Large-scale atmospheric circulations in the tropics

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Set up

- A local cartesian plane on the surface of a rotating sphere.
- Coriolis force approximated to vary linearly with lattitude: $\beta-$plane approximation.
- Large horizontal scales $L >> H$.
- In this limit, NS equations:

$$D\mathbf{u}/Dt + f \times \mathbf{u} = -\nabla p - g\hat{k} \tag{1}$$

- Equations with forcing.
- Circulations due to forcing.

Shallow Water equations (QG system)

- Local Cartesian plane.
- $\beta-$plane approximation ($f = \beta y$).
- State of rest, incompressible, hydrostatic balance ($H/L \ll 1$) as the base state.
- Small deviations from the base state.
- $\nabla_{x,y} p$ are independent of z and so are u, v.

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}
$$

QG waves f-plane

- Assume f to be constant.

$$\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}$$

Dispersion relation

Non-dimensional form:

$$\begin{aligned}
\frac{\partial u}{\partial t} - yv &= -\frac{\partial \phi}{\partial x} \\
\frac{\partial v}{\partial t} + yu &= -\frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}$$

ϕ is geo-potential.

- Assume solution of the form $(.) = (\hat{.})e^{i(kx+\omega t)}$.
- Eliminating u and ϕ, where $\hat{u} = -\dfrac{i\omega y\hat{v} + k(\partial\hat{v}/\partial y)}{(\omega - k)(\omega + k)}$ for $\omega \neq k$.

$$\frac{\partial^2 \hat{v}}{\partial y^2} + \left(\omega^2 - k^2 + \frac{k}{\omega} - y^2\right)\hat{v} = 0$$

$\hat{v} = C\exp(-y^2/2)H_n(y)$ for $\hat{v} \to 0$ as $y \to \pm\infty$ and

$$\boxed{\omega^2 - k^2 + \frac{k}{\omega} = 2n + 1}$$

QG waves

$$\boxed{\omega^2 - k^2 + \frac{k}{\omega} = 2n + 1}$$

For large (IG waves) and small (Rossby waves) regimes of ω

$$\omega_{1,2} = \pm\sqrt{k^2 + 2n + 1},\ \omega_3 = \frac{k}{k^2 + 2n + 1}$$

- For $n = 0$, the dispersion relation can be written as

$$(\omega - k)[\omega(\omega + k) - 1] = 0$$

. The valid solutions are: $\omega_1 = -\frac{k}{2} - \sqrt{\left(\frac{k}{2}\right)^2 + 1}$ and $\omega_2 = \sqrt{\left(\frac{k}{2}\right)^2 + 1} - \frac{k}{2}$. How does this look like?

Kelvin waves

The solutions where obtained by solving for v. But there exists a wave that has no meridional velocity ($v = 0$):

$$
\begin{aligned}
i\omega\hat{u} + ik\hat{\phi} &= 0 &(2)\\
y\hat{u} + \frac{d\hat{\phi}}{dy} &= 0 &(3)\\
i\omega\hat{\phi} + ik\hat{u} &= 0. &(4)
\end{aligned}
$$

- For $v = 0$, the dispersion relation can be obtained by solving for $\hat{u}$ and $\hat{\phi}$,

  $$(\omega - k)(\omega + k) = 0.$$

  Here, $\omega = -k$ is the valid solution. This represents the Kelvin wave and it propagates only in the east direction.

- This wave corresponds to $n = -1$ of $\boxed{\omega^2 - k^2 + \frac{k}{\omega} = 2n + 1}$

Dispersion diagram

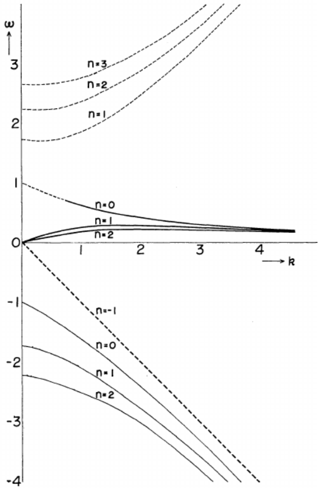


Figure: Dispersion diagram

Circulation patterns

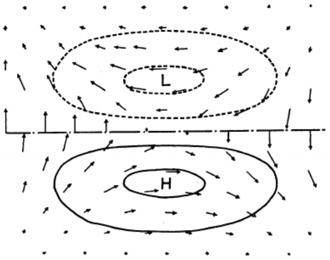
- u and ϕ in terms of v are given by

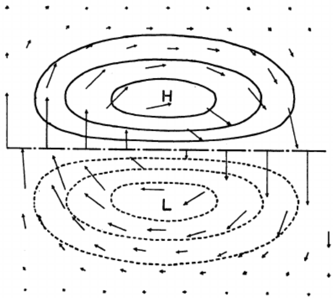
$$\hat{u} = -\frac{\mathrm{i}}{\omega^2 - k^2}(\omega y \hat{v} + k\frac{d\hat{v}}{dy}) \quad (5)$$

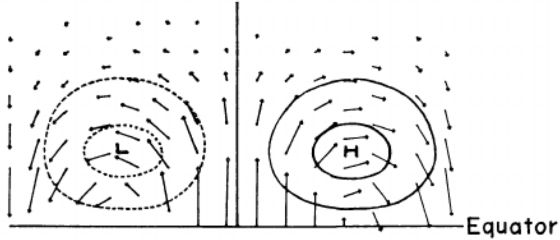
$$\hat{\phi} = \frac{\mathrm{i}}{\omega^2 - k^2}(k y \hat{v} + k\frac{d\hat{v}}{dy}) \quad (6)$$

- Recall $\hat{v} = C\exp(-y^2/2)H_n(y)$.
- Note $H_n' = 2nH_{n-1}$ and $H_{n+1} = 2yH_n - 2nH_{n-1}$.
- For $n = 0$, $\hat{v} = C\exp(-y^2/2)$ since $H_0 = 1$.

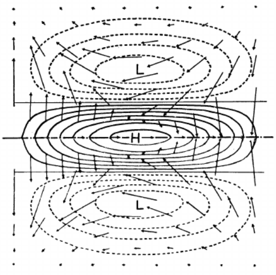
Circulation patterns ($n = 0$)

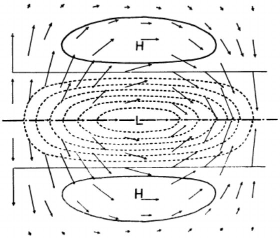


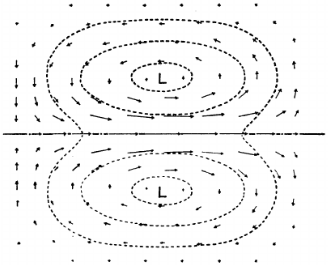




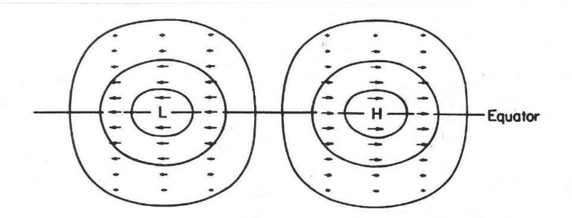
Circulation patterns ($n = 1$)







Kelvin wave



Summary

- Rossby waves are dispersive for small k but for large k their phase speed $1/(2n+1)$ times the gravity waves.
- They travel about one-third of the phase speed of Kelvin waves.
- There is mixed Rossby-Gravity wave called Yanai waves.
- For $k > 1/\sqrt{2}$ the wave behaves like a Rossby wave and for $k \leq 1/\sqrt{2}$ the wave behaves like an inertia-gravity wave.
- Rossby waves propagate only in the westward direction.
- Inertia gravity waves are almost symmetric in the east-west directions.

With a heat source

- The stratified atmosphere can be represented as a sum of vertical modes.
- Each vertical mode satisfies the shallow-water equations in the horizontal directions.
- The deep convective heating can be represented by the lowest mode.
- Response of the atmosphere to a specified heat source (one-way problem) such as Indonesia.

$$\begin{aligned}
\frac{\partial u}{\partial t} - yv &= -\frac{\partial \phi}{\partial x} \\
\frac{\partial v}{\partial t} + yu &= -\frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= -Q
\end{aligned}$$

Steady-state with friction

In the steady state the equations are given by

$$\begin{aligned} \epsilon u - yv &= -\frac{\partial \phi}{\partial x} \\ \epsilon v + yu &= -\frac{\partial \phi}{\partial y} \\ \epsilon \phi + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= -Q \end{aligned}$$

- ϵ is a scale of dissipation. It is same in x and y - an assumption!
- The above equations can be combined similarly and for small ϵ they can be approximated.

$$\epsilon \frac{\partial^2 v}{\partial y^2} + \partial vx - \epsilon y^2 v = y \frac{\partial Q}{\partial x} - \epsilon Qy$$

- This is equivalent to $yu = -\frac{\partial \phi}{\partial y}$ which is in geostrophic balance (long wave approximation).

Steady-state with friction

- The solutions are obtained by assuming

$$
\begin{aligned}
v &= \hat{v}(x)\exp(-y^2/2)H_n(y) \\
Q &= \hat{Q}(x)\exp(-y^2/2)H_n(y)
\end{aligned}
$$

- Q is prescribed.
- The analysis is done for symmetric and anti-symmetric Q (by specifying H_n).
- The resulting circulation patterns explain many of the observed circulations of winds.

Symmetric forcing

- For $n = 0$, the solutions are given by

$$\begin{aligned} u, p &= \frac{1}{2} q_0(x) \exp(-y^2/4) \\ v &= 0 \end{aligned}$$

- This Kelvin wave type solution represents Walker circulation over the Pacific when the forcing is placed on Indonesia.
- The easterly circulation is parallel to equator flowing into the heating region and then flow eastward aloft.

Circulation patterns

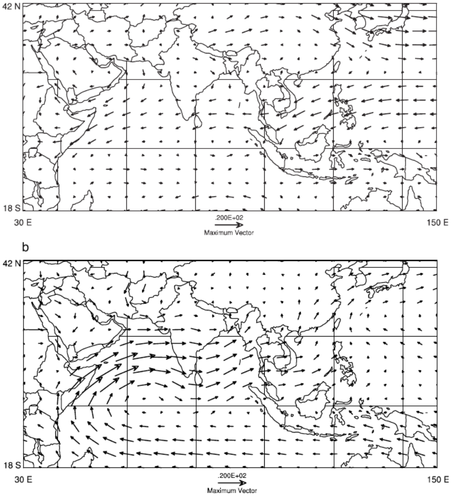


Figure: Observed wind field for January and July

The problem

- In the tropics Rossby number approaches unity, DU/Dt and $f \times U$ are comparable.
- The nonlinear terms in the momentum equations become important at the tropics.
- How do we represent vertical motion?
- Role of moisture.
- Can we construct models that can address the Indian monsoon?

Thank you!