Systematic Strategies for Stochastic Mode Reduction

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- Averaging and Homogenisation
- Quasi-Geostrophic Equations
- Stochastic Mode Reduction

The Set up

- wide variety of problems have the common feature of multiscale possession.
- Time Scales
  - Hourly- Small scale convection
  - Monthly- Intra seasonal variability of Tropics
  - Annual- El Nino-Southern oscillation
- Previous Work
  - Reduced Linear Stochastic Models
  - Approximations made on resolved mode
  - All couplings with unresolved modes dropped and replaced by ad hoc stochastic terms of linear langevin type.

Stochastic Mode Reduction Strategy

- DOF have been split into resolved and unresolved modes

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y) + h(y)$$

where $h(y)$ a non linear function of y

- Modify equation of motion of unresolved modes by representing Non-Linear self interaction terms between unresolved modes by stochastic terms.

$$h(y) \approx -\frac{\Gamma}{\epsilon} y + \frac{\sigma}{\sqrt{\epsilon}} \dot{W}(t)$$

Stochastic Mode Reduction Strategy

- Justified in coarse grained modelling on longer time scales as in climate.

$$t \to \frac{t}{\epsilon} \qquad \frac{dx}{dt} = \frac{f(x,y)}{\epsilon}$$

$$\frac{dy}{dt} = \frac{g(x,y)}{\epsilon} - \frac{\Gamma}{\epsilon^2} y + \frac{\sigma}{\epsilon} \dot{W}(t)$$

- Equation of motion of unresolved mode(s) eliminated.

Averaging and Homogenisation

- averaging and homogenization-simplification of wide range of problems possessing multiple scales
- System of Linear Equations

$$\frac{du^\epsilon}{dt} = \mathcal{L}^\epsilon u^\epsilon$$

  - averaging :

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1 \qquad\qquad \mathcal{L}_i \in \mathbb{R}^{d\times d}$$

$\mathcal{N}(\mathcal{L}_0)$ 1-Dimensional spanned by ϕ ,$\mathcal{N}(\mathcal{L}_0^T)$spanned by ψ

  - seeking solutions of the form :

$$u^\epsilon = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2)$$

- $$\mathcal{L}_0 u_0 = 0 \qquad\qquad \mathcal{O}(\frac{1}{\epsilon})$$
  $$u_0 = \alpha\phi \qquad\qquad \alpha = \alpha(t) \in \mathbb{R}$$
  $$\mathcal{L}_0 u_1 = \frac{du_0}{dt} - \mathcal{L}_1 u_0 \qquad\qquad \mathcal{O}(1)$$
- Fredholm Alternative : solution for u_1 exists if and only if :
  $$\langle \psi, \frac{du_0}{dt} - \mathcal{L}_1 u_0 \rangle = 0$$
  $$\frac{d\alpha}{dt} = \frac{\langle \psi, \mathcal{L}_1 \phi \rangle}{\langle \psi, \phi \rangle} \alpha$$
  which has non trivial solution provided
  $$\langle \psi, \mathcal{L}_1 \phi \rangle \neq 0$$
  $$\lim_{\epsilon \ll 1} u^{\epsilon}(t) \approx u_0(t) = \alpha(t)$$

- homogenisation if

$$\langle \psi, \mathcal{L}_1 \phi \rangle = 0$$

for non trivial dynamics,

$$t \to \frac{t}{\epsilon}$$

$$\mathcal{L}^{\epsilon} = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1$$

without loss of generality, we can study,

$$\mathcal{L}^{\epsilon} = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1 + \mathcal{L}_2 \qquad \mathcal{L}_i \in \mathbb{R}^{d \times d}$$

$$u_1 = -\alpha\eta \qquad \mathcal{L}_0 \eta = \mathcal{L}_1 \phi$$

$$\frac{d\alpha}{dt} = \frac{\langle \psi, \mathcal{L}_2 \phi - \mathcal{L}_1 \eta \rangle}{\langle \psi, \phi \rangle}\alpha$$

Quasi-Geostrophic Equations-An Overview

- Shallow Water Equations

$$\frac{\partial \vec{u}}{\partial t} + (\vec{v}.\nabla)\vec{u} + \vec{f} \times \vec{u} = -g\nabla_z h$$

$$\frac{\partial h}{\partial t} + H\nabla.\vec{u} = 0$$

- Rossby No.

$$R_0 = \frac{U}{fL}$$

- $R_0 = 0 \Rightarrow$ Geostrophic Balance :

$$\vec{f} \times \vec{u} = -g\nabla_z h$$

- Quasi-Geostrophy R_0 small

$$\vec{f} \times \vec{u} \approx -g\nabla_z h$$

- QG equation

$$\frac{D}{Dt}\left(\nabla^2\psi + \beta y - \frac{f_0^2}{gH}\psi\right) = 0$$

$$u = U + u^{'}$$

$$v = v^{'}$$

$$u = \frac{\partial\psi}{\partial y}$$

$$v = -\frac{\partial\psi}{\partial x}$$

$$\psi = U\psi + \psi^{'}$$

Stochastic modelling for topographic stress

- Ideal Barotropic QG equations with large scale zonal mean flow U
- $2\pi \times 2\pi$ periodic domain

$$\frac{\partial q}{\partial t} + \nabla^{\perp}.\nabla q + U\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} = 0$$

$$q = \nabla^2\psi + h(x,y)$$

$$\frac{dU}{dt} = \frac{1}{4\pi^2}\int h\frac{\partial \psi}{\partial x}dxdy$$

- Equations expanded in Fourier modes:

$$f(x, y) = \sum_{k_x, k_y} f(\vec{k}) \exp(i(k_x x + k_y y))$$

  where $f(x, y)$ is a 2π periodic function
- truncation done at $\mid k \mid^2 \leqslant \Lambda$

Continued...

- Truncated equations conserve energy and enstrophy

$$E_\Lambda = \frac{1}{2}U^2 + \frac{1}{2}\int(\nabla\psi_\Lambda)^2 d\vec{x}$$

$$\frac{dE_\Lambda}{dt} = U_\Lambda\frac{dU_\Lambda}{dt} + \frac{d}{dt}\left[\int[\nabla.(\psi_\Lambda\nabla\psi_\Lambda) - \psi_\Lambda\nabla^2\psi_\Lambda]d\vec{x}\right]$$

$$= U_\Lambda\int h_\Lambda\frac{\partial\psi_\Lambda}{\partial x}d\vec{x} + \beta\int\psi_\Lambda\frac{\partial\psi_\Lambda}{\partial x}d\vec{x} + U_\Lambda\int\psi_\Lambda\frac{\partial q_\Lambda}{\partial x}d\vec{x} + \int\psi_\Lambda(\nabla^\perp\psi_\Lambda.\nabla q_\Lambda)d$$

$$= U_\Lambda\int h_\Lambda\frac{\partial\psi_\Lambda}{\partial x}d\vec{x} + U_\Lambda\int\psi_\Lambda\frac{\partial h_\Lambda}{\partial x}d\vec{x}$$

$$= 0$$

- $$\epsilon_\Lambda = \beta U + \frac{1}{2}\int q_\Lambda^2 d\vec{x}$$

- The fully a priori strategy assumes that the climate PDF is Gaussian with following mean and variance for fixed α, μ.

$$\overline{U} = -\frac{\beta}{\mu}$$

$$var(U) = \frac{1}{\alpha\mu}$$

$$\overline{\psi_k} = -\frac{h_k}{\mu + \mid \vec{k} \mid^2}$$

$$var(\psi_k) = \frac{1}{\alpha \mid \vec{k} \mid^2 (\mu + \mid \vec{k} \mid^2)}$$

- Nondimensional variables for perturbations about the climate mean are given by

$$U^{new} = \frac{(U - \overline{U})}{\sqrt{var(U)}}$$

$$\psi_k^{new} = \frac{(\psi_k - \overline{\psi_k})}{\sqrt{var(\psi_k)}}$$

- The truncated equations in the new variables of each Fourier mode and U are given by :

$$\frac{\partial \psi_k^{new}}{\partial t} = -\frac{ik_x}{\sqrt{\alpha\mu}} U^{new}\psi_k^{new} + ik_x H_k^{'} U^{new} - i\Omega_k^{'}\psi_k^{new} + \sum_{l_x,l_y} B_{\vec{k}\vec{l}}\psi_{\vec{k}-\vec{l}}\psi_{\vec{l}} + \sum_{l_x,l_y} L_{\vec{k}\vec{l}}\psi_{\vec{l}}$$

$$\frac{dU}{dt} = 2Im\sum_{k_x,k_y} k_x H_k^{'}\psi_k$$

where,

$$H_k^{'} = h_k\sqrt{\frac{\mu}{|\vec{k}|^2(\mu + |\vec{k}|^2)}}$$

$$\Omega_k^{'} = \frac{k_x\beta}{|\vec{k}|^2} - \overline{U}k_x$$

- Invoking the approximation of stochastic consistency:

$$\sum_{\vec{k}\vec{l}} \psi_{\vec{l}}\psi_{\vec{k}-\vec{l}} + \sum_{\vec{l}} L_{\vec{k}\vec{l}}\psi_{\vec{l}} \approx -\gamma_k \psi_k^{new} + \sigma_k W_k$$

$$\frac{d\psi_k^{new}}{dt} = ik_x H_k^{'} U^{new} - \gamma_k(U^{new})\psi_k + \sigma_k W_k$$

where

$$\gamma_k(U^{new}) = \gamma_k + i\Omega_k^{'} + i\frac{k_x}{\sqrt{\alpha\mu}}U^{new}$$

- ψ_k^{new} can be eliminated provided $\gamma_k(U)$ is large enough compared to other terms.

$$\frac{dU}{dt} = -\gamma(U)U + \sqrt{2\gamma(U)}\xi(t)$$

where

$$\gamma(U) = 2\sum_{\vec{k}} \frac{k_x^2 \mid H_k^{'} \mid^2 \gamma_k}{\gamma_k^2 + (\Omega_k^{'} + k_x U(\alpha\mu)^{0.5})^2}$$

$$\xi(t) = \sqrt{\frac{2}{\gamma(U)}} Im \left(\sum_k k_x H_k^{'} \sigma_k \int_{-\infty}^{t} \exp(-\gamma_k(U)(t - t^{'})) \dot{W}_k dt^{'} \right)$$

using

$$\langle \dot{W}_k(s) \dot{W}_{k'}(s^{'}) \rangle = \delta(s - s^{'})\delta_{kk'},$$

$\xi(t)$ is delta correlated in time which implies

$$\langle \xi(t)\xi(t^{'}) \rangle \approx \delta(t - t^{'})$$

Under the assumption

$$\frac{1}{(\alpha\mu)} \mid k_x U \mid^2 \ll \gamma_k^2 + (\Omega_k^{'})^2$$

a standard predicted linear stochastic model for U emerges :

$$\frac{dU}{dt} = -2\gamma(U)U + \sqrt{2\gamma(U)}\dot{W}_k$$

where

$$\gamma(U) = \sum_k \frac{k_x^2 \mid H_k \mid^2 \gamma_k}{\gamma_k^2 + (\Omega_k^{'})^2}$$

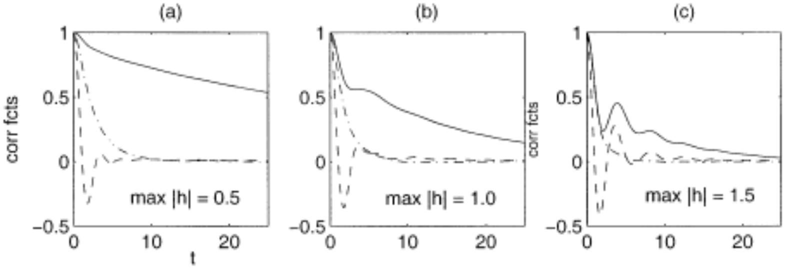

Figure: comparison of correlation function of the mean U (solid line), $Re\psi_{1,0}$ (dashed line), and $Re\psi_{0,1}$(dot-dashed line) for different values of H

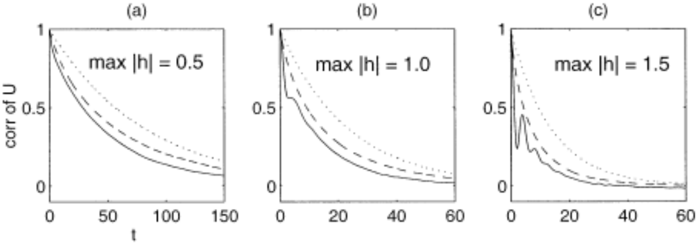

Figure: comparison of the correlation function of U DNS (solid line); nonlinear reduced stochastic model(dashed line); corresponding linear reduced stochastic model(dotted line)