On the connection between wave resonance, shear instability and oscillator synchronization

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Acknowledgements

- Neil Balmforth (UBC)
- Jeff Carpenter (HZG)
- Colm Caulfield (DAMTP, Cambridge)
- Peter Haynes (DAMTP, Cambridge)
- Eyal Heifetz (Tel Aviv)
- Greg Lawrence (UBC)
- Michael McIntyre (DAMTP, Cambridge)
- Ted Tedford (UBC)
- Firdaus Udwadia (USC)

Research has been funded by:

1. 4 year fellowship (UBC)
2. Earl R. Peterson Memorial Fellowship (UBC)
3. David Crighton Fellowship (DAMTP, Cambridge)
4. Initiation grant (IIT Kanpur)

J. Fluid Mech. (2014), *vol.* 755, *pp.* 336–364.
doi:10.1017/jfm.2014.374

A wave interaction approach to studying non-modal homogeneous and stratified shear instabilities

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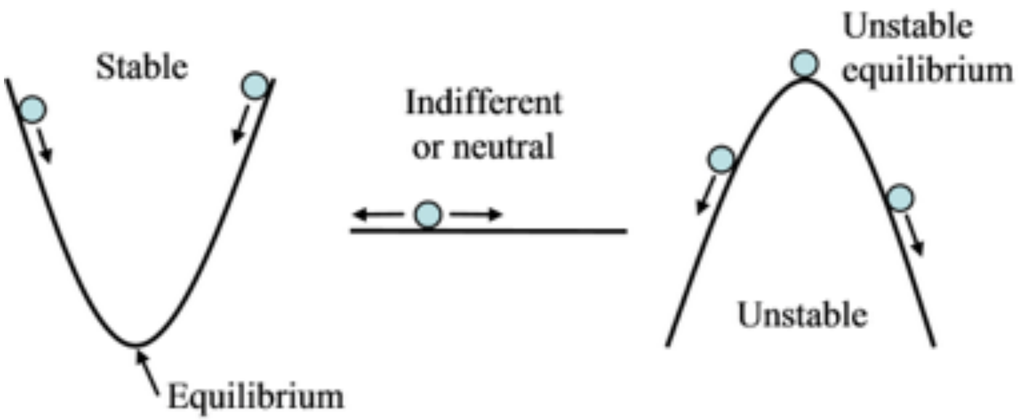
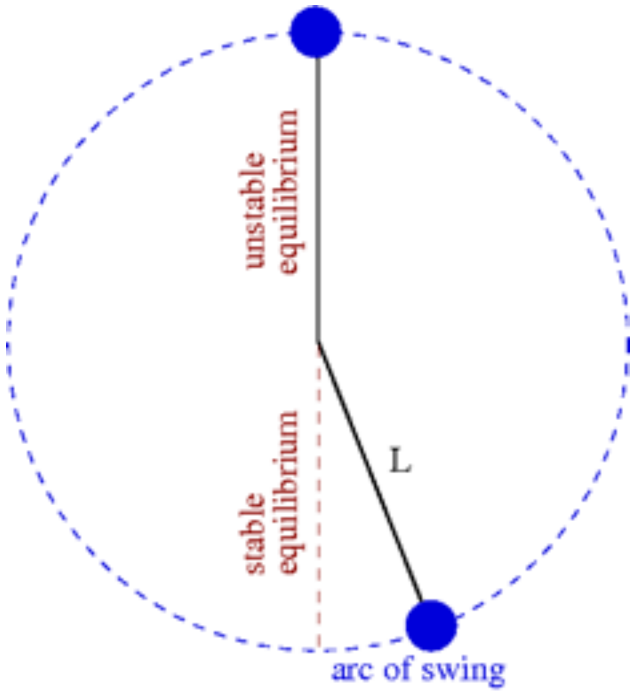
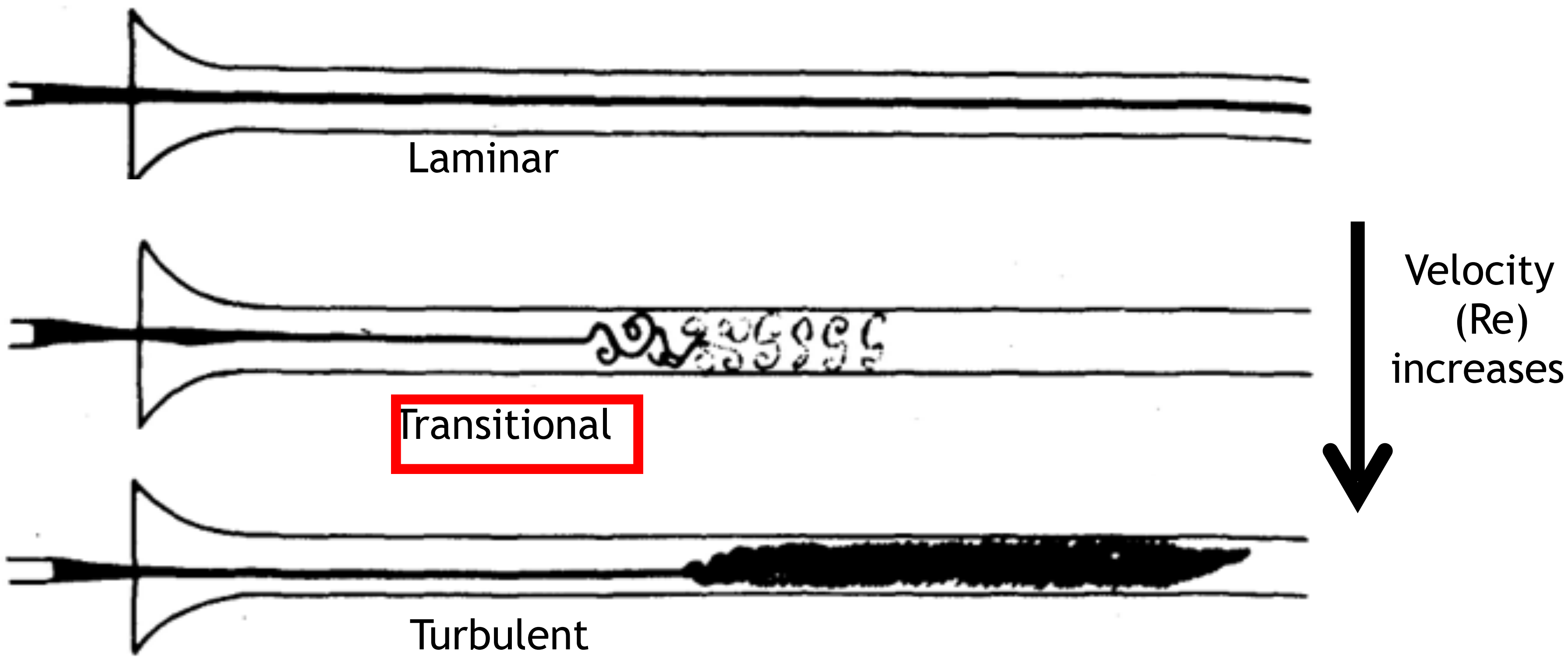
(Received 4 March 2013; revised 1 May 2014; accepted 25 June 2014)

Stability & Instability

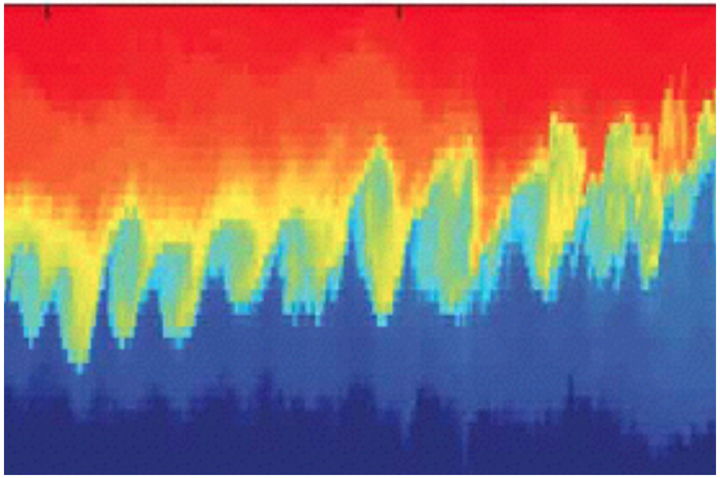


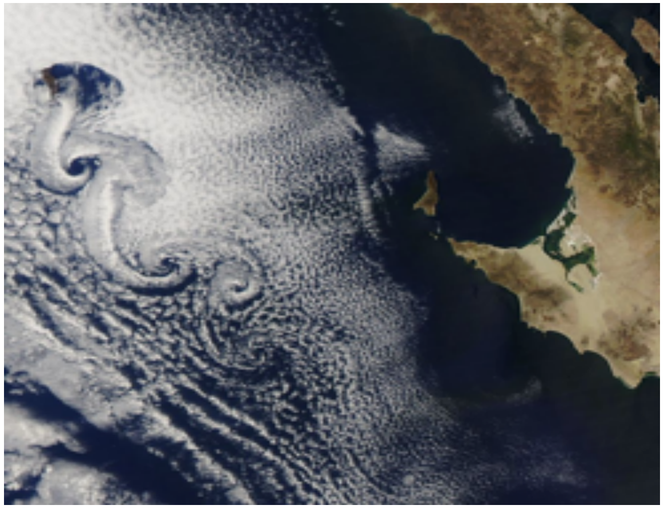


Instability (transitional flow) - the precursor of turbulence



Shear instability in geophysical problems





Traditional approach to stratified shear instability theory

Governing Navier Stokes equations (simplified)

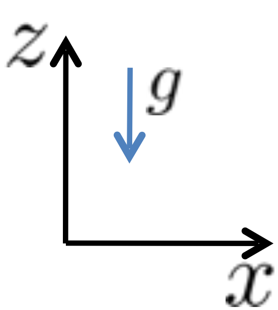
Fluid is 2D, incompressible, inviscid, Boussinesq.

Continuity:
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

X-momentum:
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \cancel{\nu \Delta u}$$

Z-momentum:
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} - g\frac{\rho}{\rho_0} + \cancel{\nu \Delta w}$$

Non-diffusive:
$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0$$



Mean flow parallel to X axis

$$u = U + u', \quad w = w', \quad \rho = \bar{\rho} + \rho', \quad P = \bar{P} + P'$$

Furthermore, assume hydrostatic background:

$$\partial \bar{P}/\partial z = -\bar{\rho} g$$

Governing Navier Stokes equations (simplified)

Define streamfunction:

$$u = \frac{\partial \psi}{\partial z} \;\; ; \;\; w = -\frac{\partial \psi}{\partial x}$$

Combine continuity and momentum equations to obtain vorticity ($\triangle \psi$) equation:

$$\frac{\partial \triangle \psi}{\partial t} + u \frac{\partial \triangle \psi}{\partial x} + w \frac{\partial \triangle \psi}{\partial z} = J \frac{\partial \rho'}{\partial x}$$

Bulk Richardson No. $J = \frac{\delta \rho \, g l}{\rho_0 \, (\delta U)^2}$

Normal-mode ansatz

$$\psi = \Psi(z) + \psi'$$

small

$$\psi'(x, z, t) = \varphi(z) e^{i\alpha(x - ct)}$$

$$\rho'(x, z, t) = \hat{\rho}(z) e^{i\alpha(x - ct)}$$

Governing Navier Stokes equations (simplified)

$$u = \frac{\partial \psi}{\partial z} \ ; \ \ w = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \triangle \psi}{\partial t} + u\frac{\partial \triangle \psi}{\partial x} + w\frac{\partial \triangle \psi}{\partial z} = J\frac{\partial \rho'}{\partial x}$$

$$\psi = \Psi(z) + \psi'$$

Normal-mode ansatz (exp growth, const speed)

Exponentially grows in time if positive

$$\psi^{'}(x,z,t) = \varphi(z)e^{i\alpha(x-ct)} = \varphi(z)e^{c_i t}e^{i\alpha(x-c_r t)}$$

$$\rho^{'}(x,z,t) = \hat{\rho}(z)e^{i\alpha(x-ct)} = \hat{\rho}(z)e^{c_i t}e^{i\alpha(x-c_r t)}$$

α is the real wavenumber (inverse of wavelength)

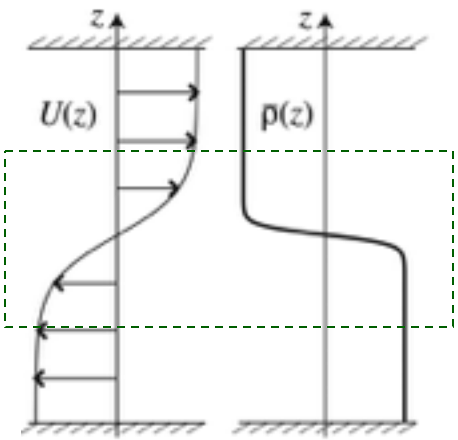
$c = c_r + ic_i$ is the complex phase-speed

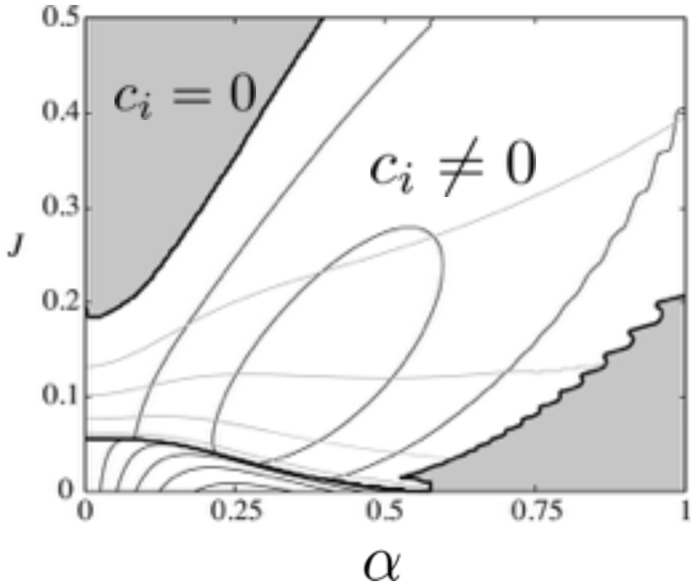
Taylor-Goldstein Equation (an Eigenvalue problem)

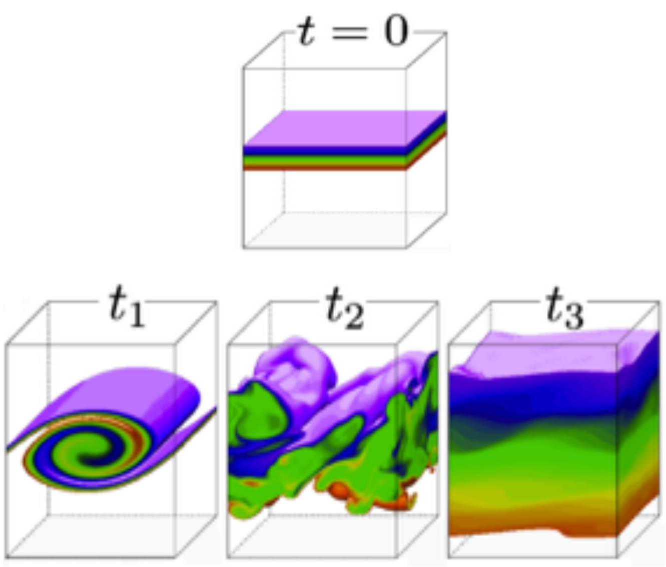
$$\frac{d^2\varphi}{dz^2} + \left[-J\frac{\frac{d\bar{\rho}}{dz}}{(U-c)^2} - \frac{\frac{d^2U}{dz^2}}{(U-c)} - \alpha^2 \right]\varphi = 0$$

Miles-Howard criterion

$$Ri(z) = -J\frac{(d\bar{\rho}/dz)}{(dU/dz)^2} < \frac{1}{4}$$

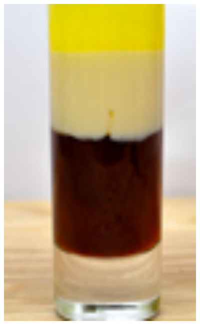


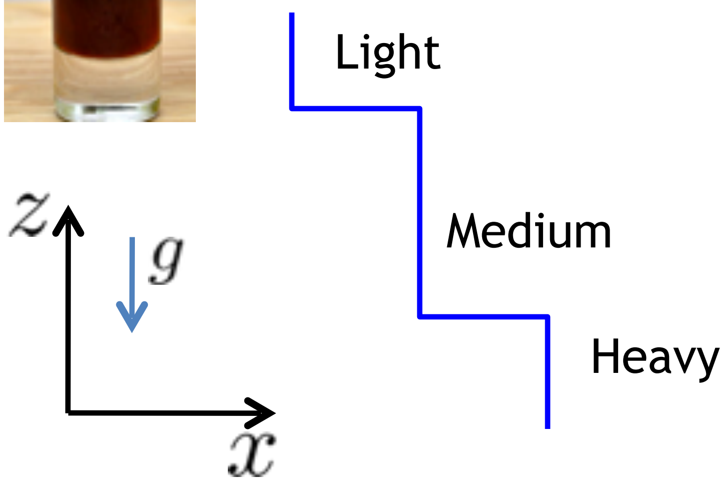




Why instability? The classic Taylor problem

Stable density ($\bar{\rho}(z)$) stratification

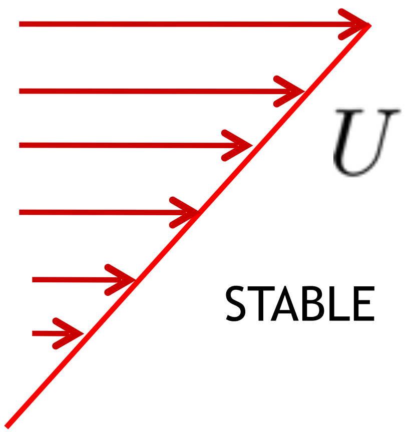




Why instability? The classic Taylor problem

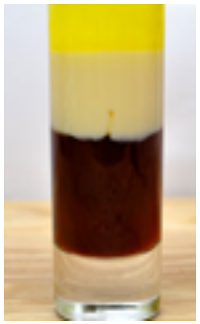
Constant Velocity Shear

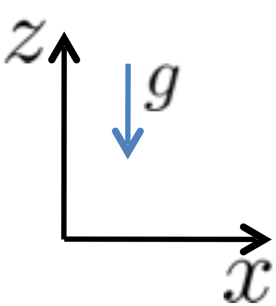
$$\frac{dU}{dz} = const$$

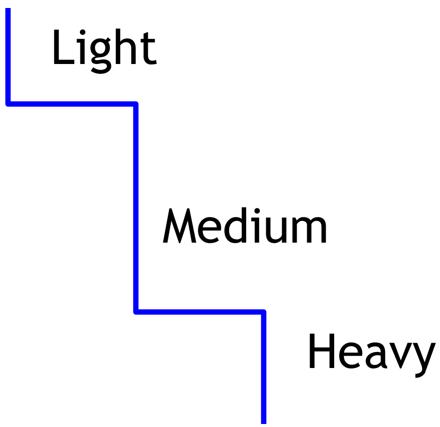


Why instability? The classic Taylor problem

Stable density ($\bar{\rho}(z)$) stratification







Constant Velocity Shear

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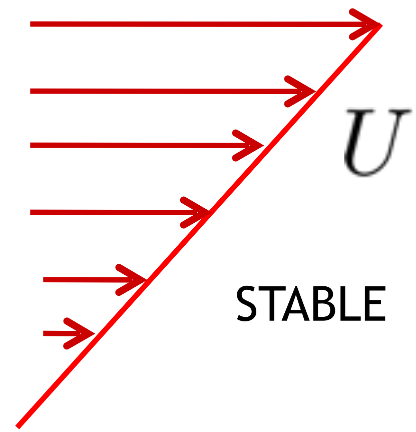
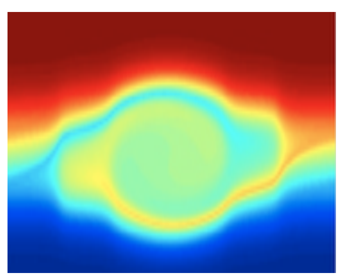


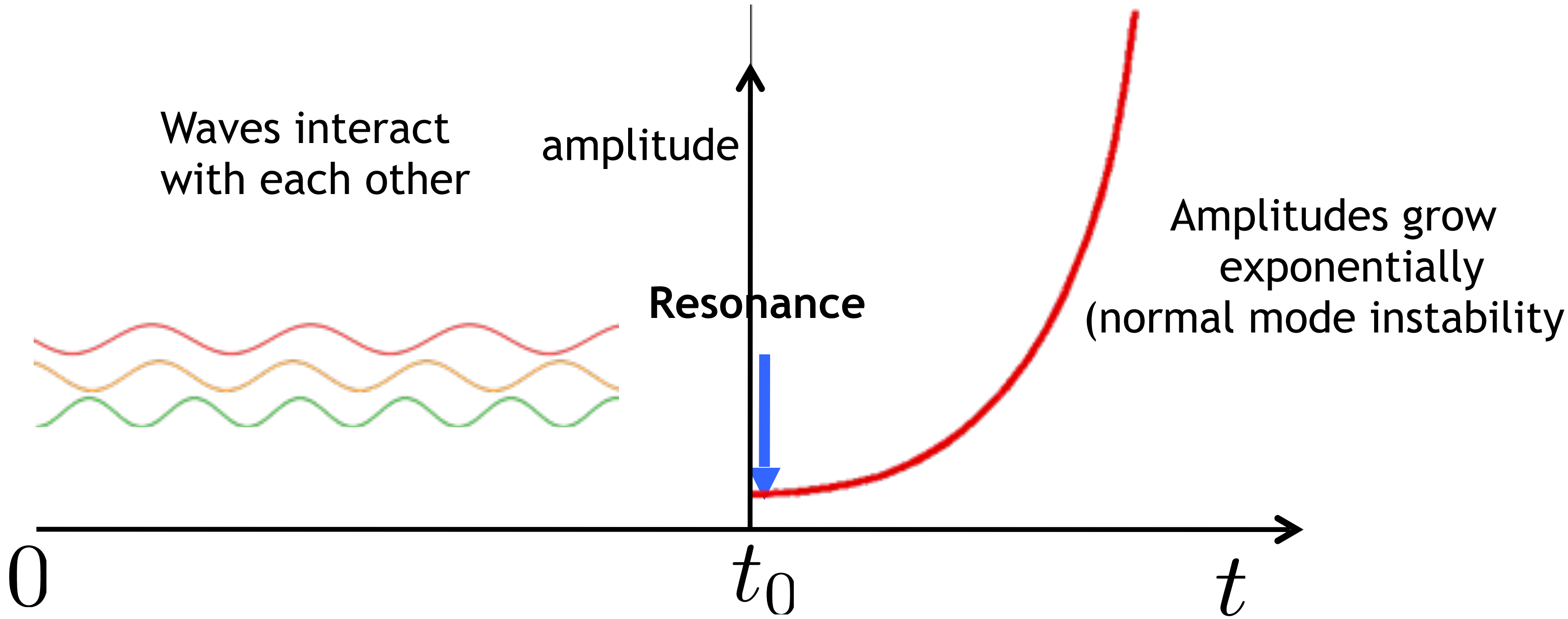
INSTABILITY !!!



'It is a simple matter to work out the equations which must be satisfied by waves in such a fluid, but the interpretation of the solutions of these equations is a matter of considerable difficulty' (Sir G.I. Taylor 1931).

Non-traditional approach to stratified shear instability theory

An intuitive explanation for shear instabilities - Resonant wave interaction



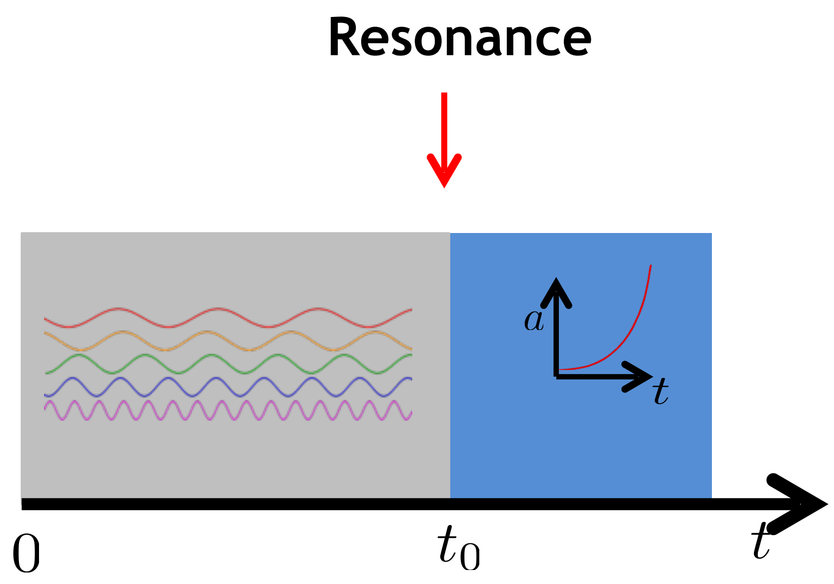
Resonance: Two waves "PHASE LOCK" and then undergo "MUTUAL GROWTH"

Goal

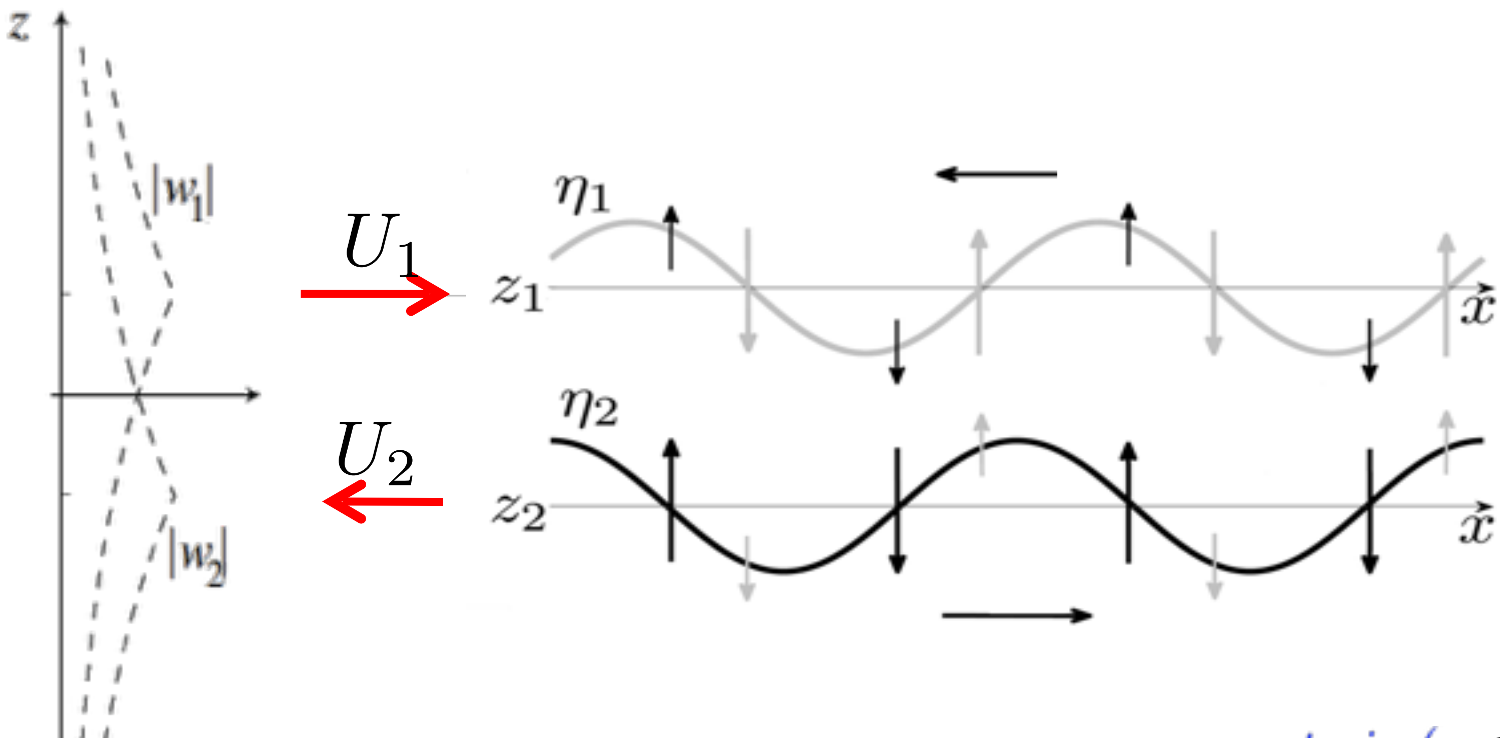
- ❑ Previous authors e.g. Holmboe (1962), Caulfield (1994) studied the normal-mode stability problem. Such problem demands the interacting waves to start from **resonant condition** (blue area).

- ❑ Our goal is to start from an **arbitrary initial condition** and see how and under what circumstances the interacting waves resonate (grey area).

- ❑ Moreover, to keep things as general as possible, we don't specify the wave type (i.e. can be vorticity wave, gravity wave, etc.)



The proposed kinematic model:
WAVE INTERACTION THEORY (WIT)



$$\frac{\partial \eta_1}{\partial t} + U_1 \frac{\partial \eta_1}{\partial x} = w_1 + e^{-\alpha|z_1 - z_2|} w_2$$

$$\frac{\partial \eta_2}{\partial t} + U_2 \frac{\partial \eta_2}{\partial x} = e^{-\alpha|z_1 - z_2|} w_1 + w_2$$

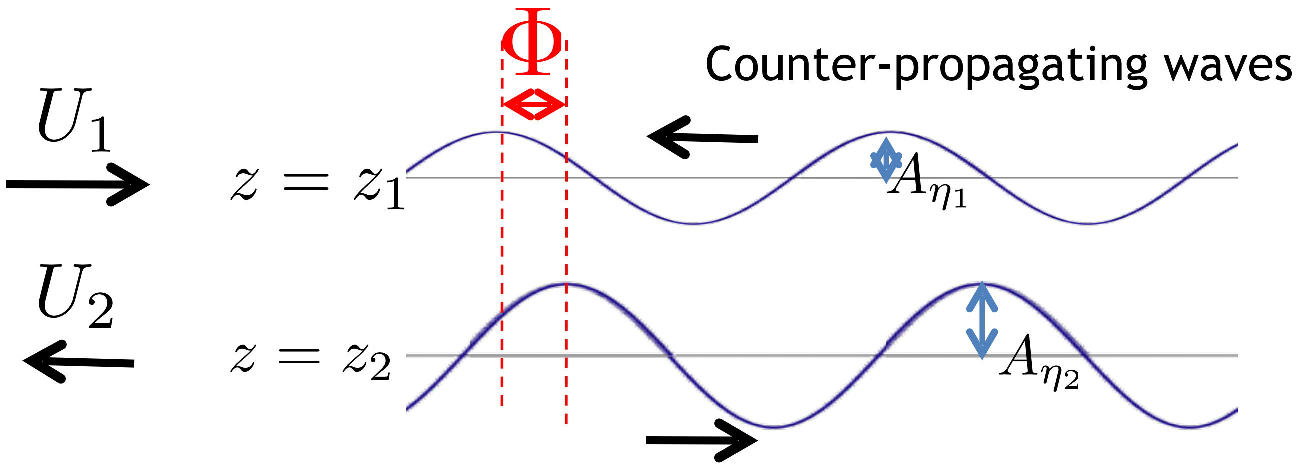
~~$e^{c_i t} e^{i\alpha(x - c_r t)}$~~

Substitute

$$\eta_i = A_{\eta_i}(t) e^{i[\alpha x + \phi_{\eta_i}(t)]}$$

$$w_i = A_{w_i}(t) e^{i[\alpha x + \phi_{w_i}(t)]}$$

Wave Interaction Theory (WIT)



$$R \equiv A_{\eta_1}/A_{\eta_2}$$

$$\Phi \equiv \phi_{\eta_2} - \phi_{\eta_1}$$

$$\omega_i \equiv A_{w_i}/A_{\eta_i}$$

$$\gamma_i \equiv \frac{1}{A_{\eta_i}}\frac{dA_{\eta_i}}{dt}$$

$$c_i \equiv -\frac{1}{\alpha}\frac{d\phi_{\eta_i}}{dt}$$

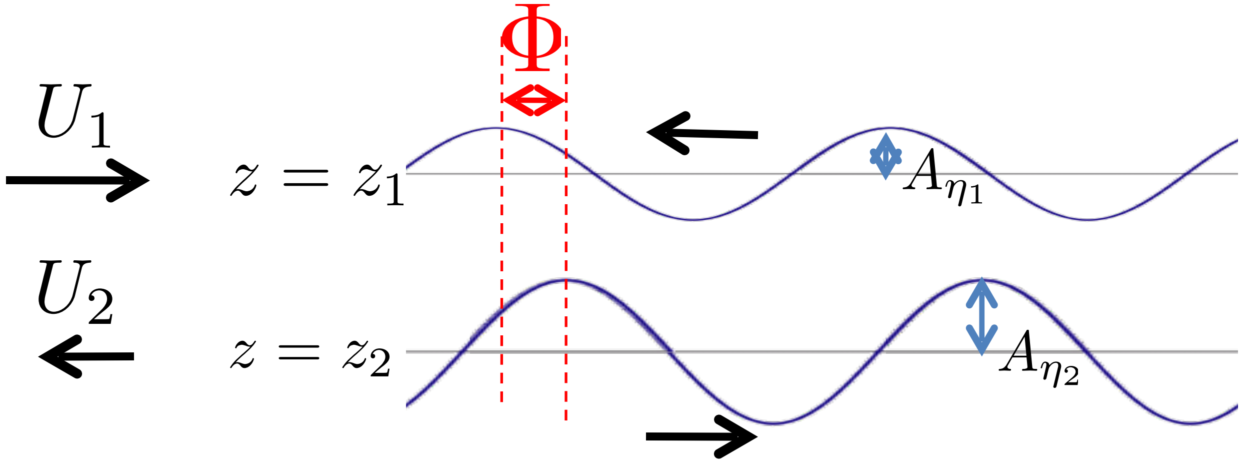
Growth rate of wave 1: $$\gamma_1 = \frac{\omega_2}{R} e^{-\alpha|z_1 - z_2|} \sin\Phi$$

Phase speed of wave 1: $$c_1 = U_1 - \frac{1}{\alpha}\left[\omega_1 - \frac{\omega_2}{R} e^{-\alpha|z_1 - z_2|} \cos\Phi\right]$$

Growth rate of wave 2: $$\gamma_2 = R\omega_1 e^{-\alpha|z_1 - z_2|} \sin\Phi$$

Phase speed of wave 2: $$c_2 = U_2 + \frac{1}{\alpha}\left[\omega_2 - R\omega_1 e^{-\alpha|z_1 - z_2|} \cos\Phi\right]$$

Introducing Dynamical Systems Perspective



$$R \equiv A_{\eta_1} / A_{\eta_2}$$
$$\Phi \equiv \phi_{\eta_2} - \phi_{\eta_1}$$
$$\omega_i \equiv A_{w_i} / A_{\eta_i}$$
$$\gamma_i \equiv \frac{1}{A_{\eta_i}} \frac{dA_{\eta_i}}{dt}$$
$$c_i \equiv -\frac{1}{\alpha} \frac{d\phi_{\eta_i}}{dt}$$

$$\frac{dR}{dt} = R\left(\gamma_1 - \gamma_2\right) = \left(\omega_2 - R^2 \omega_1\right) e^{-\alpha|z_1 - z_2|} \sin \Phi$$

$$\frac{d\Phi}{dt} = \alpha\left(c_1 - c_2\right) = \alpha\left(U_1 - U_2\right) - \left[\omega_1 + \omega_2 - \left(R\omega_1 + \frac{\omega_2}{R}\right) e^{-\alpha|z_1 - z_2|} \cos \Phi\right]$$

The Dynamical Systems Perspective

Equilibrium points: $(R,\ \Phi) = (R_{nm}, \theta_{nm})$ and $(R_{nm}, -\theta_{nm})$

$$dR/dt = 0$$

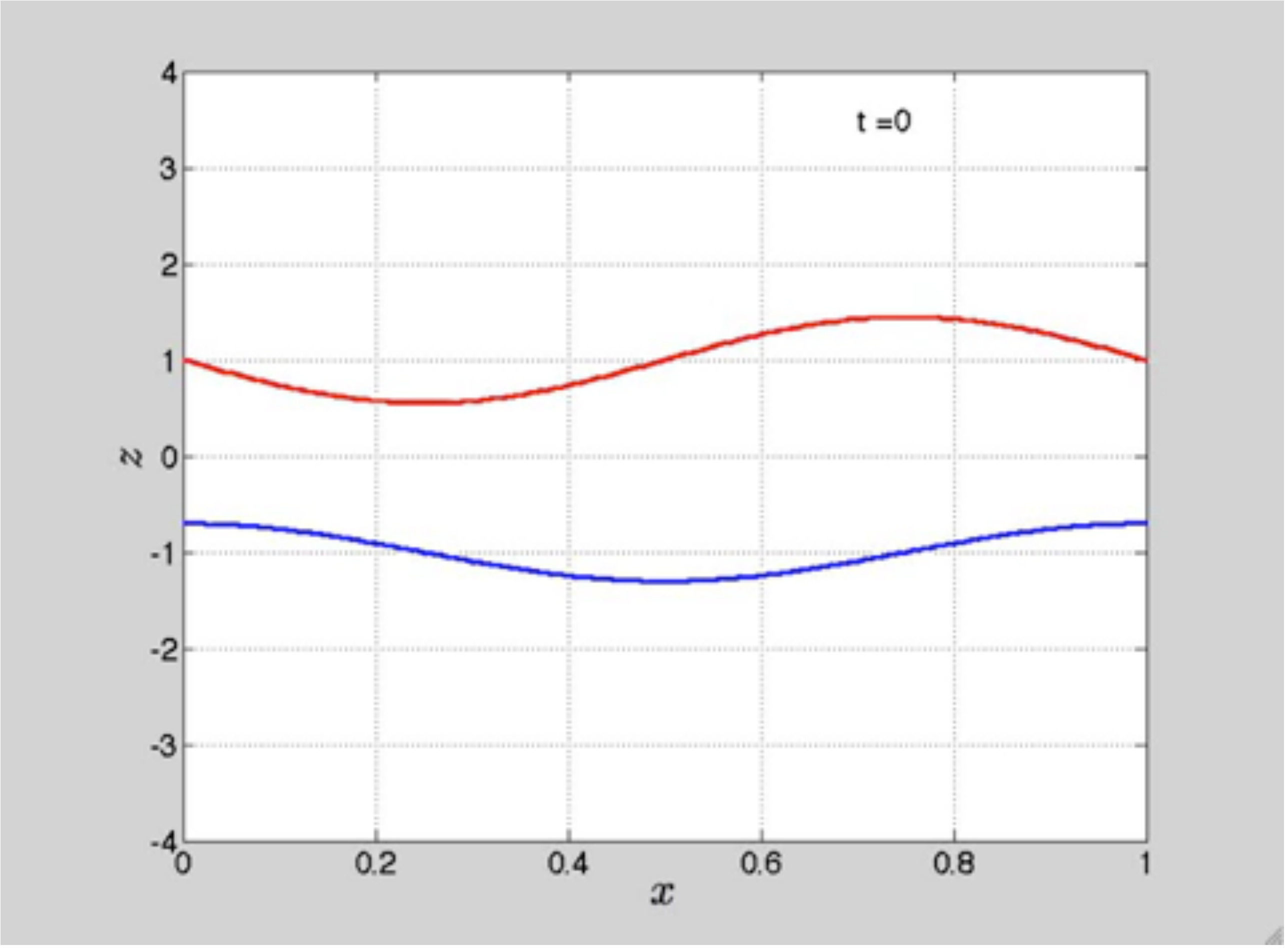
$$d\Phi/dt = 0$$

$$R_{nm} = \sqrt{\frac{\omega_2}{\omega_1}}$$

$$\theta_{nm} = \pm\cos^{-1}\left[\left\{\frac{\omega_1 + \omega_2 - \alpha\left(U_1 - U_2\right)}{2\sqrt{\omega_1\omega_2}}\right\} e^{\alpha|z_1 - z_2|}\right]$$

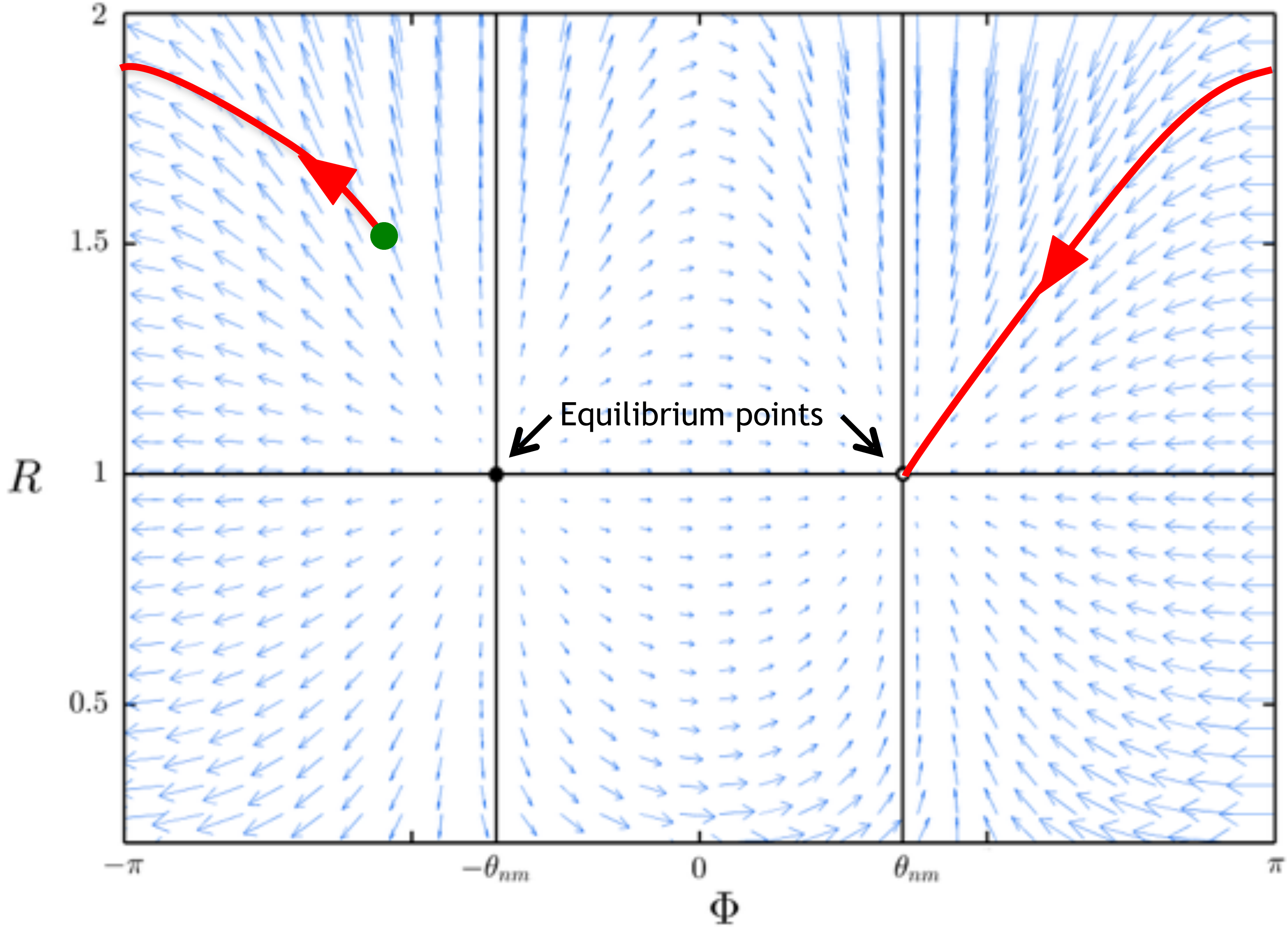
The condition for equilibrium points to exist:

$$\left|\left\{\frac{\omega_1 + \omega_2 - \alpha\left(U_1 - U_2\right)}{2\sqrt{\omega_1\omega_2}}\right\} e^{\alpha|z_1 - z_2|}\right| \leq 1$$

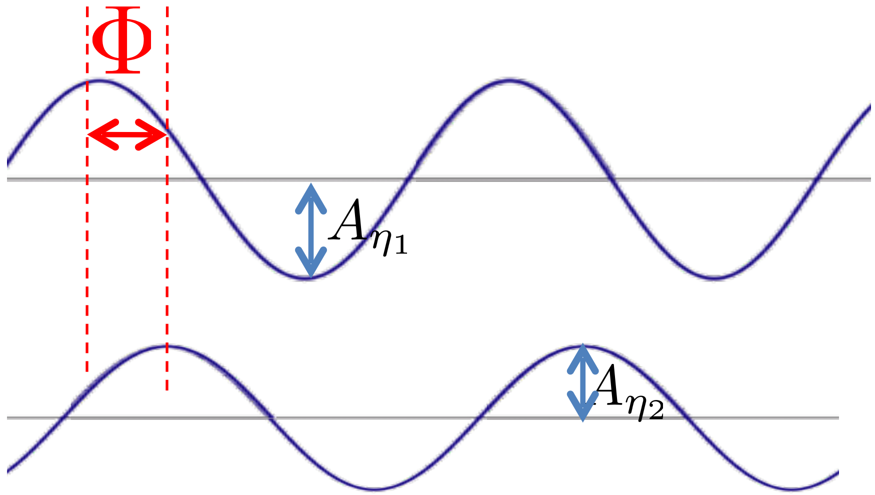


t =0
z
x

Phase Portrait

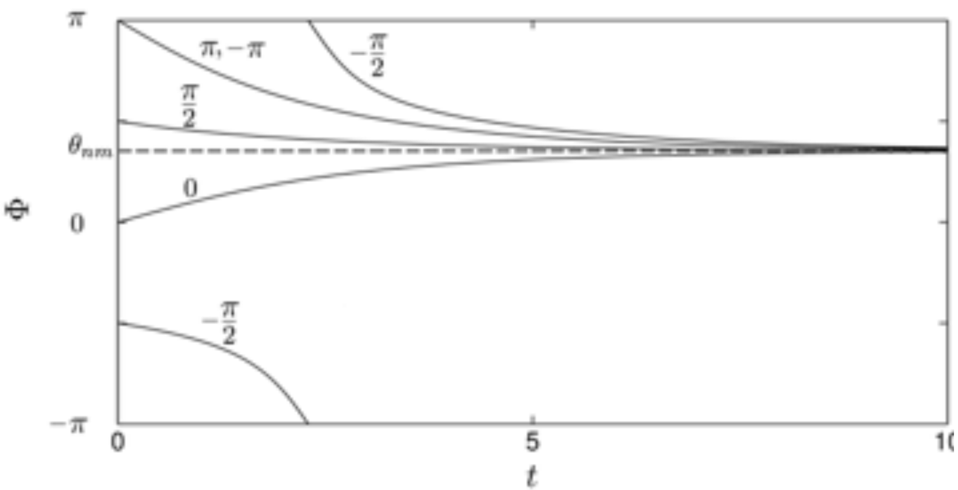


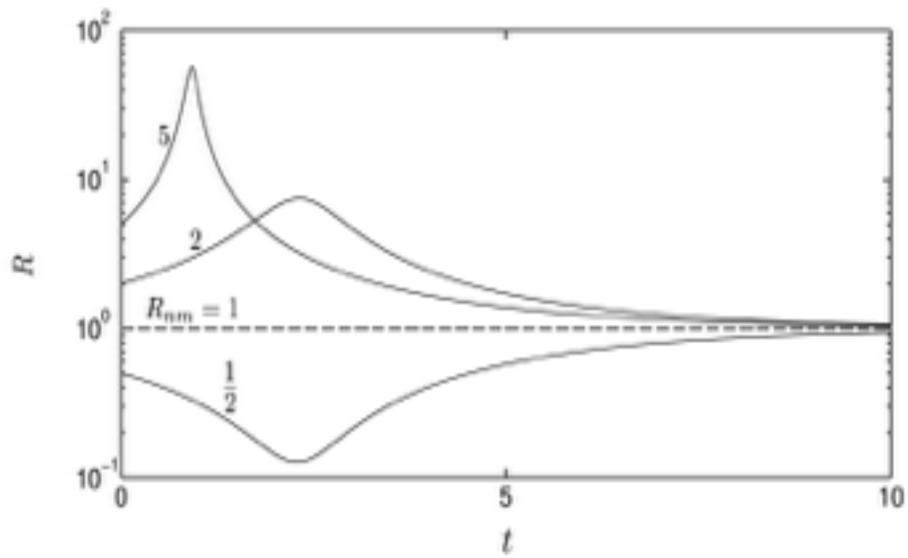
Locking happens when equilibrium points exist



$$R \equiv A_{\eta_1}/A_{\eta_2}$$

$$\Phi \equiv \phi_{\eta_2} - \phi_{\eta_1}$$





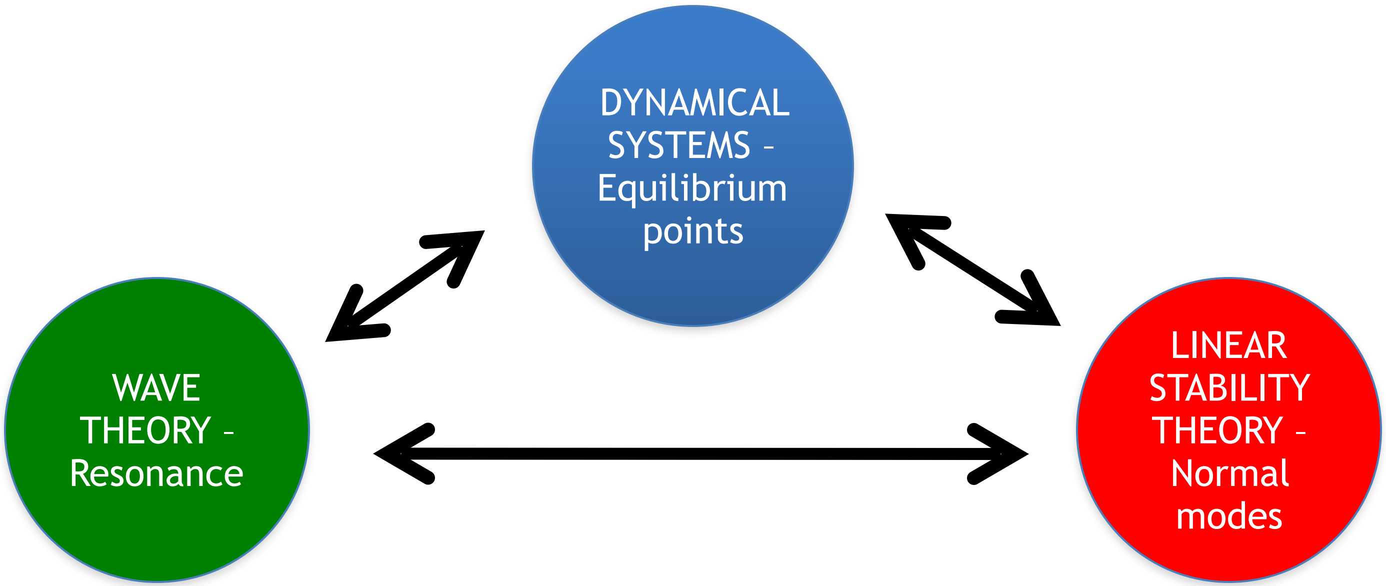
What is the meaning of equilibrium points?

$$\frac{dR}{dt} = R\left(\gamma_1 - \gamma_2\right)$$

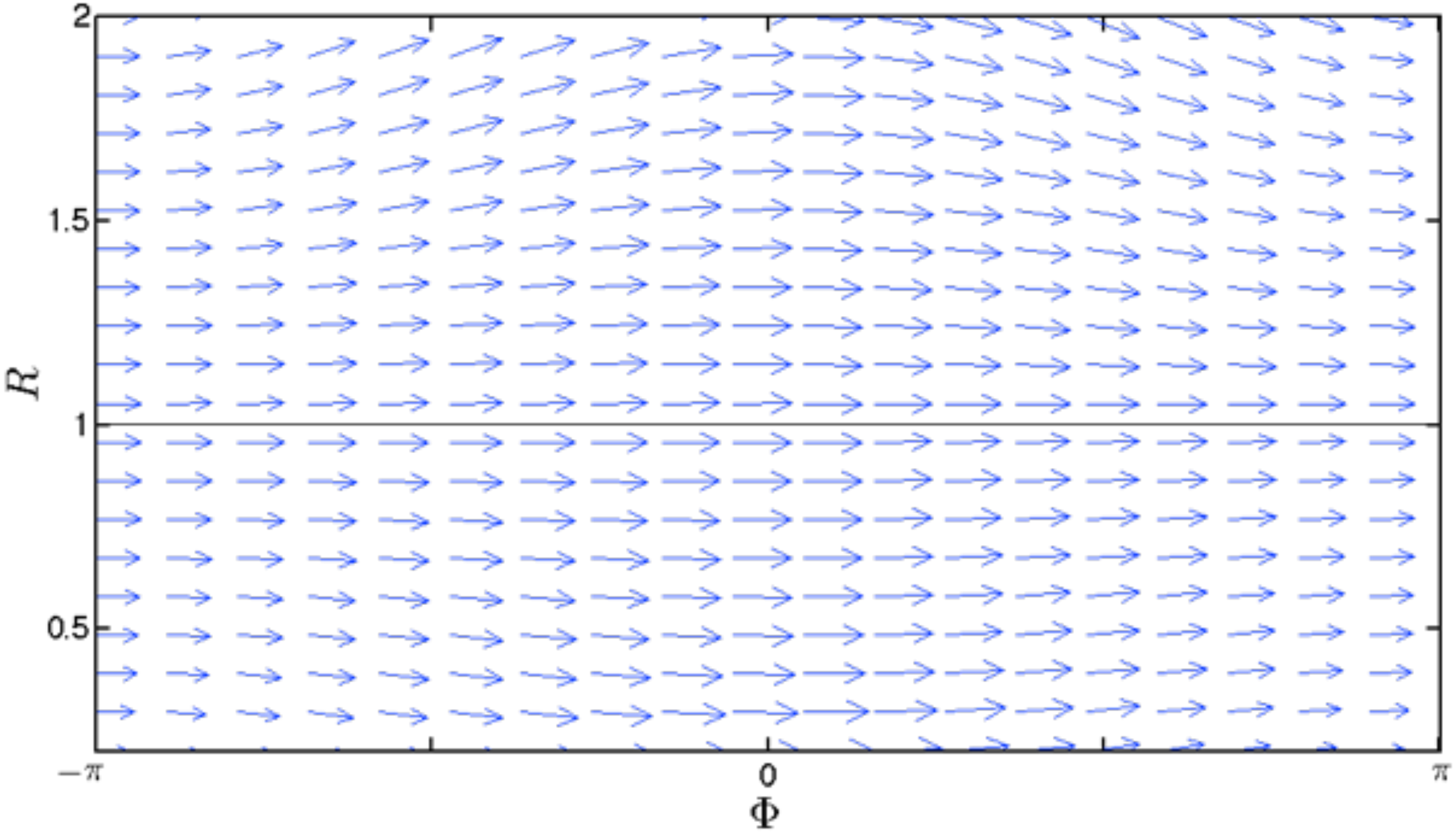
The waves start to grow at equal rate (and exponentially)

$$\frac{d\Phi}{dt} = \alpha\left(c_1 - c_2\right)$$

The waves are phase-locked



Phase Portrait (no equilibrium points)



Necessary and sufficient condition for normal-mode type instabilities to exist in idealized shear layers:

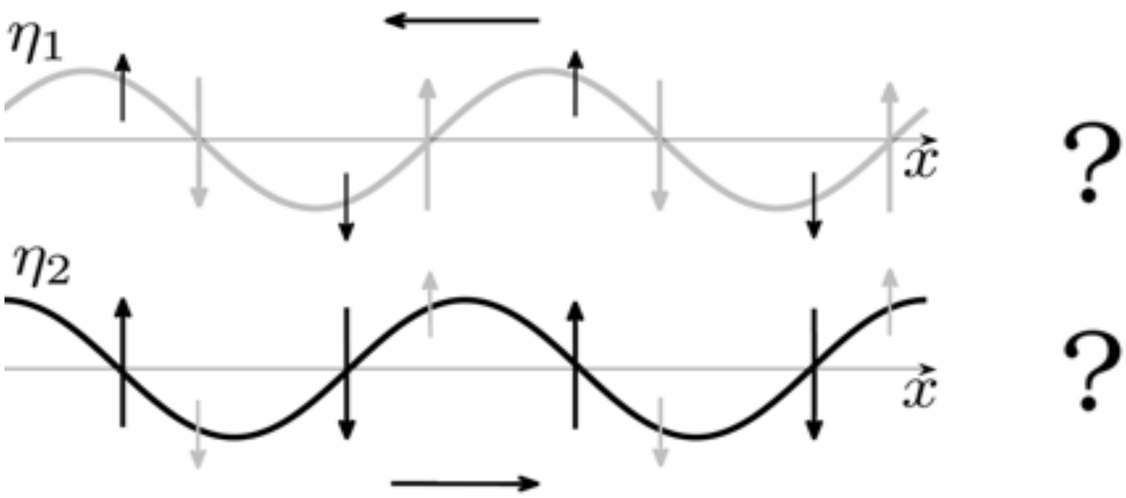
$$\left| \left\{ \frac{\omega_1 + \omega_2 - \alpha\left(U_1 - U_2\right)}{2\sqrt{\omega_1 \omega_2}} \right\} e^{\alpha |z_1 - z_2|} \right| \leq 1$$

Recall Miles-Howard criterion (**necessary condition** for stratified shear instability)

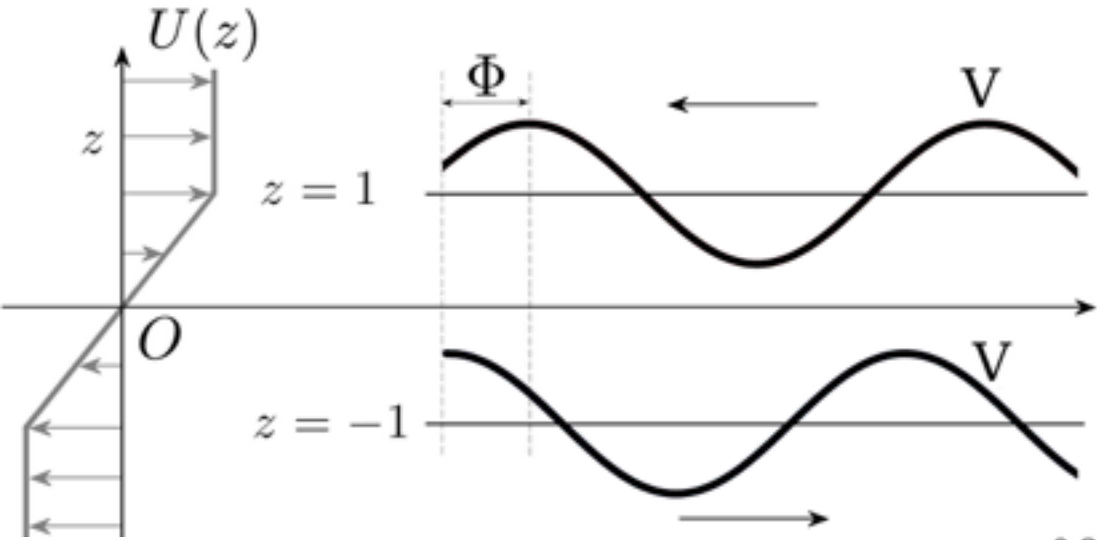
$$Ri(z) = -J \frac{(d\bar{\rho}/dz)}{(dU/dz)^2} < \frac{1}{4}$$

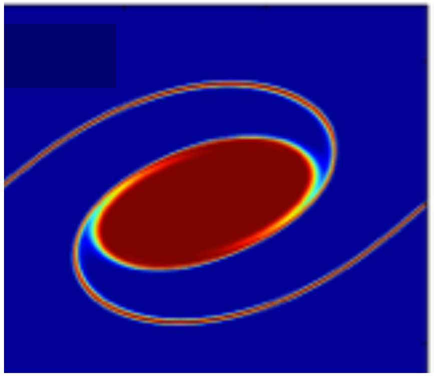
DIFFERENT TYPES OF STRATIFIED SHEAR INSTABILITIES

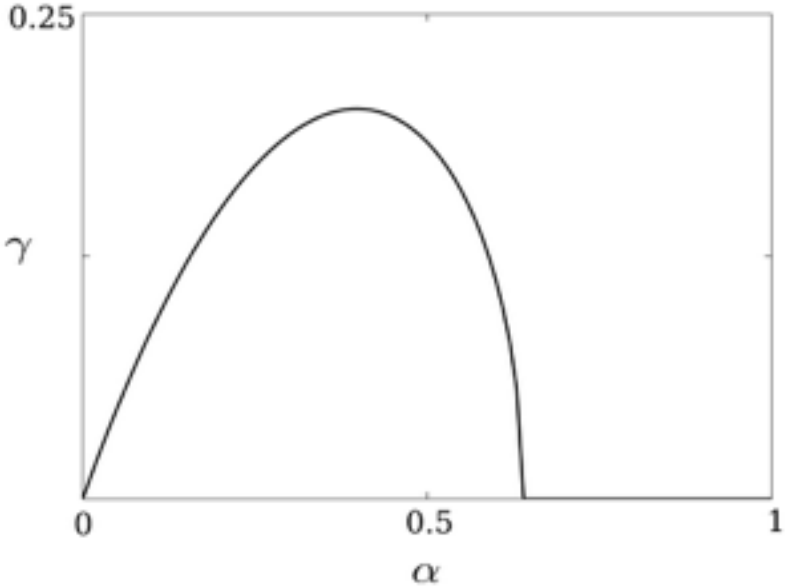
Did I mention the wave types?



CASE 1: Kelvin-Helmholtz / Rayleigh Instability

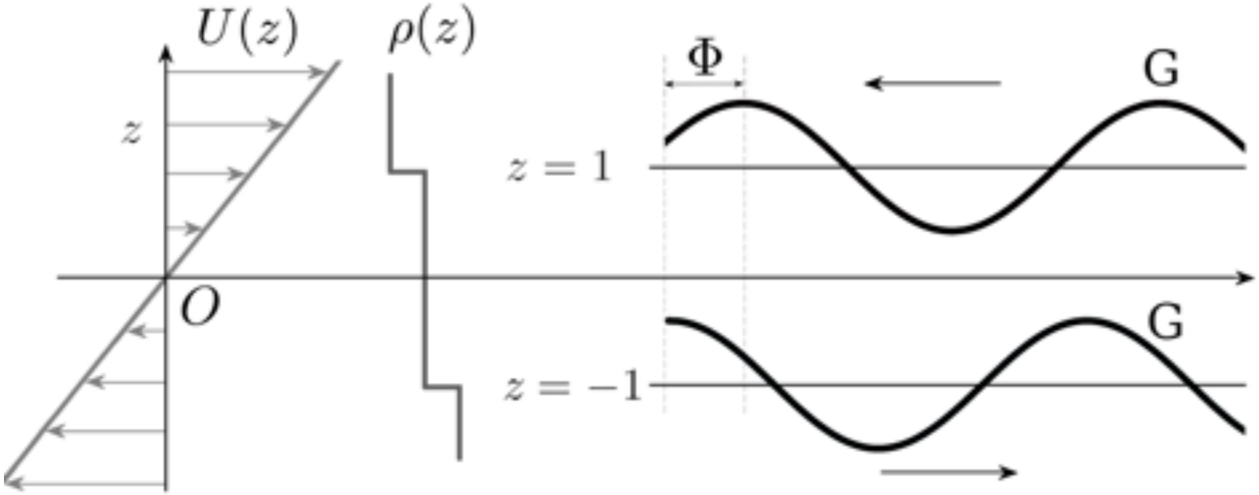


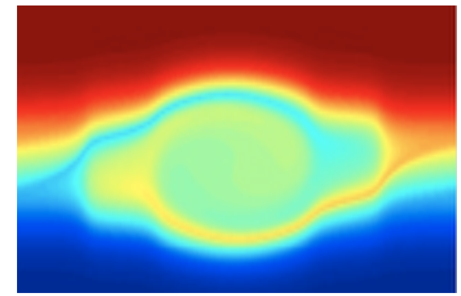




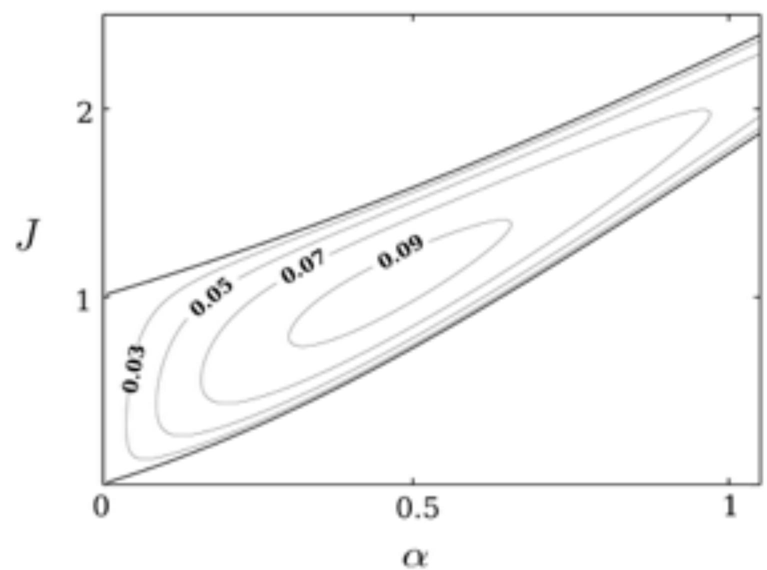
$$0 \leq \alpha \leq 0.64$$

CASE 2: Taylor / Taylor-Caulfield Instability

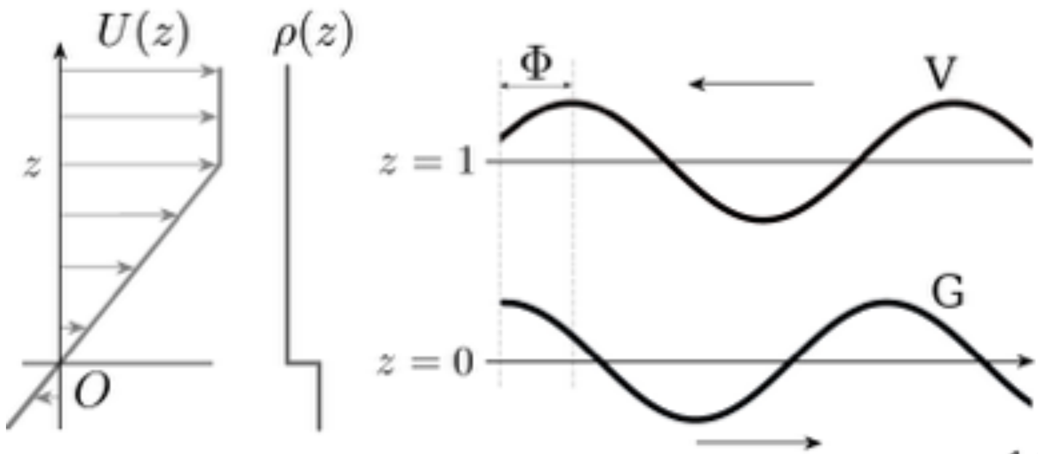


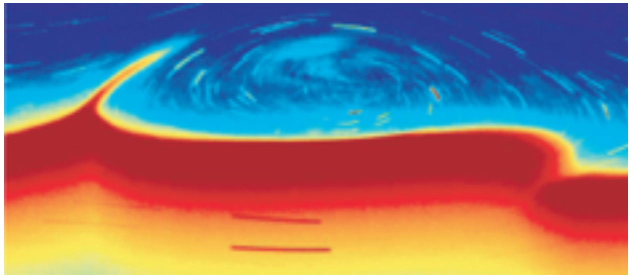


$$\frac{2\alpha}{1+e^{-2\alpha}} \leq J \leq \frac{2\alpha}{1-e^{-2\alpha}}$$



CASE 3: Holmboe Instability





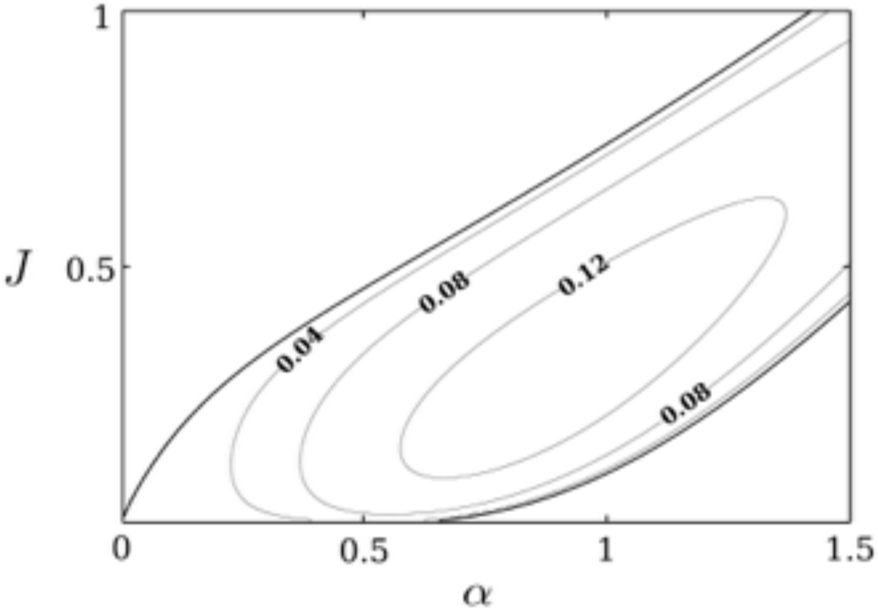
$$\frac{1}{2A}\left(-B-\sqrt{B^2-4AC}\right) \leq J \leq \frac{1}{2A}\left(-B+\sqrt{B^2-4AC}\right)$$

where

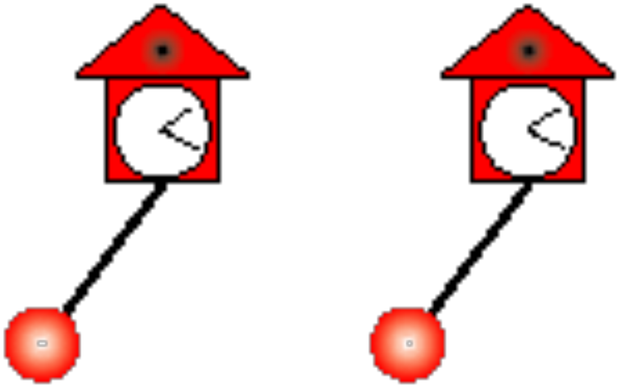
$$A = 16\alpha^2$$

$$B = -\alpha\left[8\left(2\alpha-1\right)^2+36\left(2\alpha-1\right)e^{-2\alpha}+27e^{-4\alpha}\right]$$

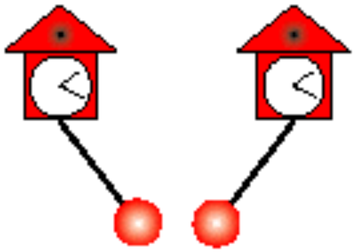
$$C = \left(2\alpha-1+e^{-2\alpha}\right)\left(2\alpha-1\right)^3$$



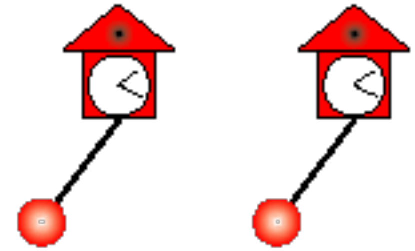
THE WORLD OF OSCILLATORS



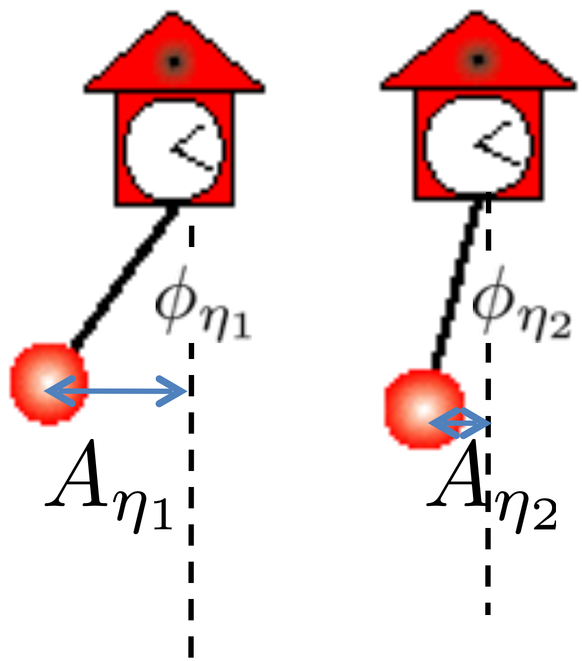
COUPLED OSCILLATORS



Anti-phase normal mode



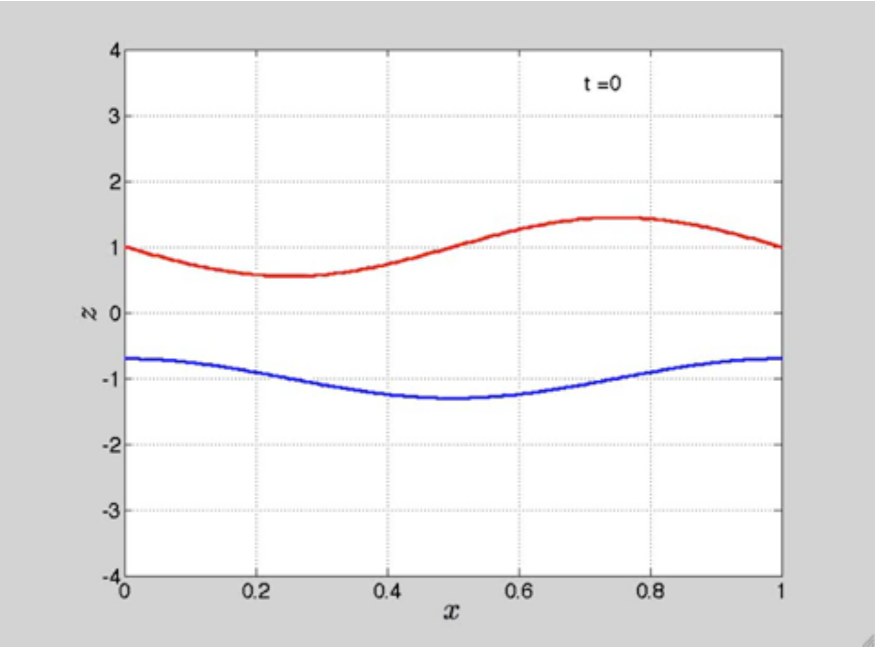
In-phase normal mode



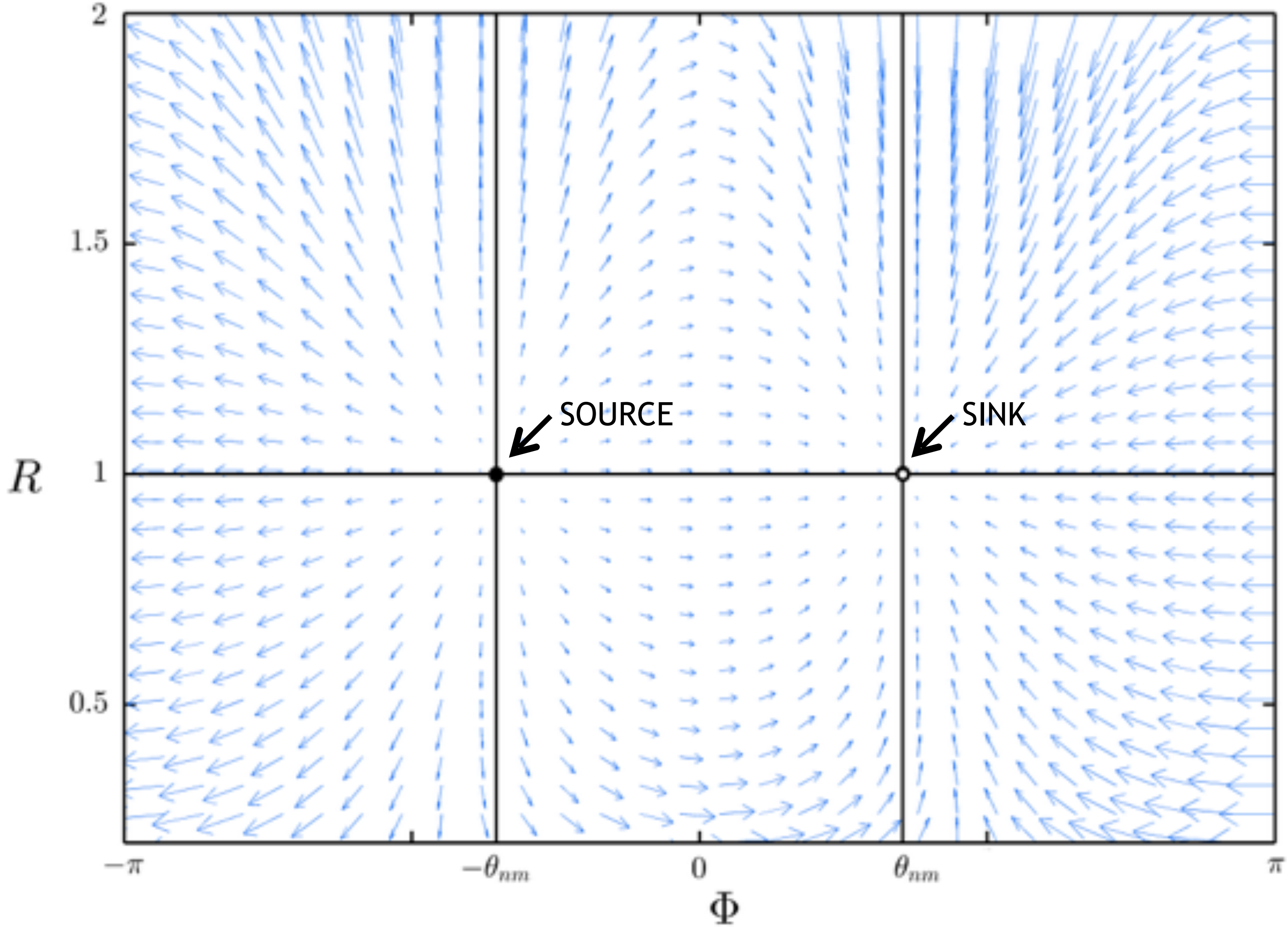
$$R \equiv A_{\eta_1} / A_{\eta_2}$$

$$\Phi \equiv \phi_{\eta_2} - \phi_{\eta_1}$$

SYNCHRONIZATION OF TWO COUPLED OSCILLATORS (and analogy with wave interaction)

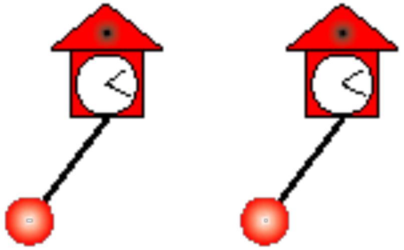


Recall WIT Phase Portrait

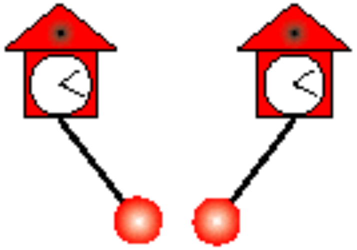


Two coupled oscillator synchronization

In-phase **normal mode**



Anti-phase **normal mode**



Sink

Source

Conclusions

- Wave Interaction Theory (WIT) provides a physical understanding of shear instabilities.

- Establishes a necessary and sufficient condition for idealized shear instabilities.

- Useful for understanding non-normal processes and transient growth mechanisms (this is what is happening prior to resonance).

- Brings normal mode theory, wave theory, and dynamical systems under one umbrella. Finds link with synchronization theory – a universal concept in nonlinear sciences.

THANK YOU*

* Linearly perturbed…