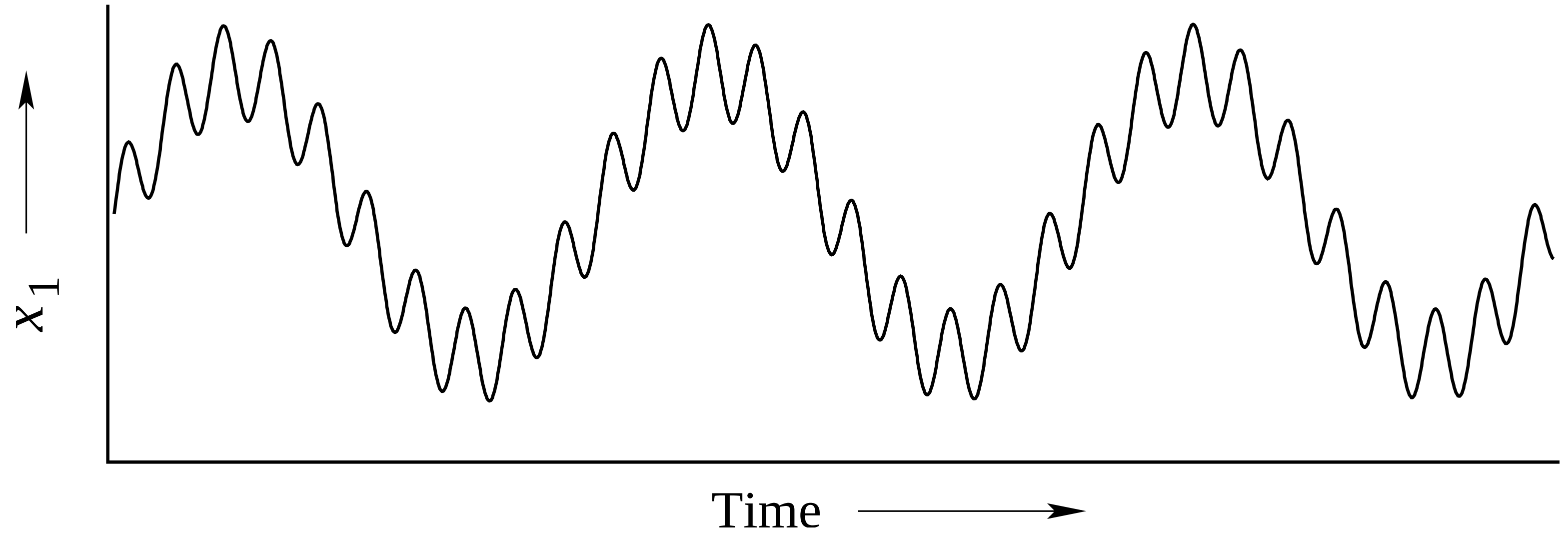
Chaos



The waveforms starting from two very close initial conditions. Firm line: the initial condition is [0, 7, 7], broken line: the initial condition is [0.0001, 7, 7].

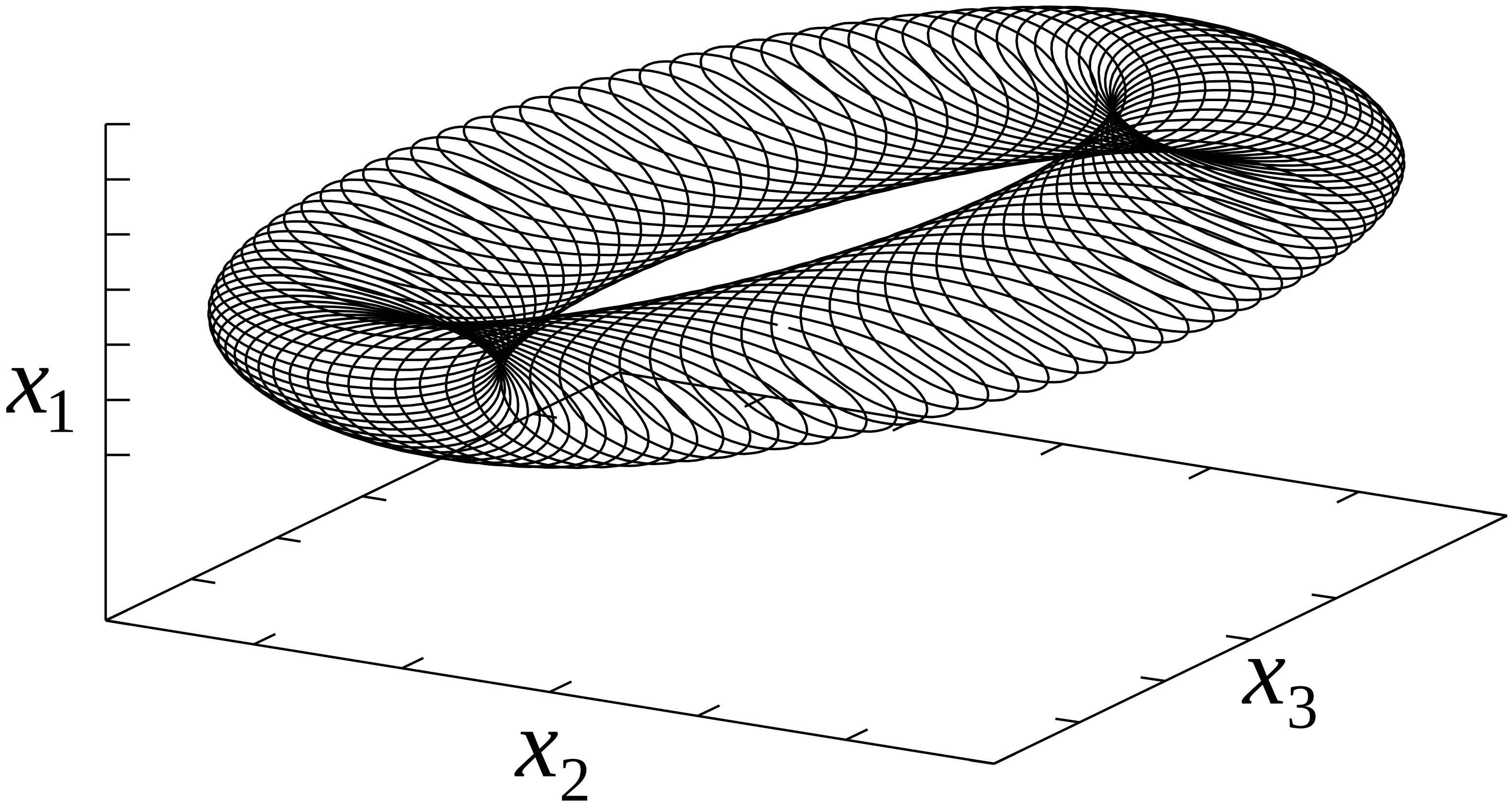
⇒ Sentitive dependence on initial condition.

Orbit on a torus

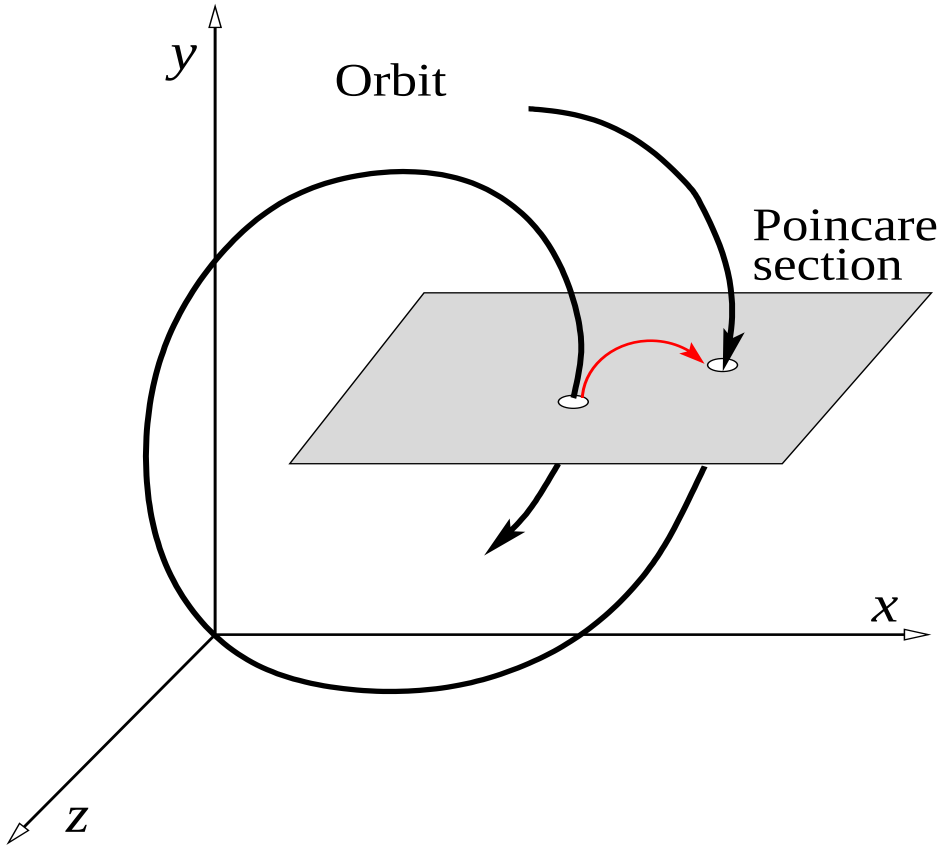


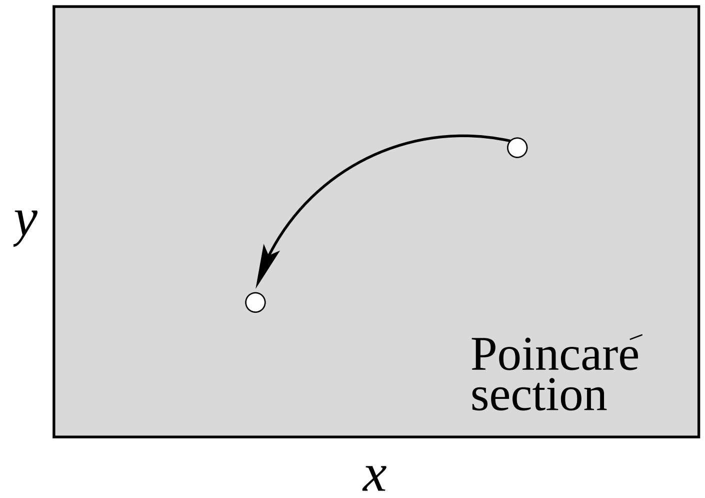
- Commensurate frequencies: mode-locked periodic orbit
- Incommensurate frequencies: quasiperiodicity

Orbit on a torus



The Poincaré section



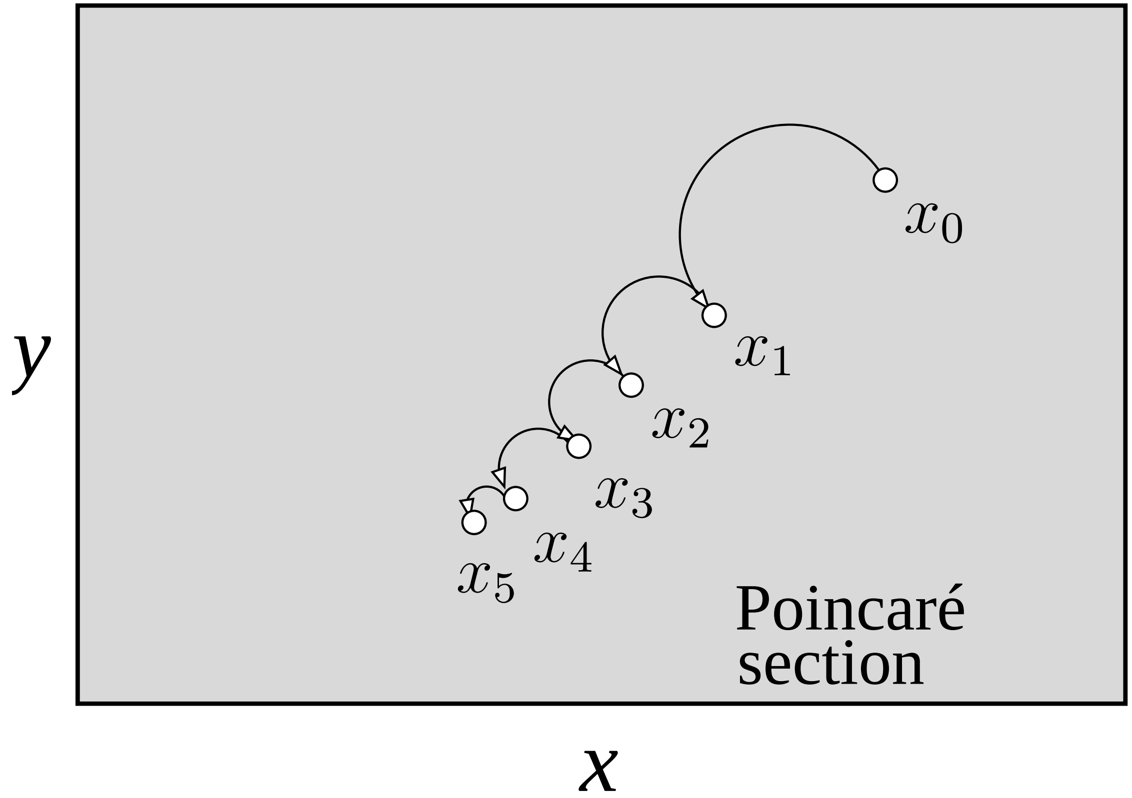


Reduces a continuous-time dynamical system into a discrete-time one:

$$x_{n+1} = f_1(x_n, y_n), \qquad y_{n+1} = f_2(x_n, y_n).$$

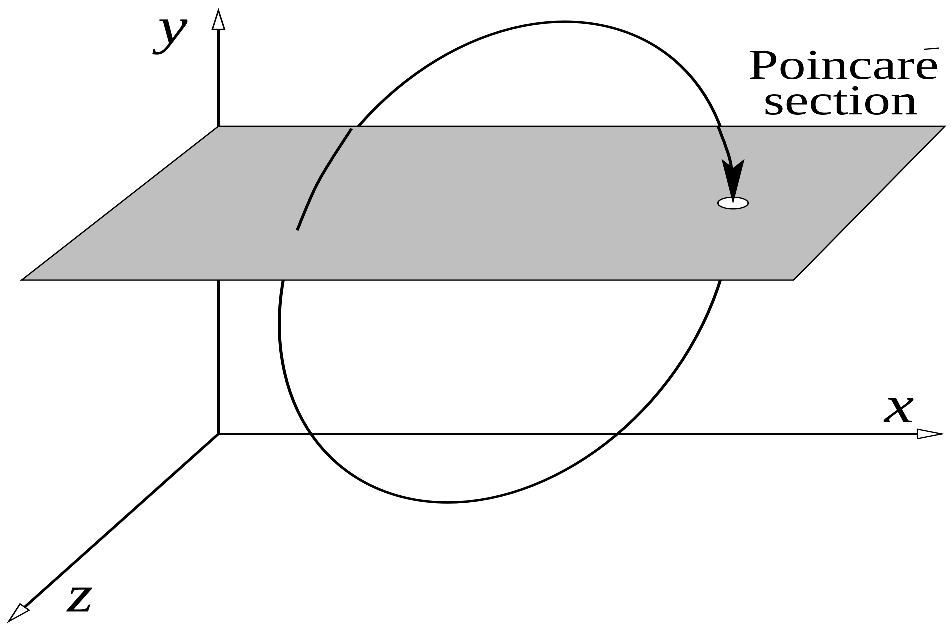
The Poincaré map

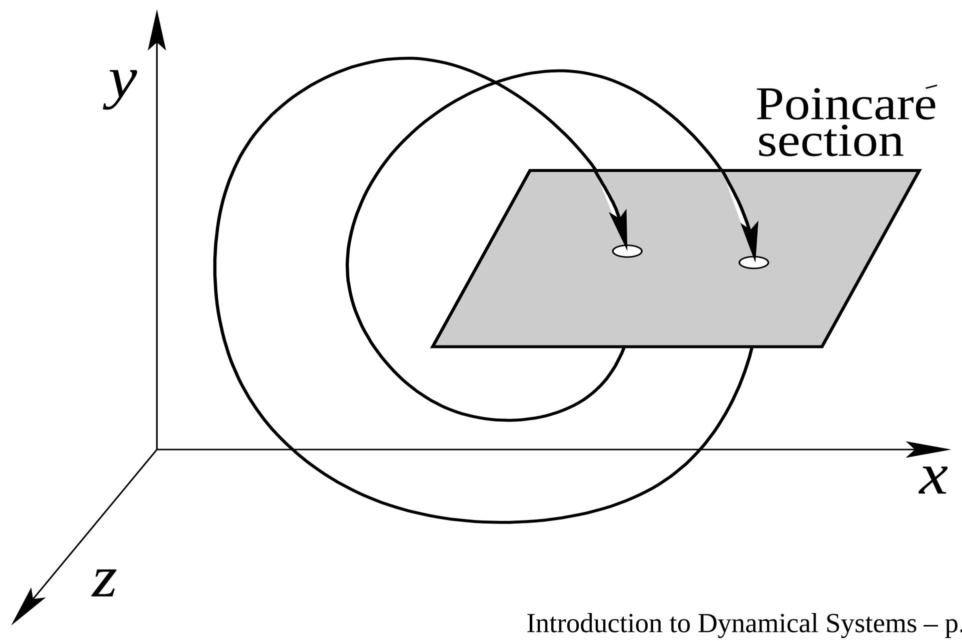
Reapeated application of the map leads to a sequence of points.



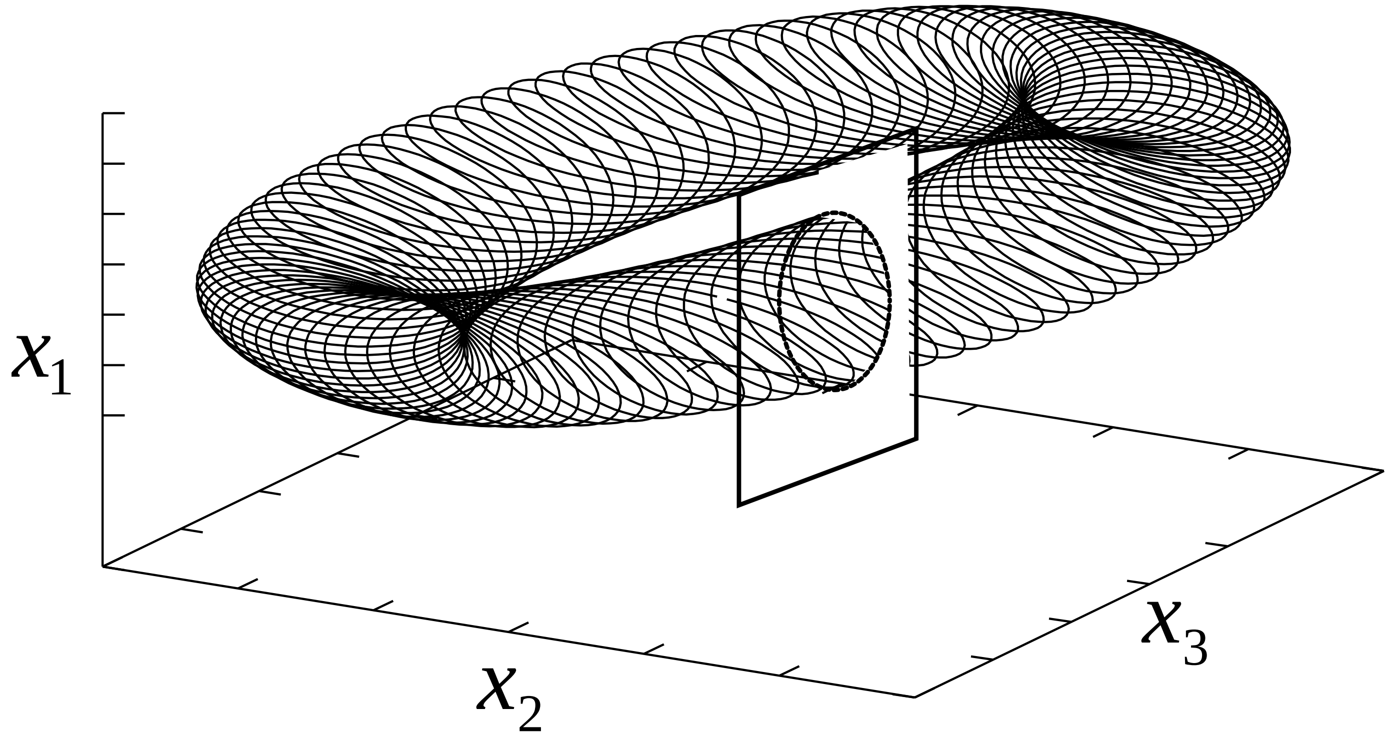
May converge to a fixed point $x_{n+1} = x_n$, $y_{n+1} = y_n$.

The Poincaré section





The Poincaré section

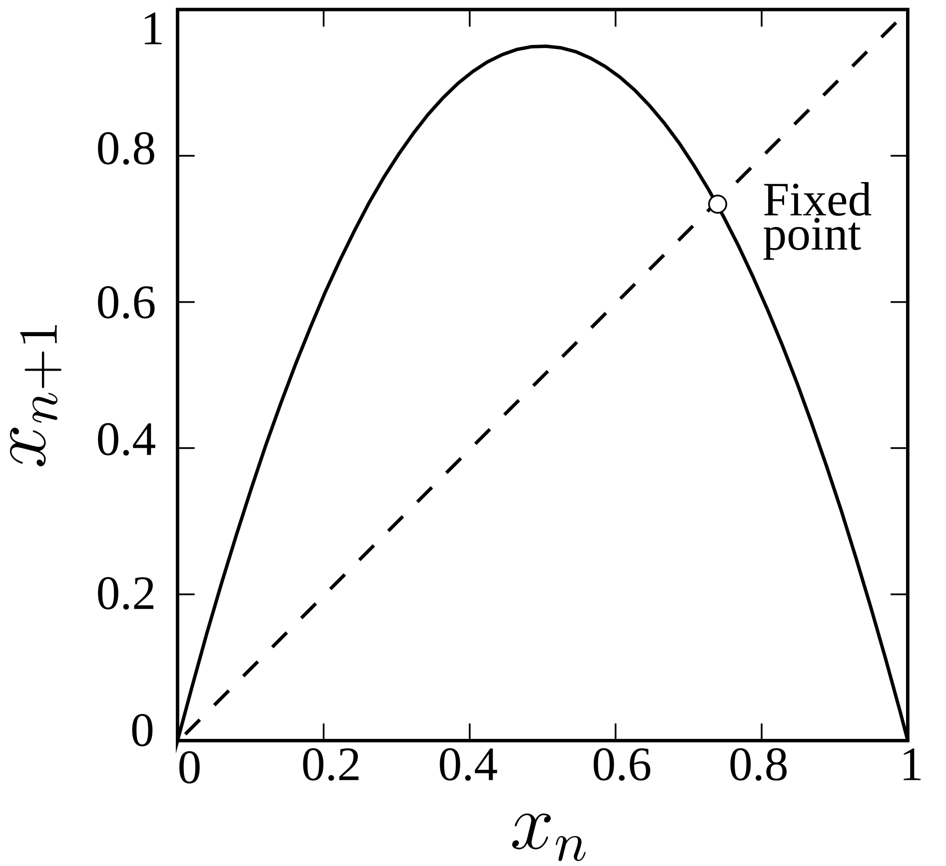


A quasiperiodic orbit appears as a closed loop in the Poincaré plane.

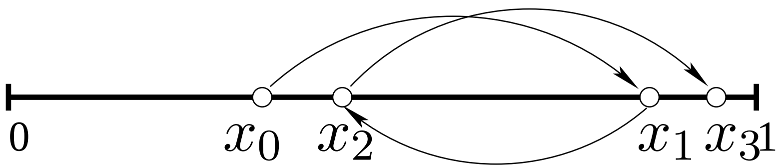
One-dimensional maps

Consider the logistic map (single population model):

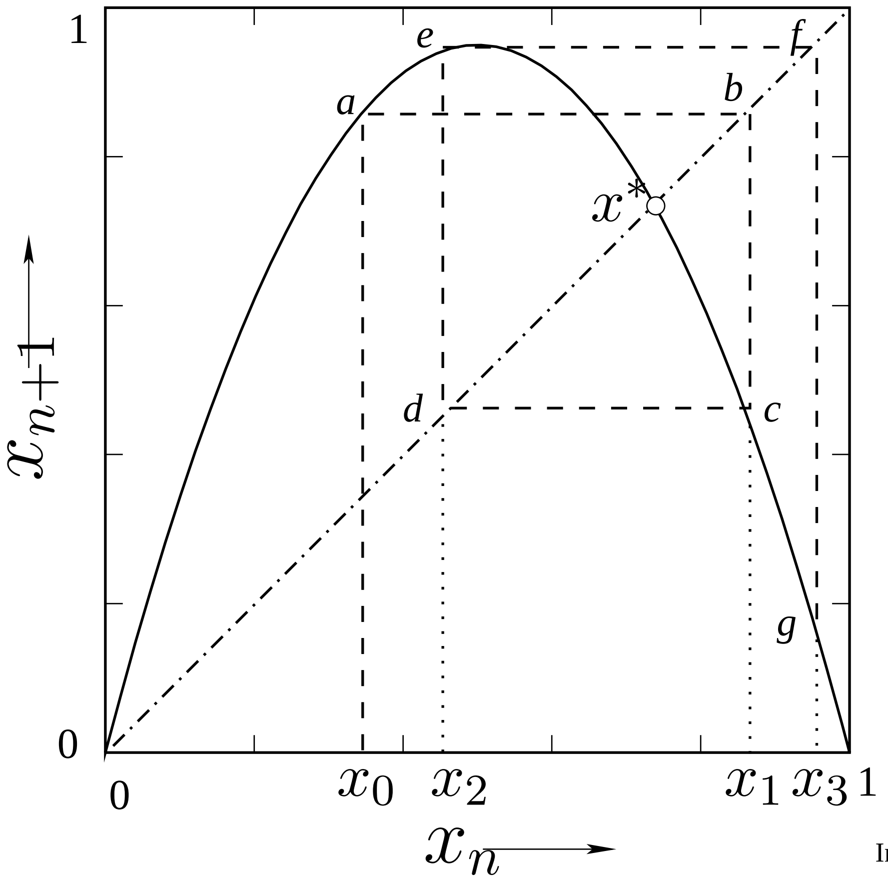
$$x_{n+1} = \mu x_n(1 - x_n)$$



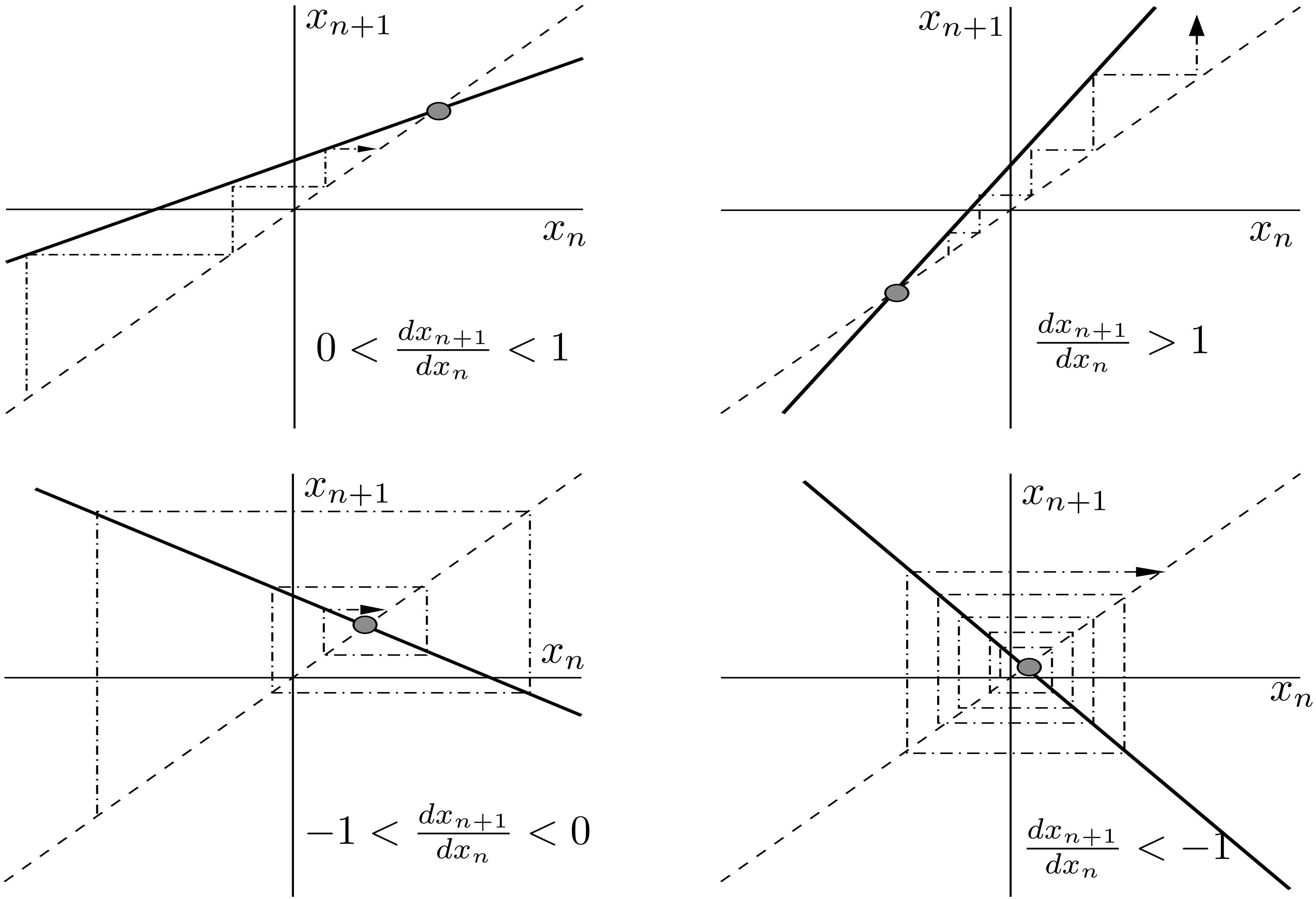
Graphical iteration



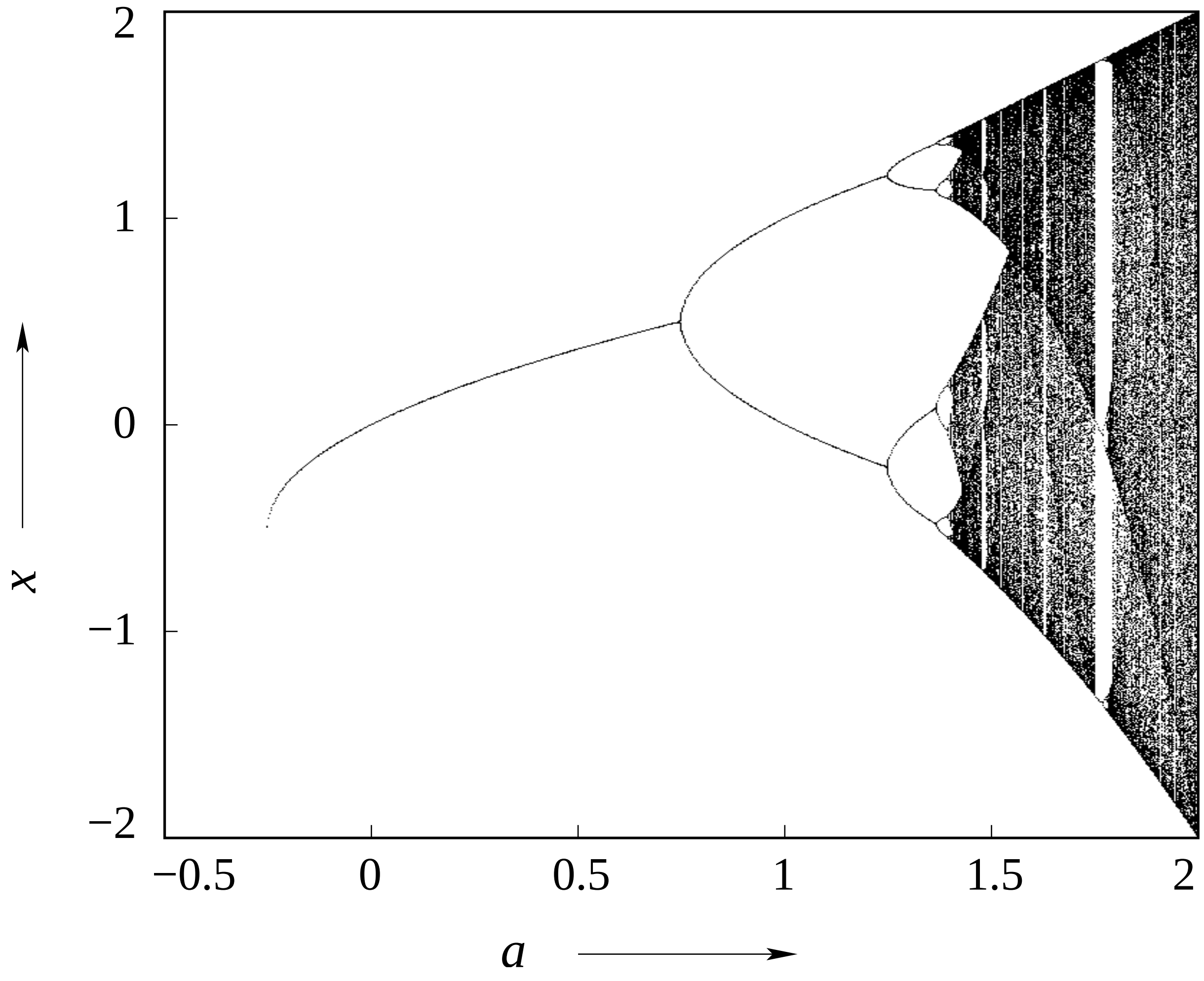
(a)



Stability of fixed points



Bifurcation diagram



The bifurcation diagram for the map $x_{n+1} = a - x_n^2$.

Saddle-node bifurcation

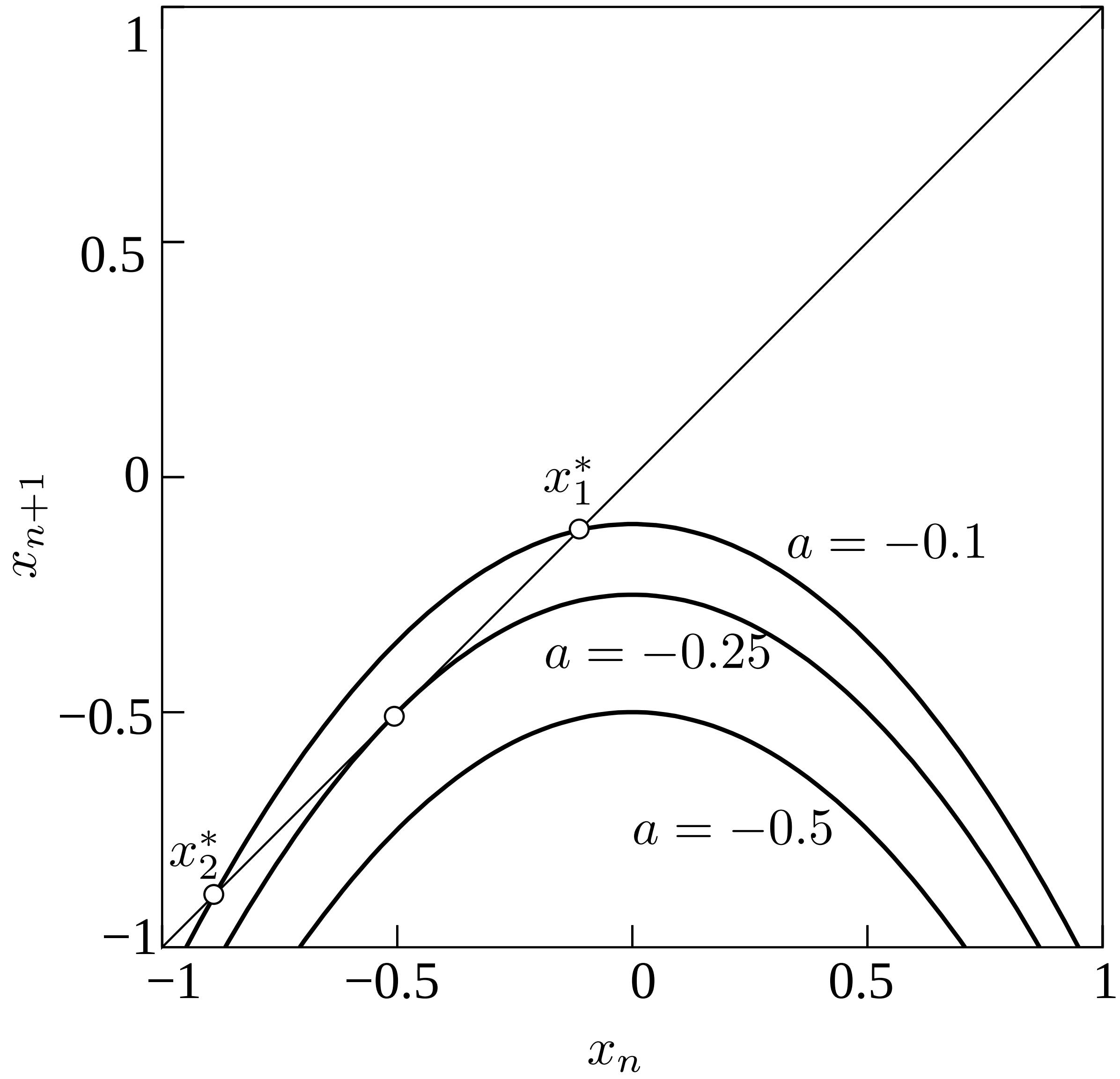
- Consider the map $x_{n+1} = a - x_n^2$.
- Locate the fixed points $x_{n+1} = x_n = x^*$.
  $\Rightarrow \quad {x^*}^2 + x^* - a = 0$, whose solutions are

$$x^* = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4a}\ .$$

- For $a < -1/4$, we do not get a real number, which implies that no fixed point exists. For $a > -1/4$ there are two fixed points

$$\begin{aligned} x_1^* &= -\frac{1}{2} + \frac{1}{2}\sqrt{1+4a}, \\ x_2^* &= -\frac{1}{2} - \frac{1}{2}\sqrt{1+4a}. \end{aligned}$$

Saddle-node bifurcation



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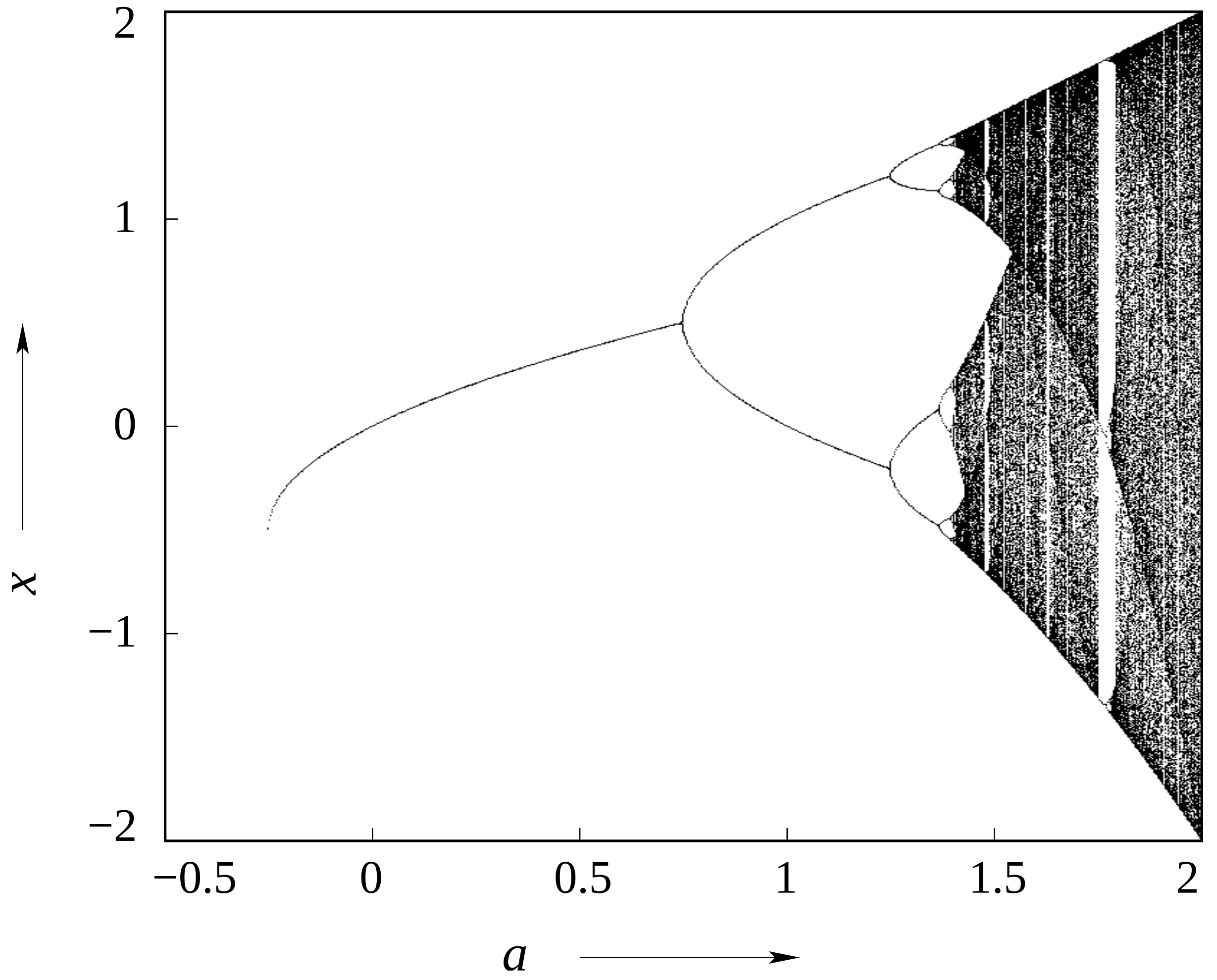

Period doubling bifurcation

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Slope of the graph at x_1^*:

$$\left.\frac{dx_{n+1}}{dx_n}\right|_{x_1^*} = 1 - \sqrt{1 + 4a}.$$

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For $-1/4 < a < 3/4$, the magnitude of the derivative is less than unity, and hence the fixed point x_1^* is stable. For $a > 3/4$ the fixed point becomes unstable.

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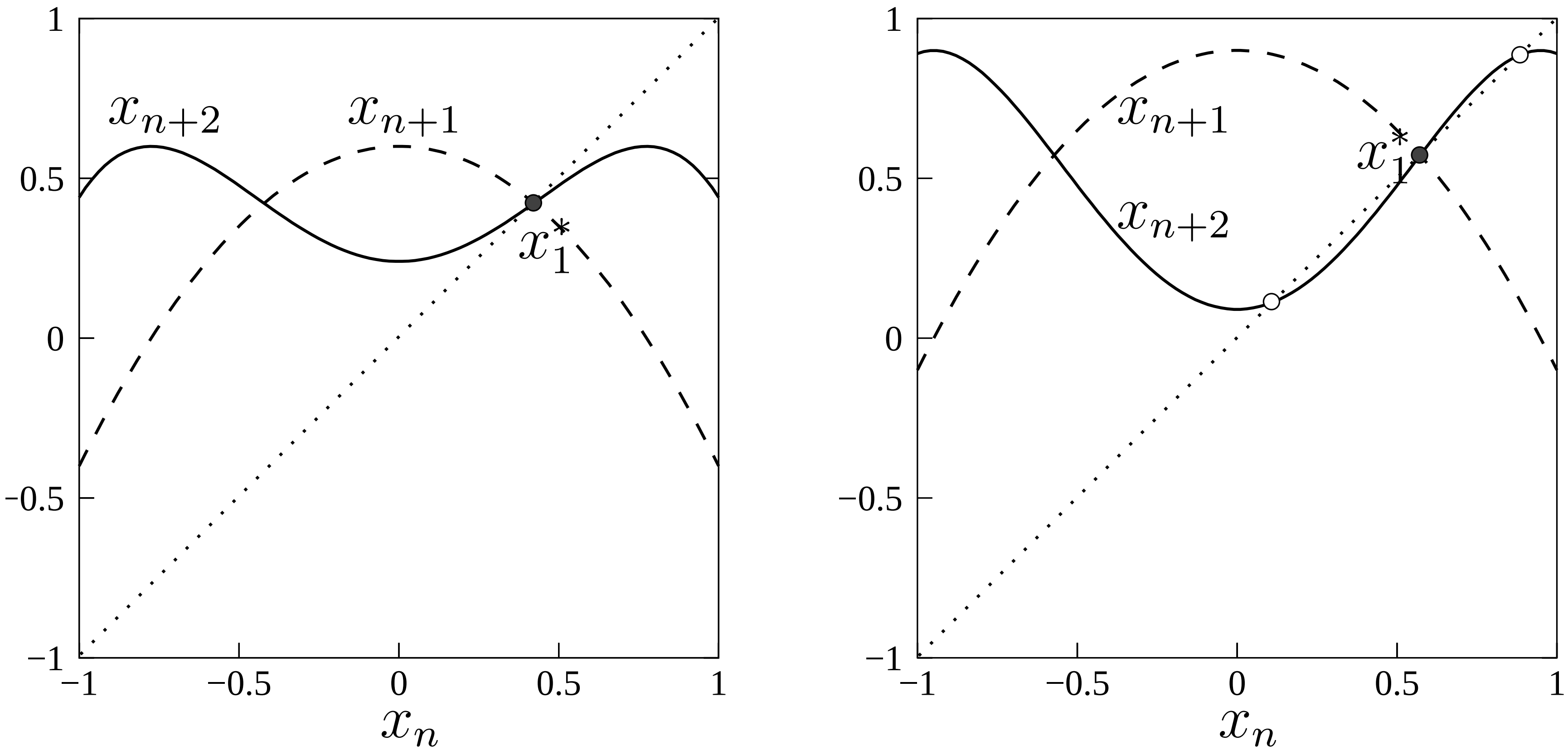
$$\begin{aligned} x_{n+2} &= a - x_{n+1}^2 \\ &= a - \left(a - x_n^2\right)^2 \\ &= -x_n^4 + 2ax_n^2 - a^2 + a. \end{aligned}$$

Period doubling bifurcation



For $a > 0.75$ the period-1 fixed point becomes unstable, and two stable fixed points appear in the graph of x_{n+2}.

Two-dimensional maps

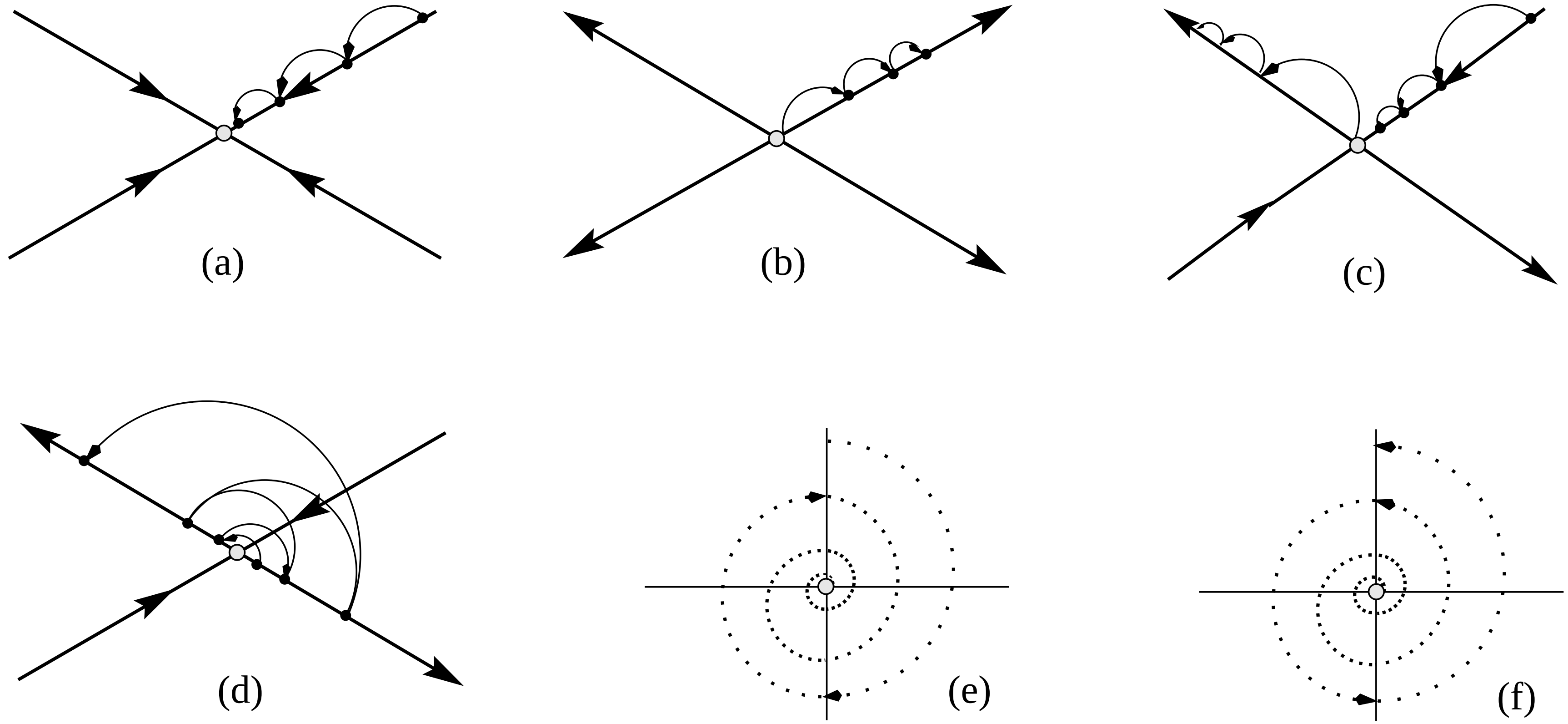
$$\begin{aligned} x_{n+1} &= f_1(x_n,\ y_n), \\ y_{n+1} &= f_2(x_n,\ y_n), \end{aligned}$$

The local linear approximation is then given by

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

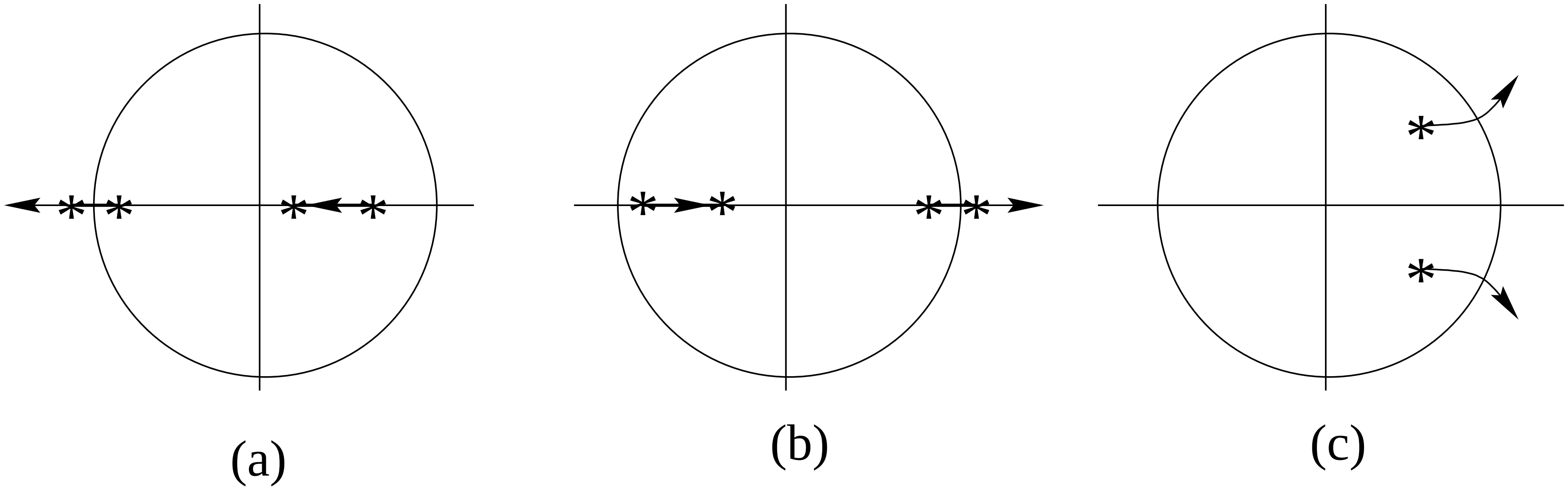
The stability of the fixed point is given by the eigenvalues of the Jacobian matrix.

Local dynamics



(a) Attracting node: $0 < \lambda_1, \lambda_2 < 1$. (b) Repelling node: $\lambda_1, \lambda_2 > 1$. (c) Regular saddle: $0 < \lambda_1 < 1$, $\lambda_2 > 1$. (d) Flip saddle: $0 < \lambda_1 < 1$, $\lambda_2 < -1$. (e) Spiral attractor: eigenvalues complex, $|\lambda_1|, |\lambda_2| < 1$. (f) Spiral repeller: eigenvalues complex, $|\lambda_1|, |\lambda_2| > 1$.

Three basic bifurcations



(a) A period doubling bifurcation: eigenvalue crosses the unit circle on the negative real line,
(b) A saddle-node bifurcation: an eigenvalue touches the unit circle on the positive real line, and
(c) A Neimark-Sacker bifurcation: a complex conjugate pair of eigenvalues cross the unit circle.

An example

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Consider the Hénon map

$$x_{n+1} = A - x_n^2 + 0.4\, y_n, \qquad y_{n+1} = x_n.$$

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The fixed points are

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$$x^* = \frac{1}{2}\left(-0.6 \pm \sqrt{0.36 + 4A}\right) \quad \text{and} \quad y^* = x^*.$$

For $A < -0.09$, the fixed point does not exist. For $A > -0.09$ there are two fixed points

$$\begin{aligned} x_1^* = y_1^* &= -0.3 + \sqrt{0.09 + A}, \\ x_2^* = y_1^* &= -0.3 - \sqrt{0.09 + A}. \end{aligned}$$

The Hénon map

The Hénon map

At the fixed points, local linearization yields the Jacobian

$$\begin{bmatrix} -2x^* & 0.4 \\ 1 & 0 \end{bmatrix}.$$

whose eigenvalues are

$$\lambda = -x^* \pm \sqrt{x^{*2} + 0.4}.$$

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At $A = -0.09$, the two solutions have the same value, $x^* = y^* = -0.3$. At that point one of the eigenvalues is $+1$. $A > -0.09$, one FP is a saddle, and the other is a node. $\Rightarrow$ saddle-node bifurcation.

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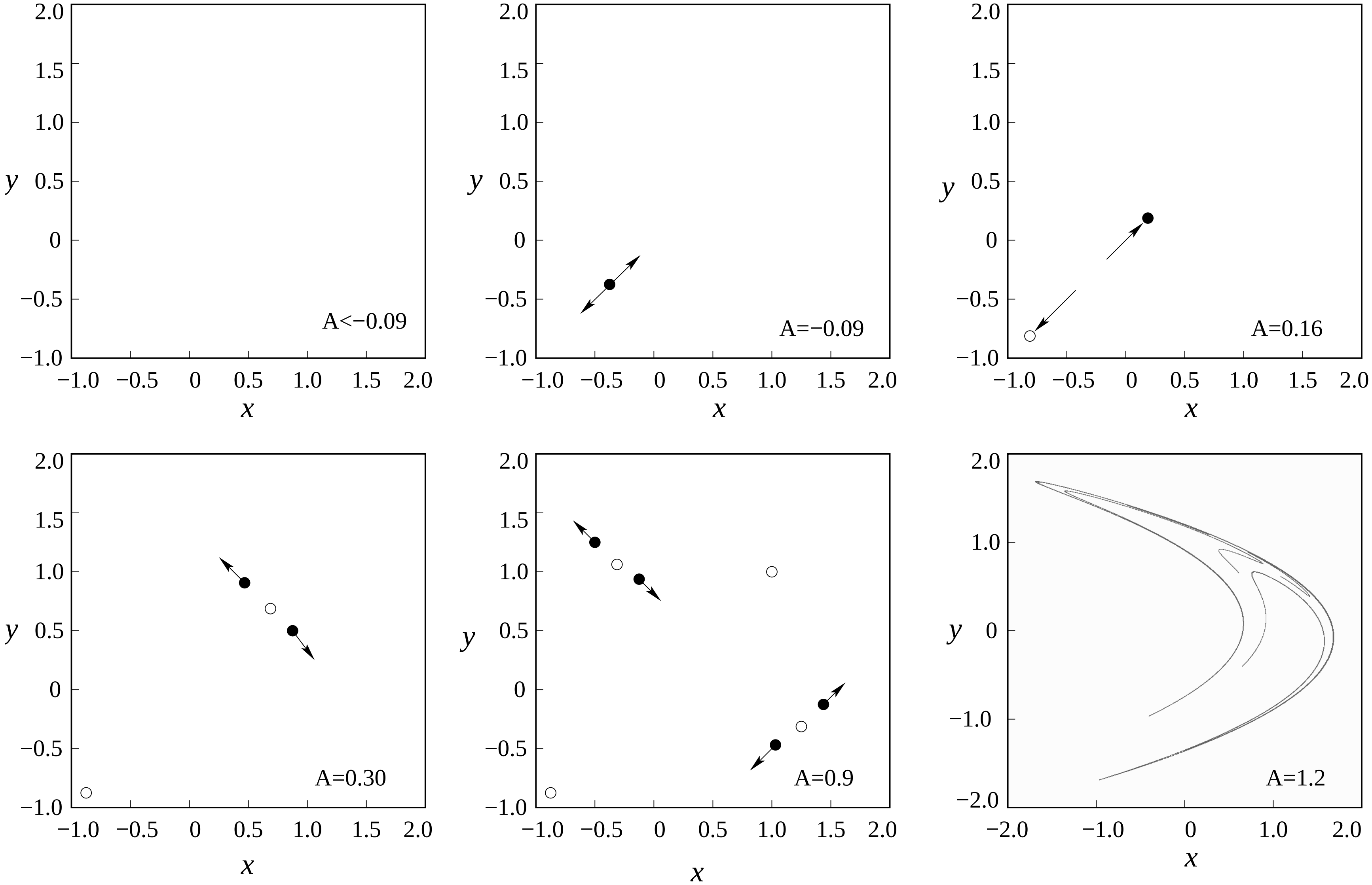
The Hénon map



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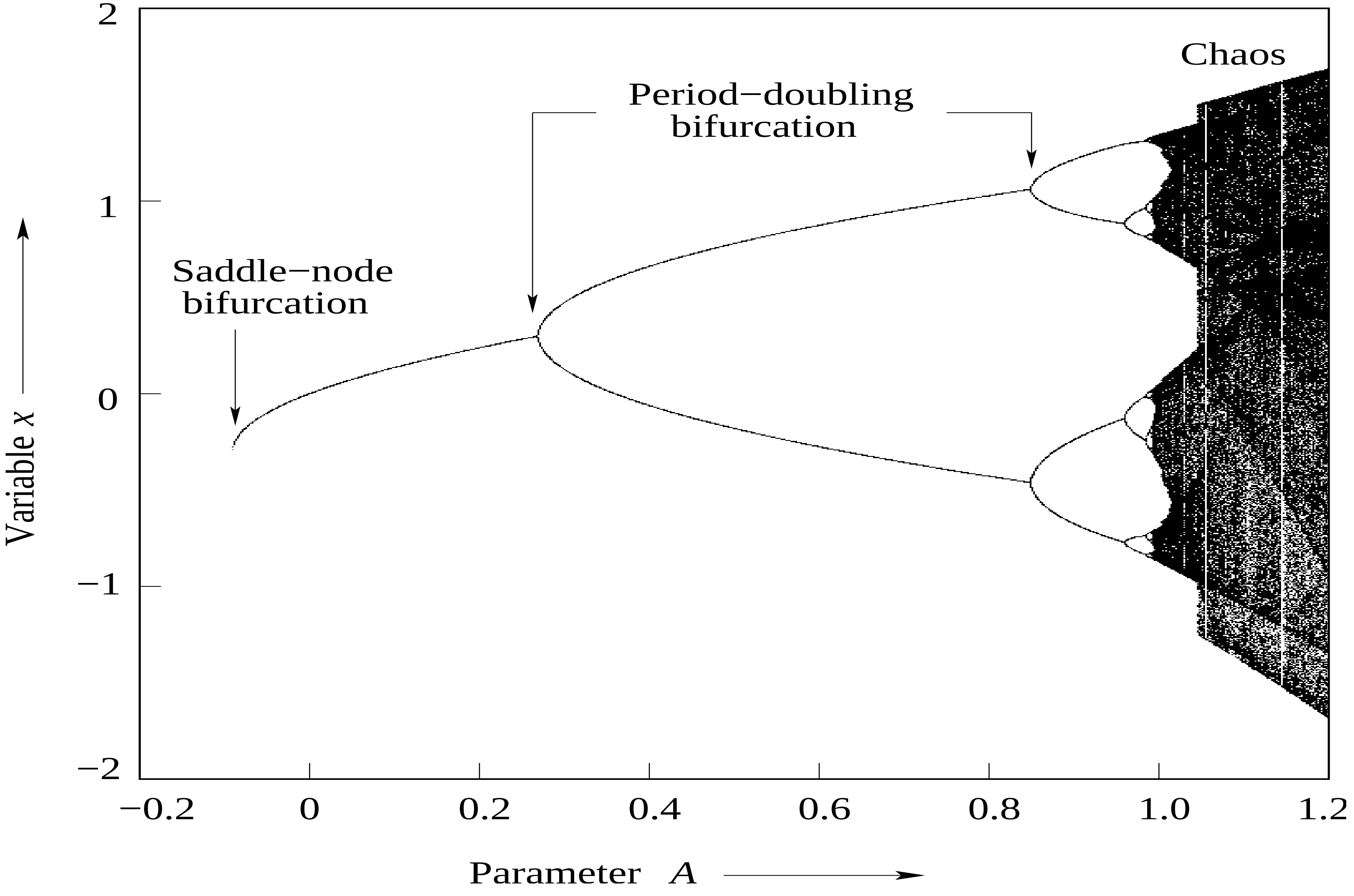
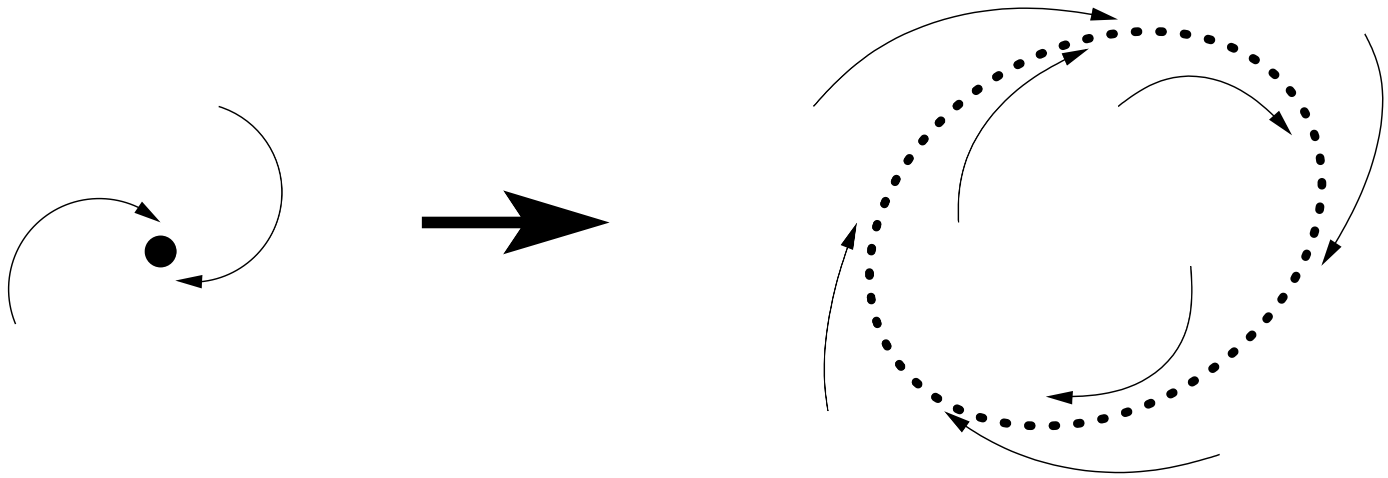


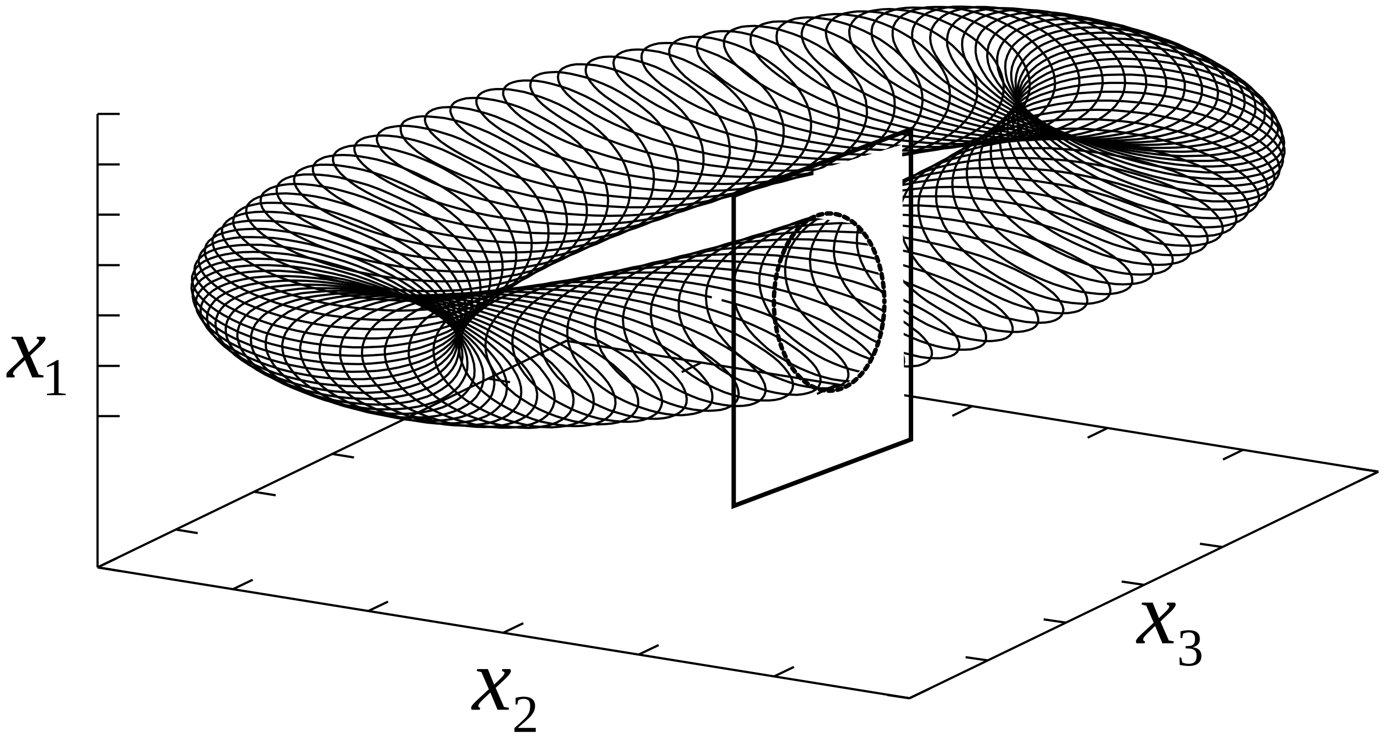
Neimark-Sacker bifurcation

If a pair of complex conjugate eigenvalues exit the unit circle, a closed loop is born in discrete time



Neimark-Sacker bifurcation

Closed loop in discrete time = torus in continuous time



Thank You