Revisiting the slow manifold of the Lorenz-Krishnamurthy quintet.

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$$
\begin{aligned}
dV/dt &= UW - bUZ - aV + aF,\\
dU/dt &= -VW + bVZ - aU,\\
dW/dt &= -VU - aW,\\
dX/dt &= -Z - aX,\\
dZ/dt &= bVU + X - aZ.
\end{aligned}
$$

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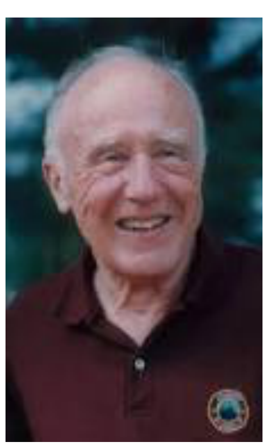
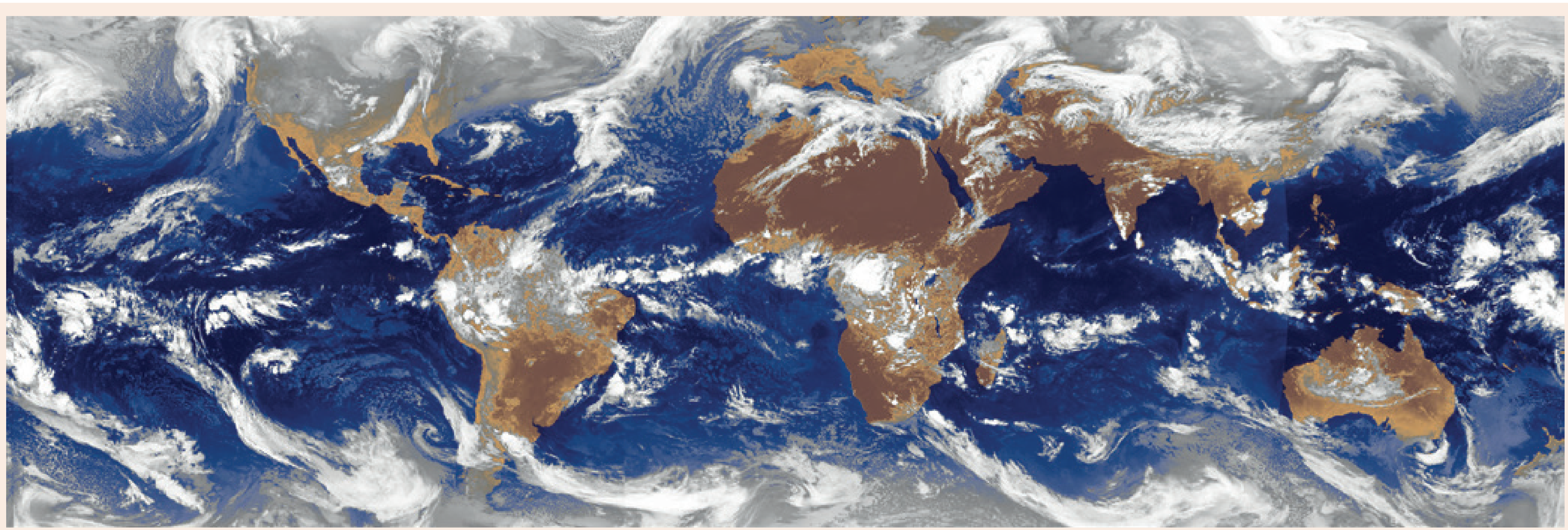


Edward Norton Lorenz

Discoverer of Chaos

V Krishnamurthy

The atmosphere is a forced and dissipative nonlinear system featuring nontrivial dynamics on a vast range of spatial and temporal scales. It is an outstanding example of a high-dimensional forced and dissipative complex system.



March 29th 2004 at 12:00 GMT

Clouds, circulation and climate sensitivity

Sandrine Bony[1*], Bjorn Stevens[2], Dargan M. W. Frierson[3], Christian Jakob[4], Masa Kageyama[5], Robert Pincus[6,7], Theodore G. Shepherd[8], Steven C. Sherwood[9], A. Pier Siebesma[10], Adam H. Sobel[11], Masahiro Watanabe[12] and Mark J. Webb[13]

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Can we capture the large scales?

This type of large scale motion is primarily horizontal because on large-scale, the fluid is confined to a relatively thin spherical shell.

The density stratification resulting from the near hydrostatic equilibrium discourages vertical motion.

The essential scales for describing large-scale motion are horizontal velocity and horizontal length.

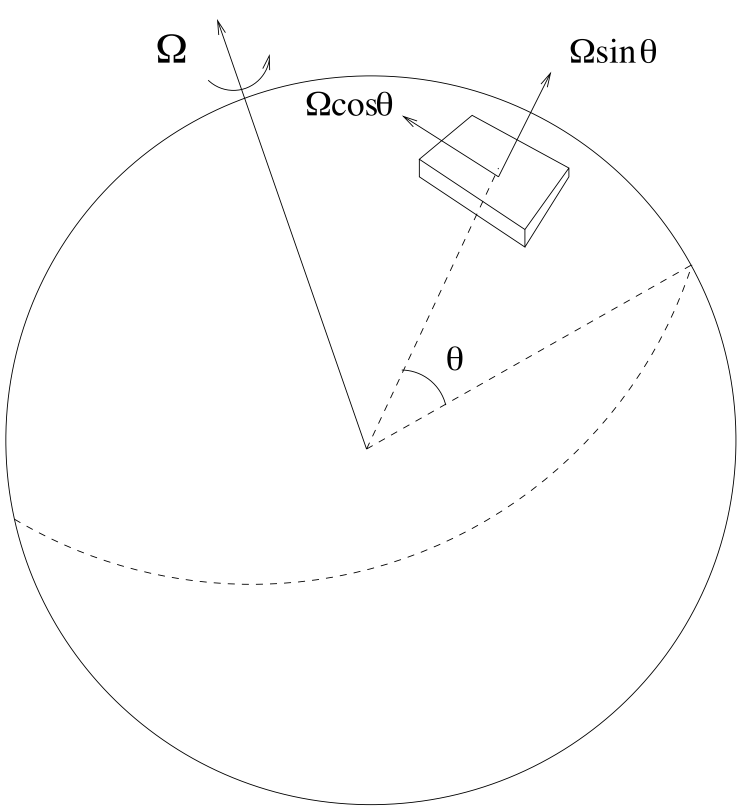
Thus large scale flow can be thought of as flow in which the Coriolis force is significant.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + 2\Omega w\, cos\theta - 2\Omega v\, sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u\, sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2\Omega u\, cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z$$

A region of the atmosphere at latitude θ.



$$\Omega = \frac{2\pi}{86400}\, sec^{-1}$$

Jule Charney (1917–1981)

Based on observations and scale analysis

$$-fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p$$

Geostrophic balance.

Balanced equation.

$$f \equiv 2\Omega sin\theta$$

In 1955 Charney introduced this nonlinear balance equation as an initialization method to determine wind from pressure fields.

Using a simple model, Charney showed that the motion, initialized with the balance equation, continued to be approximately balanced for some time; the balance equation thus describes a nearly invariant manifold.

I personally regard the successful reduction of the dynamic equations to a single prognostic equation by means of the geostrophic relationship, entirely apart from any applicability to NWP, as the greatest single achievement of twentieth-century dynamic meteorology. Consideration of the processes described by the new equation enabled me to see why cyclones and anticyclones and other weather systems move as they do—an understanding that the primitive equations never conveyed.

Reflections on the Conception, Birth, and Childhood of Numerical Weather Prediction

Edward N. Lorenz

Annu. Rev. Earth Planet. Sci. 2006. 34:37–45

Lorenz 1980

If we choose initial conditions that are in geostrophic balance ...!?!

Will the solutions do the same?

Attractor Sets and Quasi-Geostrophic Equilibrium

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(Manuscript received 9 January 1980, in final form 8 April 1980)

ABSTRACT

The attractor set of a forced dissipative dynamical system is for practical purposes the set of points in phase space which continue to be encountered by an arbitrary orbit after an arbitrary long time. For a reasonably realistic atmospheric model the attractor should be a bounded set, and most of its points should represent states of approximate geostrophic equilibrium.

A low-order primitive-equation (PE) model consisting of nine ordinary differential equations is derived from the shallow-water equations with bottom topography. A low-order quasi-geostrophic (QG) model with three equations is derived from the PE model by dropping the time derivatives in the divergence equations.

For the chosen parameter values, gravity waves which are initially present in the PE model nearly disappear after a few weeks, while the quasi-geostrophic oscillations continue undiminished. The states which are free of gravity waves form a three-dimensional stable invariant manifold within the nine-dimensional phase space. Points on this manifold are readily found by an algorithm based on the separation of time scales. The attractor set consists of a complex of two-dimensional surfaces embedded in this manifold. The geostrophic equation is a good approximation on most of the attractor, while the balance equation is better. The attractors of the PE and QG models are qualitatively similar.

Some speculations regarding the invariant manifold and the attractor in a large global circulation model are offered.

The physical laws which govern the behavior of a fluid system are commonly expressed as a set of partial differential equations (PDE's). It is often assumed that we may replace these equations by a large set of ordinary differential equations (ODE's), with time as the independent variable, without seriously altering the properties which interest us most. Such a substitution may in fact be a necessary step in preparing the PDE's for numerical integration. For various reasons, however, we sometimes choose to replace the PDE's by a *small* set of ODE's, hoping that some of the gross qualitative properties of the solutions will not be lost.

A fundamental property of a dynamical system is its *attractor set* A. A point Q is in A if the points for which Q is a limit point together form a set of nonzero volume in phase space. It is evident that each point on the orbit through Q is then in A, so that the attractor is composed of orbits.

If a system is a reasonably realistic model of the earth's atmosphere, we can anticipate some of the properties of its attractor from our experience with weather maps. Thus, we would expect most of the dependent variables to be bounded; we do not, for example, encounter sealevel maps with 1200 mb high-pressure centers or 800 mb lows. Likewise, there are combinations of variables which seldom if ever occur; we do not find maps where the wind blows the wrong way about the principal highs and lows in middle and higher latitudes. Points in the attractor, then, should not be too far removed from the origin, and should correspond to states where the bulk of the atmosphere is in approximate geostrophic equilibrium.

It is thus evident that except in very simple models a sufficiently precise description of the attractor to allow one to project onto it may be next to impossible. We shall therefore not attempt to study a particularly realistic atmospheric model, and seek instead the simplest model which retains pressure and the two wind components as separate variables and includes the nonlinear interactions which give rise to aperiodicity. For the latter purpose we should represent each variable by at least three functions of time alone. We thus anticipate a system of nine ODE's, and our problem will be to describe the attractor in nine-dimensional phase space.

During a visit to IISc in 1994

On the Nonexistence of a Slow Manifold

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(Manuscript received 29 September 1986, in final form 13 April 1987)

ABSTRACT

We define the slow manifold S in the state space of a primitive-equation model as a hypothetical invariant manifold on which there is no gravity-wave activity, and on which unique velocity-potential and streamfunction fields correspond to each isobaric-height field. We introduce a five-variable forced damped model, and show that for this model the point H representing the Hadley circulation and the two orbits forming the unstable manifold of H must lie in S if S exists. We then show that in traveling along one of these orbits one eventually encounters gravity waves, whereupon it follows that S does not exist.

A measure G of gravity-wave activity is found to decrease very rapidly as the external forcing F decreases. An approximate formula is derived for G as a function of F.

We show that a particular nine-variable forced damped model with orography also fails to possess a slow manifold, and we speculate as to the existence of slow manifolds in larger and more realistic models.

Slow manifolds for singularly perturbed ODE's

$$\begin{cases} \vec{x}' = \vec{f}\,(\vec{x}, \vec{z}, \varepsilon) \\ \vec{z}' = \varepsilon \vec{g}\,(\vec{x}, \vec{z}, \varepsilon) \end{cases} \tag{1}$$

where $\vec{x} \in \mathbb{R}^m$, $\vec{z} \in \mathbb{R}^p$, $\varepsilon \in \mathbb{R}^+$, and the prime denotes differentiation with respect to the independent variable t. The functions $\vec{f}$ and $\vec{g}$ are assumed to be C^∞ functions[3] of $\vec{x}$, $\vec{z}$ and ε in $U \times I$, where U is an open subset of $\mathbb{R}^m \times \mathbb{R}^p$ and I is an open interval containing $\varepsilon = 0$.

$$\tau = \varepsilon t$$

$$\begin{cases} \varepsilon \dot{\vec{x}} = \vec{f}\,(\vec{x}, \vec{z}, \varepsilon) \\ \dot{\vec{z}} = \vec{g}\,(\vec{x}, \vec{z}, \varepsilon) \end{cases} \tag{2}$$

The independent variables t and τ are referred to the *fast* and *slow* times, respectively, and (1) and (2) are called the *fast* and *slow* systems, respectively. These systems are equivalent whenever $\varepsilon \neq 0$, and they are labeled *singular perturbation problems* when $\varepsilon \ll 1$, i.e., is a small positive parameter. The label "singular" stems in part from the discontinuous limiting behavior in the system (1) as $\varepsilon \to 0$.

In such case, the system (1) reduces to an m-dimensional system called *reduced fast system*, with the variable $\vec{z}$ as a constant parameter:

$$\begin{cases} \vec{x}' = \vec{f}\,(\vec{x}, \vec{z}, 0) \\ \vec{z}' = \vec{0} \end{cases} \tag{3}$$

System (2) leads to the following differential-algebraic system called *reduced slow system* which dimension decreases from $m + p$ to p:

$$\begin{cases} \vec{0} = \vec{f}\,(\vec{x}, \vec{z}, 0) \\ \dot{\vec{z}} = \vec{g}\,(\vec{x}, \vec{z}, 0) \end{cases} \tag{4}$$

By exploiting the decomposition into *fast* and *slow* reduced systems (3) and (4), the geometric approach reduced the full *singularly perturbed system* to separate lower-dimensional regular perturbation problems in the *fast* and *slow* regimes, respectively.

Slow manifold

$$\vec{z} = h(\vec{x})$$

Existence of a Slow Manifold in a Model System of Equations

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(Manuscript received in final form 7 March 1990)

ABSTRACT

A model system of equations proposed by Lorenz and Krishnamurthy is analyzed. The Hartman-Grobman theorem is employed to prove that the equations of the model admit a slow manifold devoid of gravity-wave activity, and the theory of normal forms is used to construct the manifold and to determine when the manifold is stable. The study disproves a conjecture by Lorenz and Krishnamurthy that a slow manifold does not exist for their model.

$Hartman - Grobman$

$$\frac{dy}{dt} = Ay + f(y)$$

$$A \ is \ hyperbolic$$

$$f(0) = 0$$

$$\exists \ \Phi$$

$$x = \Phi(y)$$

$$\frac{dx}{dt} = Ax$$

$$X = f(U, V, W), \quad Z = g(U, V, W),$$

in the neighborhood of the equilibrium point

$$U = V - \epsilon = W = X = Z = 0.$$

$$f(u, v, w) = \sum_{N=1}^{\infty} f^{[N]}(u, v, w),$$

$$g(u, v, w) = \sum_{N=1}^{\infty} f^{[N]}(u, v, w),$$

$$X \approx AU + (V - \epsilon)(pU + qW),$$

$$Z \approx Bw + (V - \epsilon)(rU + sW),$$

The Slow Manifold—What Is It?

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(Manuscript received 17 June 1991, in final form 17 March 1992)

ABSTRACT

Two studies that disagree as to whether a slow manifold is present in a particular low-order primitive equation model are compared. It is shown that the discrepancy occurs because of a difference of opinion as to what constitutes a slow manifold.

There are fast oscillations and so not a slow manifold!

$$\varepsilon\, dy/dt = ay + e^{it}, \qquad t \geq 0, \quad y(0) = y_0. \tag{1}$$

Here ε, a are constants with $0 < \varepsilon \ll 1$, $|a| = O(1)$, and Real $a \leqslant 0$. The solution of (1) is given by

$$y(t) = y_S(t) + y_R(t), \tag{2}$$

$$y_S(t) = e^{it}(-a + i\varepsilon)^{-1}, \qquad y_R(t) = e^{(a/\varepsilon)t}(y_0 - y_S(0)).$$

Thus, it consists of the slowly varying part $y_S(t)$ and the rapidly changing part $y_R(t)$. There are two fundamentally different situations.

1. $a = -1$

In this case $y_R(t)$ decays rapidly and outside a boundary layer the solution varies slowly

2. $a = i$

Now $y_R(t)$ does not decay and $y(t)$ is highly oscillatory everywhere. In many applications one is not interested in the fast time scale. Therefore, it is of interest to develop methods of preparing the initial data such that the fast time scale is suppressed. There are two ways to do this.

INITIALIZATION. One prepares the initial data in such a way that the fast time scale is not activated. In the preceding example we need only to choose

$$y_0 = y_S(0) = (-a + i\varepsilon)^{-1} = -a^{-1}(1 + i\varepsilon/a - (\varepsilon/a)^2 + \cdots). \tag{3}$$

Then $y_R(t) \equiv 0$, and the solution of our problem consists only of the slowly varying part $y_S(t)$. For more complicated problems one can determine $y_S(0)$ only approximately. The rapidly changing part will always be present, but we can reduce its amplitude to the size $O(\varepsilon^p)$, p a natural number. An effective way to do this is to use the "bounded derivative principle," which is based on the following observation:

If $y(t)$ varies on the slow time scale, then $d^\nu y/dt^\nu \sim O(1)$ for $\nu = 1, 2, \ldots, p$ where $p > 1$ is some suitable number. Therefore our principle is

Choose the initial value $y(0) = y_0$ such that for $t = 0$

$$d^\nu y/dt^\nu|_{t=0} \sim O(1), \qquad \nu = 1, 2, \ldots, p. \tag{4}$$

Using the differential equation, we can express the derivatives at $t = 0$ by $y(0)$. Therefore, we can determine $y(0)$ such that (4) is satisfied without solving the differential equations.

Let us apply this principle to our example: $dy/dt|_{t=0} = O(1)$ if and only if

$$ay(0) = -1 + O(\varepsilon);$$

i.e.,

$$y(0) = -1/a + \varepsilon y_1, \qquad dy/dt|_{t=0} = ay_1, \quad y_1 = O(1). \tag{5}$$

If we choose $y(0)$ according to (5), then

$$y_R(0) = y(0) - y_S(0) = -1/a + \varepsilon y_1 - 1/(-a + i\varepsilon) = O(\varepsilon);$$

i.e., the amplitude of $y_R(t)$ is $O(\varepsilon)$ for all times. We consider now the second derivative. The differential equation gives us

$$\varepsilon\, d^2y/dt^2 = a\, dy/dt + ie^{it}.$$

Thus $d^2y/dt^2|_{t=0} = O(1)$ if and only if

$$a\, dy/dt|_{t=0} = a^2 y_1 = -i + O(\varepsilon);$$

i.e.,

$$y_1 = -i/a^2 + \varepsilon y_2, \qquad d^2y/dt^2|_{t=0} = a^2 y_2,$$

and by (5)

$$y(0) = -1/a(1 + i\varepsilon/a) + \varepsilon^2 y_2. \tag{6}$$

In this case we obtain for the amplitude

$$y_R(0) = y(0) - y_S(0) = O(\varepsilon^2).$$

The above procedure can be continued. If we choose the initial data such that the first p time derivatives are $O(1)$, then the amplitude of the fast part of the solution is $O(\varepsilon^p)$. We shall prove that results of this kind are valid for very general systems of linear and nonlinear ordinary and partial differential equations.

$$u = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \qquad x = \begin{bmatrix} X \\ Z \end{bmatrix}$$

$$\frac{du}{dt} = Au + f(x, u) + F_0$$

$$\frac{dx}{dt} = Bx + g(u)$$

$$A = -\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \qquad a > 0.$$

$$f(x, u) = \begin{pmatrix} bVZ - VW \\ -bUZ + UW \\ -UV \end{pmatrix}$$

$$B = \begin{pmatrix} -a & -1 \\ 1 & -a \end{pmatrix}$$

$$g(u) = \begin{pmatrix} 0 \\ bUV \end{pmatrix}$$

$$\sigma(B) = -a \pm i$$

$$u(0) = (U_0, V_0, W_0)$$

$$x(0) = (x_0, Z_0)$$

Slow Manifold

If

$$X_0 = F(U_0, V_0, W_0)$$
$$Z_0 = G(U_0, V_0, W_0)$$

Then

$$X(t) = F(U(t), V(t), W(t))$$
$$Z(t) = G(U(t), V(t), W(t))$$

Difficulty!

$$\frac{\partial F}{\partial t} + aF + G + F_1 \frac{\partial F}{\partial U} + F_2 \frac{\partial F}{\partial V} + F_3 \frac{\partial F}{\partial W} = 0$$

$$\frac{\partial G}{\partial t} + aG - F + F_1 \frac{\partial G}{\partial U} + F_2 \frac{\partial G}{\partial V} + F_3 \frac{\partial G}{\partial W} = f_0 UV$$

$$F(0, u) = X_0, \quad G(0, u) = Z_0$$

Solutions breakdown in finite time!
Difficult to compute F & G
Defeat's the original goal of simplifying!

Simple ansatz (if any) based on universal transient behaviour is the only way out!

We use an approach proposed by

Sharath Girimaji

Professor of Aerospace Engineering

Texas A&M University

Consider

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}), \quad \text{where} \quad \mathbf{z} = (z_1, z_2, \cdots z_n).$$

$$\mathbf{z} = (\mathbf{x}, \mathbf{y});\ \mathbf{x} \to \textbf{fast},\ \mathbf{y} \to \textbf{slow}$$

Minimization of Evolution rate (MER)

For dissipative ODE's there is a three stage behaviour;

- **Initial transient (fast processes getting exhausted).**
- **Solutions bunch together in a lower dimension space.**
- **Final equilibrium state**

Time scales at each state in phase space is proportional to the amount of time a solution trajectory resides in an infinitesimal neighbourhood of that point and is inversely proportional to the local evolution rate $\sqrt{\sum_{\mathrm{i}} \mathrm{g}_{\mathrm{i}}^2}$.

Solutions stagnate near longtime scale states and pass quickly through short time scale states.

An arbitrary trajectory is most likely to be found at the state with the largest residence time that is smallest evolution rate.

Proposal for slow manifold

$$E(z) = E(x, y) = \sum_i g_i^2(x, y)$$

$$x(y) = \mathrm{Min}_x E(x, y)$$

One could also naively set

$$\frac{dx}{dt} = 0.$$

For the LK system we obtain

$$x = B^{-1} g(u)$$

- Its possible that close to the slow manifold fast variable evolution can be slower than that of slow variables!

- A better gauge is convergence rate between two neighbouring trajectories.

These issues are still to be explored ?!

A linear example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda = -1 \quad x = -y\, cot\theta$$

$$\lambda = -\epsilon \quad x = y\, tan\theta$$

(slow manifold)

$$-a = cos^2\theta + \epsilon\, sin^2\theta$$

$$b = (1-\epsilon) sin 2\theta / 2$$

$$x = y\, tan\theta\, [1 - \epsilon\, sec^2\theta + \epsilon^2 sec^2\theta\, tan^2\theta + \cdots]$$

(steady state)

$$E = \dot{x}^2 + \dot{y}^2$$

$$x(y) = y\, tan\theta\, [1 - (1 + tan^2\theta)\epsilon^2 + (..)\epsilon^4 + \cdots]$$

(MER)

MER more accurate than steady state

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REVISITING THE SLOW MANIFOLD OF THE LORENZ-KRISHNAMURTHY QUINTET

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$$F = \left[\frac{dU}{dt}\right]^2 + \left[\frac{dV}{dt}\right]^2 + \left[\frac{dW}{dt}\right]^2 + \left[\frac{dX}{dt}\right]^2 + \left[\frac{dZ}{dt}\right]^2$$

For the LK system

$$X = \frac{-bUV}{1+a^2}$$

$$Z = \frac{bW(U^2+V^2) + abU(V+f_0)}{b^2(U^2+V^2) + (1+a^2)}$$

Further work

- MER needs to be established on a more rigorous framework.

“Slow manifold” for the LK quintet using MER

$$X = -\frac{bUV}{1+a^2}$$

$$Z = \frac{bW(U^2+V^2) + abU(V+f_0)}{b^2(U^2+V^2) + (1+a^2)}$$

$$
\begin{aligned}
dU/dt &= -VW + bV\left[\frac{bW(V^2+U^2)+abU(V+F)}{b^2(U^2+V^2)+1+a^2}\right] - aU \\
dV/dt &= UW - bU\left[\frac{bW(V^2+U^2)+abU(V+F)}{b^2(U^2+V^2)+1+a^2}\right] - aV + aF \\
dW/dt &= -UV - aW.
\end{aligned}
$$

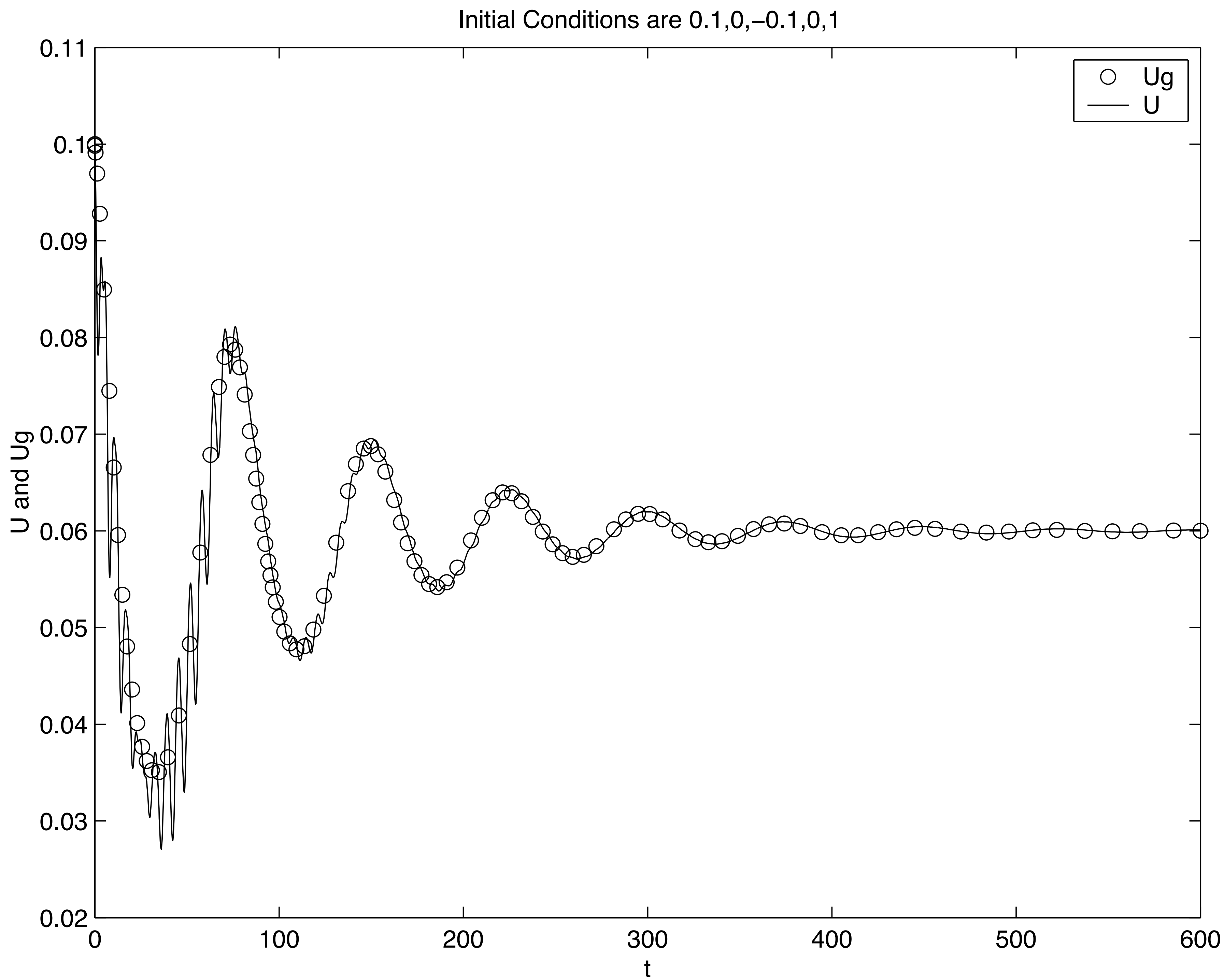


FIGURE 1. Graph showing evolution of by solving the original LK-model eqn(1-5) and the solution U_g obtained by solving the reduced set of equations (8-10) by minimization of evolution rate. The graph is for $a = 0.02; F = 0.2; b = 0.5$ (same values as used by [3])

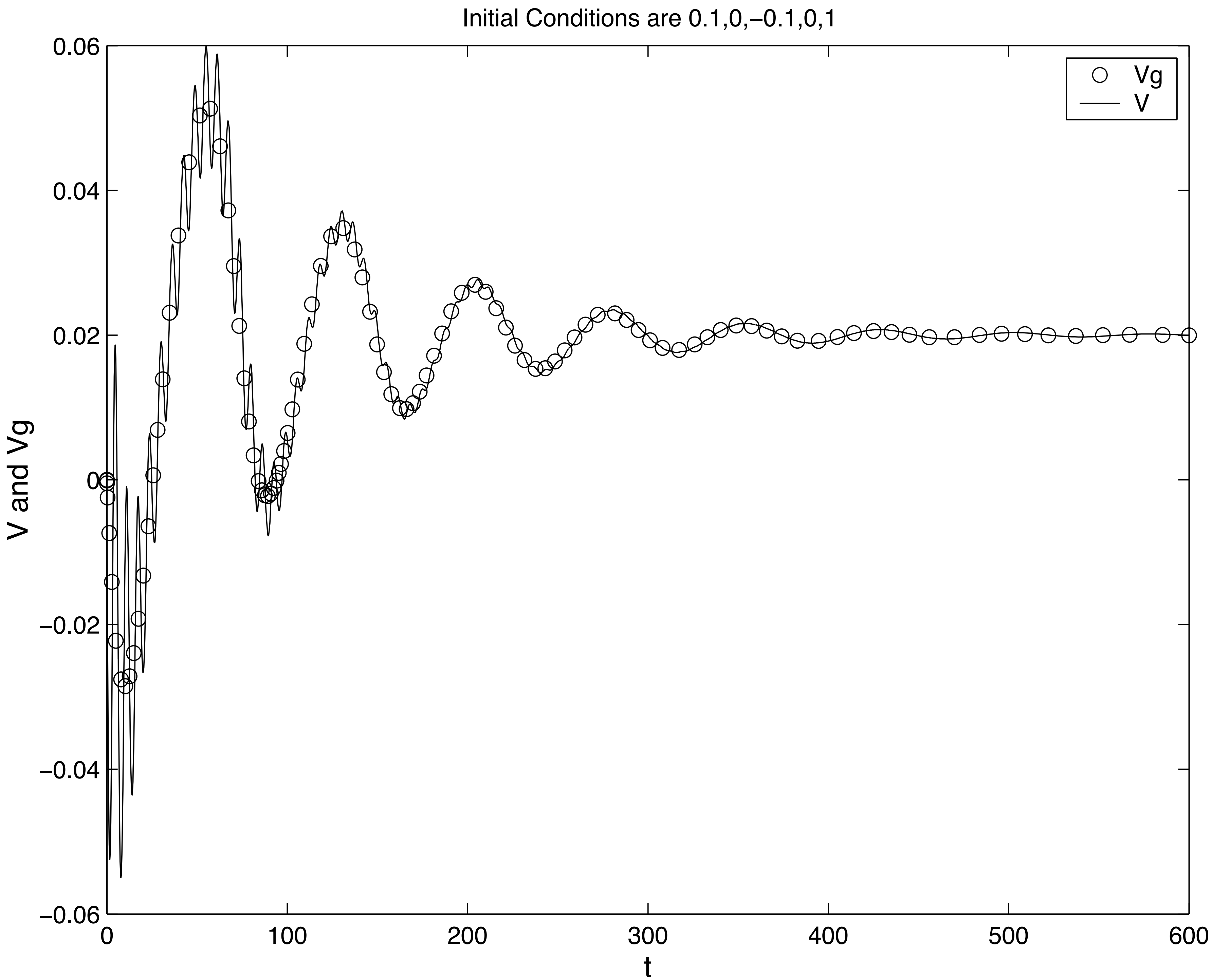
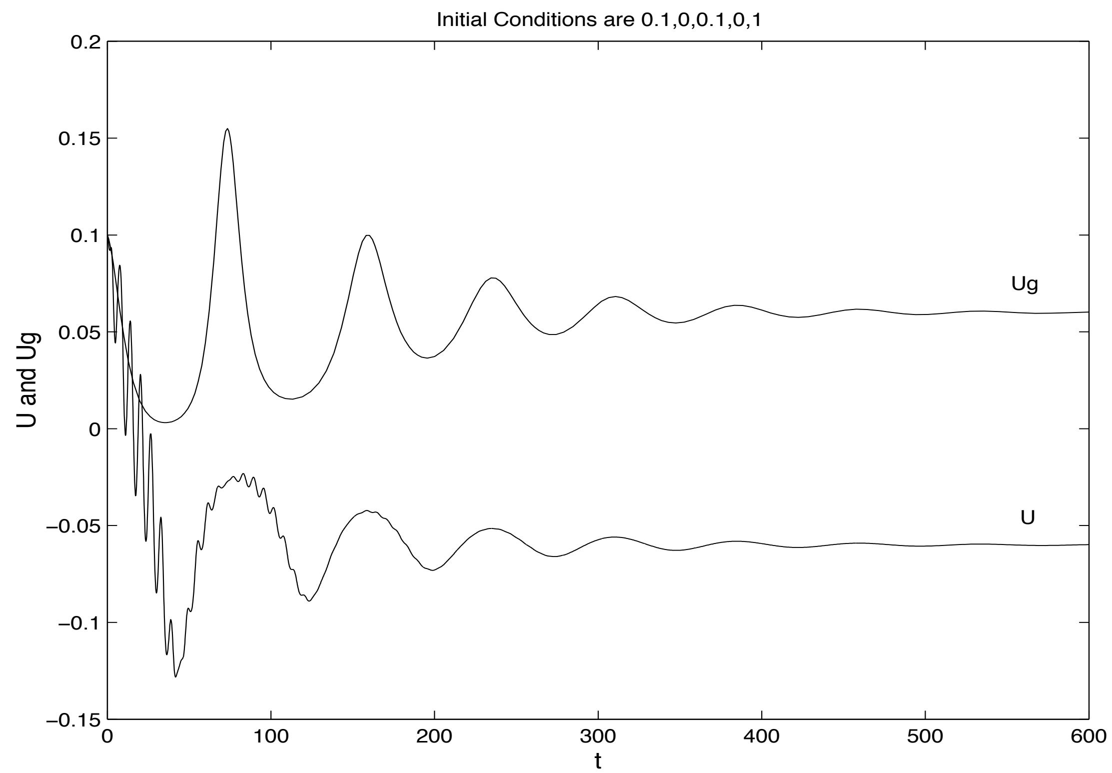
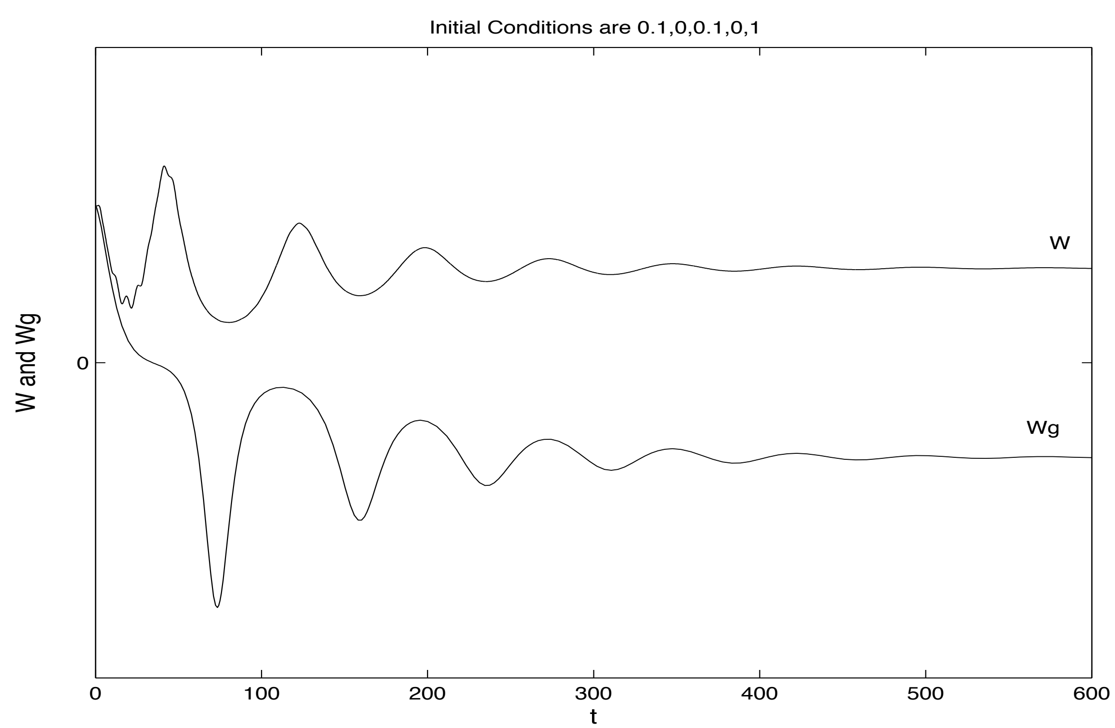


FIGURE 2. Same as fig 1 but for V



Imbalance leading to discrepancy?

FIGURE 3. Evolution of U and U_g for initial conditions $U_0 = W_0$



Exact slow manifold $\nexists$?

FIGURE 4. Evolution of W and W_g when $U_0 = W_0$

Nonlinear Galerkin Methods

$$\frac{du}{dt} + Au = f(u)$$

$$u(0) = u_0$$

A is a self-adjoint operator (> 0) on a Hilbert space H

$$Aw_j = \lambda_j w_j \quad j = 1, 2, \cdots$$

$$0 < \lambda_1 < \lambda_2 \cdots \quad \lambda_j \to \infty \text{ as } j \to \infty$$

$$P_N : H \to \text{Span}\{w_1, w_2, \cdots w_N\}$$

$$Q_N = I - P_N$$

$$u = p + q = P_N u + Q_N u$$

Does there $\exists$ a "manifold"

$$q = \Phi(p)?$$

Present Status?

Qual. Theory Dyn. Syst. (2014) 13:19–37
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Qualitative Theory of Dynamical Systems

The Slow Invariant Manifold of the Lorenz–Krishnamurthy Model

Jean-Marc Ginoux

Flow Curvature Method is used to provide an eighteenth-order approximation of the slow manifold for the LK model.

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}); \quad \mathbf{z} = (\mathbf{x}, \mathbf{y})$$

flow curvature manifold

$$\boldsymbol{\Phi}(\mathbf{z}) = \mathbf{det}[\dot{\mathbf{z}}, \ddot{\mathbf{z}}, \dddot{\mathbf{z}} \cdots \overset{(\mathbf{n})}{\mathbf{z}}] = \mathbf{0}$$

$$\mathbf{x} \approx \phi(\mathbf{y})$$

Based on the use of local metrics properties of curvatures inherent to Differential Geometry,

Thank You