Stochastic Mode Reduction in Climate

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Outline

- Averaging and Homogenization.
- Quasi-geostrophic Equation.
- Stochastic Mode Reduction

Consider a chemical immersed in an incompressible fluid; a pollutant in the atmosphere, or a dye (such as ink) in water. Advection-diffusion equation,

$$\frac{\partial T}{\partial t} + v.\nabla T = D\nabla^2 T \tag{1}$$

$v(x,t)$: Velocity field.
$T(x,t)$: Concentration field.
D: Molecular diffusion coefficient.
Assume,

$$\begin{aligned} v(x,t) &= -b(x) \\ \nabla.b(x) &= 0 \end{aligned}$$

Taking, initial concentration $T_0 = g(\epsilon x)$, here $\epsilon <<< 1$ and rescaling,

$$\begin{aligned} x &\rightarrow \frac{x}{\epsilon} \\ t &\rightarrow \frac{t}{\epsilon^a} \end{aligned}$$

Advection-diffusion equation becomes,

$$\frac{1}{\epsilon^{2-a}} \frac{\partial T}{\partial t} - \frac{1}{\epsilon} b^{\epsilon}.\nabla T = D \nabla^2 T$$

In case of $D = 0$ and $a = 1$, using the method of characteristics, one gets,

$$\frac{dx^{\epsilon}}{dt} = b(\frac{x^{\epsilon}}{\epsilon})$$

$b(x)$ is periodic and x/ϵ varies rapidly on the scale of the period. Average the equation to eliminate these rapid oscillations. Eliminating fast scales in a time-dependent transport PDE is intimately related to averaging for ODEs. In case of $D > 0$ and $a = 2$, We get,

$$\frac{dx^\epsilon}{dt} = \frac{1}{\epsilon} b(\frac{x^\epsilon}{\epsilon}) + \sqrt{2D}\frac{dW}{dt}$$

here $W(t)$ is a standard Brownian motion. After eliminating the rapidly varying quantity x/ϵ, for periodic, divergence free and zero mean velocity field, one gets,

$$\frac{dX}{dt} = \sqrt{2\kappa}\frac{dW}{dt}$$

$x^\epsilon(t)$ converges to $X(t)$ in the limit of $\epsilon \to 0$

Averaging Versus Homogenization

The unifying principle underlying these techniques is the formal perturbation expansions for linear operator equation of the form,

$$\frac{\partial u^\epsilon}{\partial t} = L^\epsilon u^\epsilon$$

The operator L^ϵ has the form,

$$\begin{aligned} L^\epsilon &= \frac{1}{\epsilon} L_0 + L_1 \\ L^\epsilon &= \frac{1}{\epsilon^2} L_0 + \frac{1}{\epsilon} L_1 + L_2 \end{aligned}$$

We will refer to the first case as averaging, or first-order perturbation theory. The second case will be referred to as homogenization or second-order perturbation theory.