Dynamics of polynomial shift–like maps

Sayani Bera

Indian Association for the Cultivation of Science, Kolkata

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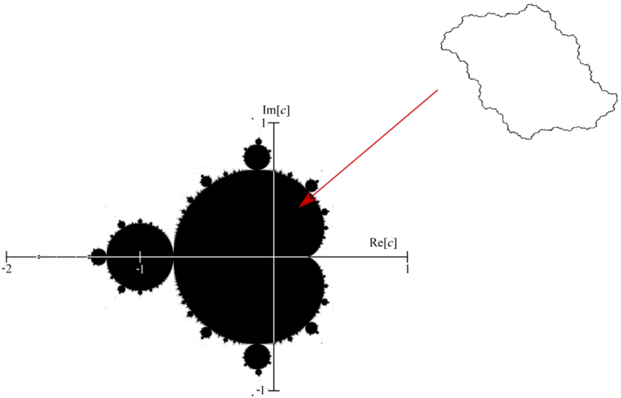

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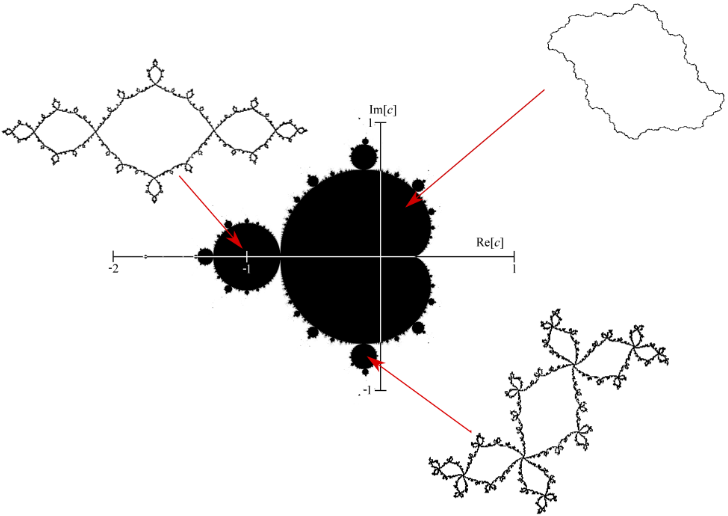

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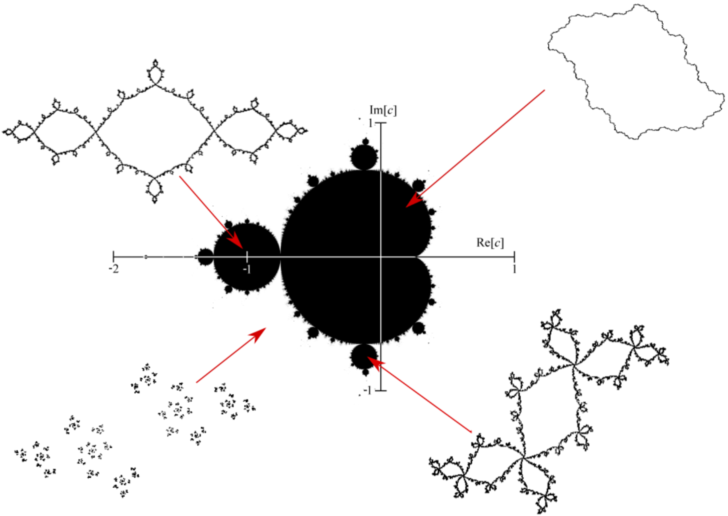

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- A map of the form $\mathbf{G(x, y) = (ax + p(y), by)}$ is also an automorphism of $\mathbb{C}^2$ for $a, b \neq 0$ and p an entire map in $\mathbb{C}$. These are called **elementary maps**.

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- By *Jung's Theorem* it is known that **elementary maps** and **Hénon maps** generate the polynomial automorphisms of $\mathbb{C}^2$.
- By a result of *Friedland–Milnor*, Hénon maps or finite composition of Hénon maps are the only polynomial automorphisms that have interesting dynamics in $\mathbb{C}^2$.

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- If $|\mathrm{Det}DH(x, y)| = |a| > 1$ then the Hénon map has exactly one Fatou component.

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A polynomial map p in $\mathbb{C}$ is said to be dynamically **hyperbolic** if there exists a conformal metric ρ on a neighbourhood of U of the Julia set $\mathcal{J}_p$ such that p is expanding on $\mathcal{U}$.

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(iii) And there exist constants $\lambda > 1$, $C > 0$ such that $|DF^n(x)v| \leq C\lambda^{-n}|v|$ for $v \in E_x^s$ and $|DF^n(x)v| \geq C^{-1}\lambda^n|v|$ for $v \in E_x^u$.

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Theorem (Shafikov–Wolf, 2003, Michigan Math. J.)

A hyperbolic and regular polynomial automorphism of $\mathbb{C}^k$, $k \geq 2$ has finitely many Fatou–Components, i.e., it do not admit non–wandering phenomenon.

Shift–like maps in $\mathbb{C}^3$

- (1–Shift) For $\nu = 1$ and $a \in \mathbb{C}^*$, $S_a(z_1, z_2, z_3) = (z_2, z_3, az_1 + p(z_3))$.
- (2–Shift) For $\nu = 2$ and $a \in \mathbb{C}^*$, $S_a(z_1, z_2, z_3) = (z_2, z_3, az_1 + p(z_2))$.
- Either case S_a is **not** *regular* but S_a^2 is *regular*.
- If p is a *hyperbolic* polynomial then there exists $A > 0$ such that if $0 < |a| < A$, then the $1-$shift S_a is *hyperbolic* on J.

Theorem (Shafikov–Wolf, 2003, Michigan Math. J.)

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** A sufficiently small $1-$shift of a hyperbolic polynomial in $\mathbb{C}^3$ does not have wandering domains.

Non–wandering result for shift–like maps

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Let p be a hyperbolic polynomial, then there exists $A > 0$ such that for every $\nu-$shift S_a, $1 \leq \nu \leq k-1$, the number of Fatou component for S_a is finite.

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- For a 2−shift-like map as $a \to 0$,

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Idea of proof for a type 2−shift-like map in $\mathbb{C}^3$.

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  - (ii) There exists a conformal metric ρ_i on U_i such that S_a^2 is hyperbolic on J_a^i.
  - (iii) The Julia sets $J_a^{\pm}$ can be recovered as stable unstable sets of J_a^i's, $1 \leq i \leq 3$, i.e.,

$$J_a^+ = W^s(J_a^1 \sqcup J_a^2 \sqcup J_a^3) \text{ and } J_a^- = W^u(J_a^1 \sqcup J_a^2 \sqcup J_a^3).$$

Thank You