Quasi geostrophic dynamics near the equator

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Outline

Coriolis force

$$\vec{f} = -2\vec{\Omega} \times \vec{u} \quad (1)$$

Resolve to components at latitude ϕ:
$(0, f\cos(\phi), f\sin(\phi))$
ϕ - latitude.

For small excursions around ϕ_0

Taylor expand f around ϕ_0
For example, the vertical component,

$$f_z = f \sin(\phi_0$$

For small deviations,

$$\begin{aligned} f_z &= 2\Omega \sin\left(\phi_0\right) + 2\Omega \cos(\phi - \phi_0)(\phi - \phi_0) + \mathcal{O}\left((\phi - \phi_0)^2\right) \\ &= 2\Omega \sin\left(\phi_0\right) + 2\Omega \cos(\phi - \phi_0)\frac{y}{R} + \mathcal{O}\left((\phi - \phi_0)^2\right) \end{aligned}$$

Approximations to f

f-plane

$$f = 2\Omega \sin(\phi_0) = f_0$$

- tangent plane at ϕ_0.
- not valid for large length scales in meridianal plane.
- not valid near the equator.

β**-plane**

$$\begin{aligned} f &= 2\Omega \sin(\phi_0) + 2\Omega \cos(\phi_0)\frac{y}{R} \\ &= f_0 + \beta y. \end{aligned}$$

Shallow Water Equation with rotation

linearlized around $(0, 0, H_0)$

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0.
\end{aligned}
$$

Geostrophic balance

- No acceleration.
- Pressure gradient = Coriolis force.

$$
\begin{aligned}
u &= -\frac{g}{f}\frac{\partial \eta}{\partial y} \\
v &= \frac{g}{f}\frac{\partial \eta}{\partial x} \\
\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0.
\end{aligned}
$$

Plane wave solutions in f-plane, $f = f_0$

- geostrophic mode,

$$\sigma = 0$$

- external inertial-gravity waves,

$$\sigma = \pm\sqrt{f_0^2 + \left(k_x^2 + k_y^2\right) gH_0}$$

Quasi-geostrophic dynamics - evolve Geostrophic mode

Assumptions

- Rotation is significant
- Motion is slow compared to the wavespeed
- height perturbations are small compared to the mean height

Steps:Non-dimensionalization.

$$\frac{\partial u}{\partial t} - f_0 v = g\frac{\partial \eta}{\partial x}.$$

$$\begin{aligned} u &\rightarrow U\tilde{u} \\ \eta &\rightarrow N\tilde{\eta} \\ x &\rightarrow L\tilde{x} \end{aligned}$$

Steps: Identify small, non-dimensional parameters

$$\frac{\partial u}{\partial t} - \frac{1}{Ro} v = \frac{-\Gamma}{Fr^2} \frac{\partial \eta}{\partial x},$$

where

$$\begin{aligned} Ro &= \frac{u}{f_0 L} \\ Fr &= \frac{u}{c} \\ \Gamma &= \frac{N}{H_0}, \end{aligned}$$

Steps:What are the strengths of the parameters?

- Rotation is significant, $Ro < 1$.
- Motion is slow compared to the wavespeed, $Fr < 1$
- height perturbations are small compared to the mean height, $\Gamma < 1$
- the easiest choice, $Ro = Fr = \Gamma = \epsilon < 1$.

Steps:Rewrite in terms of small parameter, ϵ

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{v}{\epsilon} &= \frac{-1}{\epsilon}\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + \frac{u}{\epsilon} &= \frac{-1}{\epsilon}\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + \left(1 + \frac{1}{\epsilon}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0.
\end{aligned}$$

Steps:Expand in terms of ϵ

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \dots$$
$$v = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \dots$$
$$\eta = \eta^{(0)} + \epsilon \eta^{(1)} + \epsilon^2 \eta^{(2)} + \dots$$

Balancing terms at $\mathcal{O}\left(\frac{1}{\epsilon}\right)$

$$
\begin{aligned}
v^{(0)} &= \frac{\partial eta^{(0)}}{\partial x} \\
u^{(0)} &= -\frac{\partial eta^{(0)}}{\partial y} \\
\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} &= 0.
\end{aligned}
$$

geostrophic balance

Balancing terms at $\mathcal{O}(1)$

$$\frac{D\left(\Delta\eta^{(0)} - \eta^{(0)}\right)}{Dt} = 0.$$

- evolution of the geostrophic mode, $\sigma = 0$
- inertial-gravity waves are filtered out.

In β-plane where $|f_0| > \beta$

$$\frac{Dg\left(\Delta\psi - \frac{\psi}{L_D^2} + \beta\psi\right)}{Dt} = 0.$$

a material conservation law.

Linearizing with respect to the base state of rest,

$$\frac{\partial\left(\Delta\psi - \frac{\psi}{L_D^2}\right)}{\partial t} + \beta\frac{\partial\psi}{\partial x} = 0.$$

Plane wave solution is given by

$$\psi \sim \hat{\psi}_k e^{i(kx-\sigma t)},$$

$$\sigma = \frac{-\beta k_x'}{K^2 + 1/L_D^2}.$$

Rossby wave

Quasi geostrophic theory near the equator?

- f-plane approximation is not valid

$$f_0 = 2\Omega \sin(\phi_0) = 0.$$

- β-plane approximation is possible
- Problem: $|\beta y| > |f_0|$.
- Nature of solution is different from mid-latitude β-plane where $|\beta y| < |f_0|$.

It is possible to write a relation

$$\sigma = 0$$

$$\sigma = \pm\sqrt{H_0 g(k_x^2 + k_y^2) + \beta^2 y^2}$$

What are the valid assumptions for a QG theory near the equator?