# Study of Momentum Equation in One and Two Dimensions

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• In one dimension, the equation becomes (also called Burger Equation):

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} = \nu \frac{\partial^2 u_x}{\partial x^2} + f_x$$

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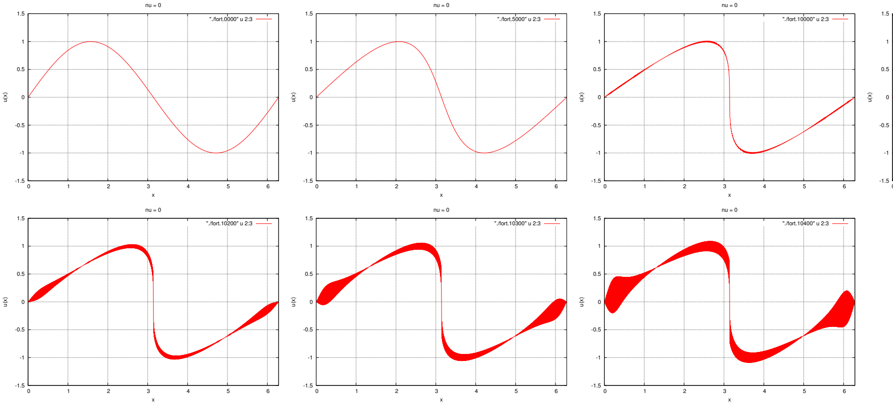


Figure: $\nu = 0.0$, Fig 1, PRE,84,016301, Fig 1, PNAS, 97, 12413

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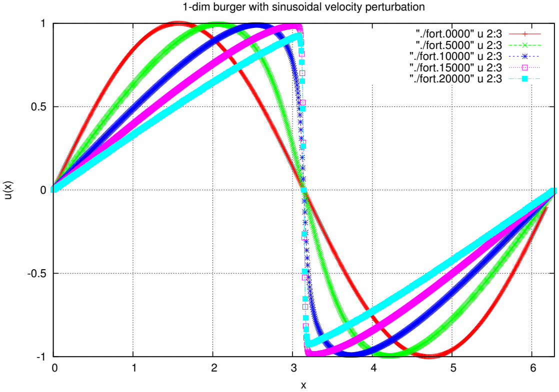


Figure: $\nu = 0.01$, Fig 11, Pramana,73,1,2009

Order-p velocity structure function

$$S_p(l) = \langle [v(x+l) - v(x)]^p \rangle$$

where $v(x)$ is the velocity at the point x and the angular brackets denote and average over the statistical steady state of the turbulent fluid.
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For $l$ in inertial range, $S_p(l) \sim l^{\xi}(p)$
for small $\Delta x$,

$$S_p(\Delta x, t) \sim C_p|\Delta x|^p + C'_p|\Delta x|$$

For $p$<1, the first term dominates and for $p$>1 the second term.

- Energy Spectra should follow a power law with slope $\frac{1}{k^2}$

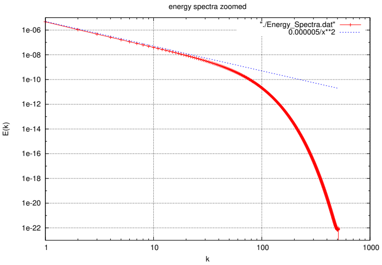


Figure: a. with zero viscosity, b. with different viscosity, Fig 12, Pramana,73,1,2009

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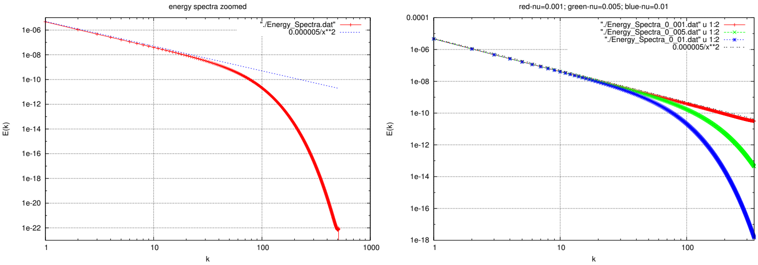


Figure: a. with zero viscosity, b. with different viscosity, Fig 12, Pramana,73,1,2009

- Viscosity acts on higher modes of wavenumber i.e. in small scales.

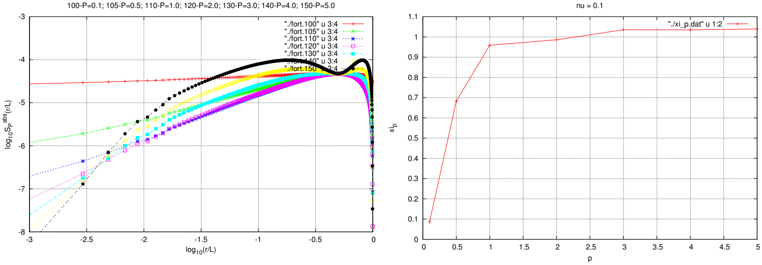


Figure: 1(b), 2(a), PRL,94,194501

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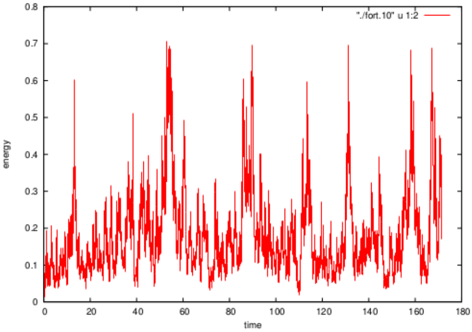

The order-p velocity structure function looks like,

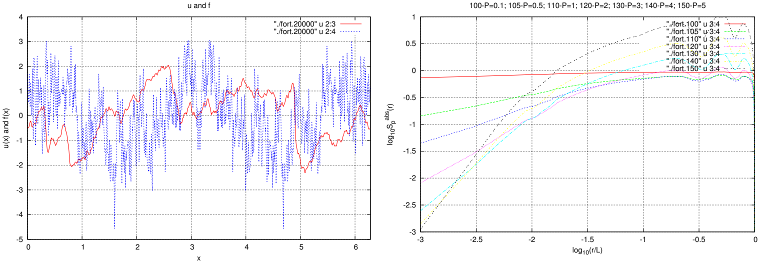


Figure: 1(a),(c),PRL,94,194501

• The vorticity($\omega$) - stream-function($\psi$) formulation of Navier-Stokes Equation is::

$$
\begin{aligned}
&\partial_t \omega - J(\psi, \omega) = \nu \nabla^2 \omega + f_\omega - \mu\omega \\
&J(\psi, \omega) = (\partial_x \psi)(\partial_y \omega) - (\partial_x \omega)(\partial_y \psi) \\
&\omega = \nabla^2 \psi, \quad \vec{u} = (\partial_y \psi, -\partial_x \psi)
\end{aligned}
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• Initial Condition : $\omega(k) = \frac{e^{-k^2}}{k^2}$
• The Energy spectra should be k-shell averaged.
• The Energy spectra should fall with slope $k^{-3}$

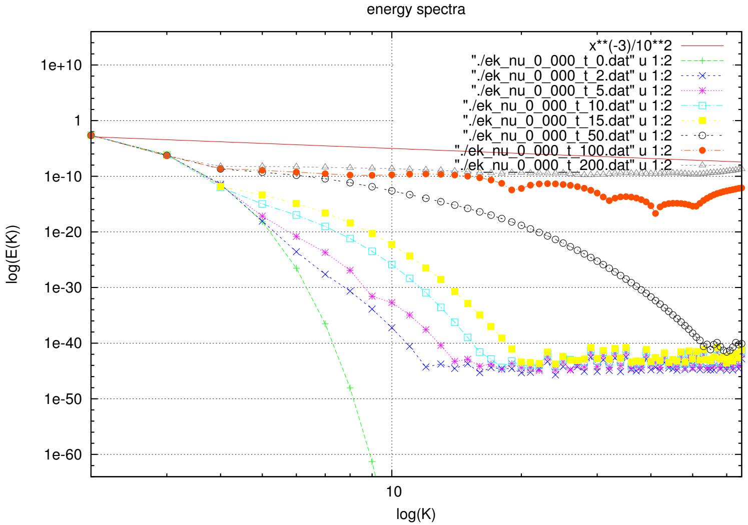


Figure: Zero Viscosity, Energy Transfer with time within modes, Fig 2, Physica D, 51,533

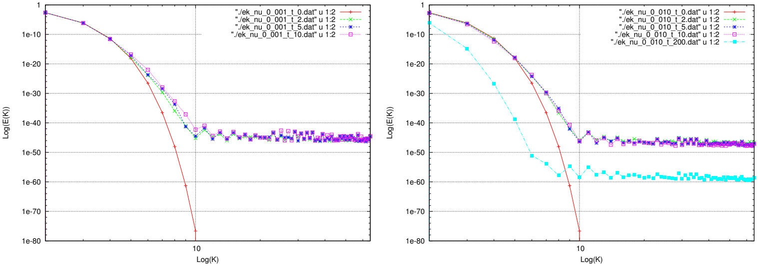


Figure: With viscosity, Energy Transfer with time within modes

# Initial Condition:: $\omega(x, y) = \sin(x)\sin(y)$; $\nu = 0.0001$

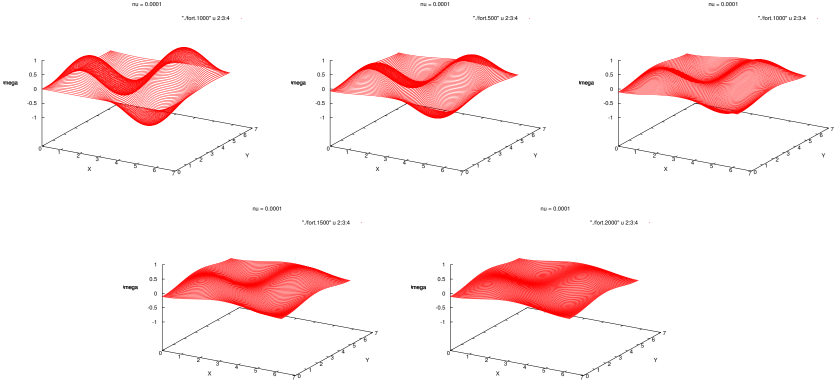


Figure: Decay of vorticity with time

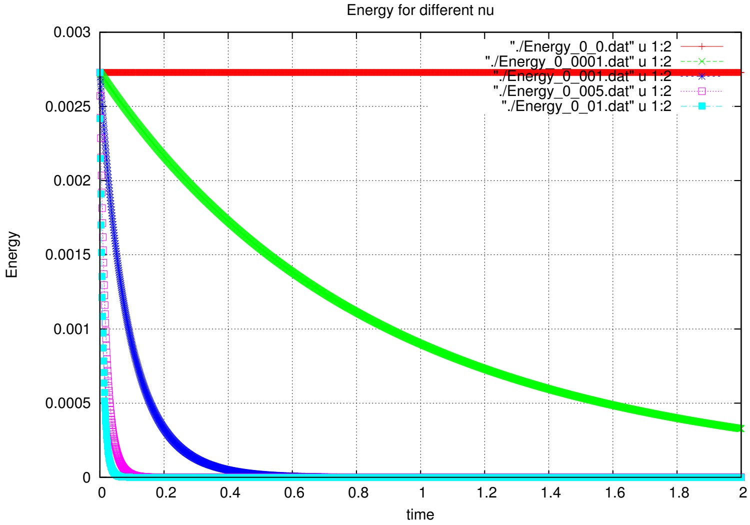


Figure: Energy vs time with different viscosity

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