Local polynomial convexity of certain class of surfaces in $\mathbb{C}^2$ with isolated CR singularity

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Definition: For $K \subset_{cpt} \mathbb{C}^n$, define $\widehat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p|, p \in \mathbb{C}[z_1, \ldots, z_n]\}$. K is said to be *polynomially convex* if $\widehat{K} = K$.

Notation: $\mathscr{P}(K)$ denotes the uniform closure of polynomials in $z_1, \ldots, z_n$ in $\mathcal{C}(K)$.

Motivations:

Theorem (Oka-Weil)

Let $K \subset \mathbb{C}^n$ be a compact polynomially convex set. Then any function that is holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials in $z_1, \ldots, z_n$.

Theorem (O'Farrell-Preskenis-Walsh)

Let $K \subset \mathbb{C}^n$ be a compact, $\widehat{K} = K$ and $E \subset K$ be such that $K \setminus E$ is locally contained in totally-real submanifolds of $\mathbb{C}^n$. Then $\mathscr{P}(K) = \{f \in \mathcal{C}(K) : f|_E \in \mathscr{P}(E)\}$.

Examples: Surfaces in $\mathbb{C}^2$

- $M = \{(z, f(z) \in \mathbb{C}^2 : f(z) = az + b\overline{z}\}$. Every compact subset of M is polynomially convex iff $b \neq 0$ (iff M is totally-real) . In this case we have $\mathscr{P}(K) = \mathcal{C}(K)$.

- (Wermer) Totally real surface is locally polynomially convex at each of its points.

  For $\mathcal{C}^2$-smooth real surface M in $\mathbb{C}^2$ a each point $a \in M$, there exists a $\delta > 0$ such that $M \cap B(a;\delta) = \left\{(z, f(z)) \in \mathbb{C}^2 : f \in \mathcal{C}^2(B(0;\delta)) \cap \mathbb{C}\right\}$. Such a surface M is totally real at $a \in M$ if and only if $\dfrac{\partial f}{\partial \overline{z}}(a) \neq 0$.

- We assume $0 \in M$ is an isolated CR-singularity. In this case, locally near the origin, M is of the form

$$M \cap B(0;\delta) = \left\{(z, f(z)) \in \mathbb{C}^2 : f(z) = az^2 + b\overline{z}^2 + cz\overline{z} + o(|z|^2)\right\}.$$

Non-degenerate/degenerate CR singularity: Bishop (1965), Forstnerič-Stout (1993), Jöricke (1997)

Bishop's normal form:

$$M \cap B(0;\delta) = \left\{(z, f(z)) \in \mathbb{C}^2 : f(z) = \gamma(z^2 + \overline{z}^2) + |z|^2 + o(|z|^2)\right\},$$

where $\gamma \geq 0$. γ is a biholomorphic invariant.

Types of CR singularity:

- Elliptic if $\gamma < 1/2$: Bishop (1965) showed, by attaching a one parameter family of analytic discs, that the surface is not locally polynomially convex at the origin.
- Hyperbolic if $\gamma > 1/2$: Forstnerič-Stout (1991) showed that the surface is locally polynomially convex at the origin.
- Parabolic if $\gamma = 1/2$: Higher order terms determine the polynomial convexity property. The situation is not completely clear. Jöricke (1997) showed that there is a bifurcation.

Isolated CR singularity at $0 \in \mathbb{C}^2$ of higher order

- Locally at the origin $M \cap U = \{(z, f(z,\overline{z})) \in \mathbb{C}^2 : z \in D(0;r)\}$, where $f(z,\overline{z}) = p_k(z,\overline{z}) + o(|z|^k)$ and p_k is a homogenous polynomial of degree $k, k \geq 3$.
- The main question here: *Does there exist Bishop-type trichotomy in this case?*
- Some works in this direction: Forstnerič (1987), Harris (1991), Wiegerinck (1995), Bharali (2005, 2006, 2012)
- Maslov-type index: A curve $\widetilde{\gamma} \subset M \longleftrightarrow \gamma \subset \mathbb{C}$ with γ simple closed. $I(\gamma) :=$ winding number of $\dfrac{\partial f}{\partial \overline{z}}$ along γ. Near a CR-singularity the index is independent of the curve; we denote it by I_M. elliptic: $I_M = 1$, hyperbolic: $I_M = -1$, parabolic: $I_M = -1, 0, 1$.
- It was thought for sometime that $I_M > 0 \Longrightarrow$ non polynomial convexity and $I_M < 0 \Longrightarrow$ polynomial convexity. Wiegerink gave example of surface with $I_M < 0$ but have nontrivial local hull. No counterexample is known for the first implication.

Isolated CR singularity of order 3

Let M be a surface in $\mathbb{C}^2$ with an isolated degenerated CR-singularity at $0 \in \mathbb{C}^2$ with the order of degeneracy 3. Therefore, there exists a neighbourhood U of $0 \in \mathbb{C}^2$ and $r > 0$ such that

$$M \cap U = \{(z, f(z)) \in \mathbb{C}^2 : z \in D(0; r)\}.$$

$f(z, \overline{z}) = p(z, \overline{z}) + o(|z|^3)$, where $p(z, \overline{z}) = a_1 z^2 \overline{z} + a_2 z \overline{z}^2 + a_3 \overline{z}^3$.

$S := \{(z, p(z, \overline{z}) \in \mathbb{C}^2 : z \in \mathbb{C}\}$.

Assumption: Consider only those surfaces M such that under a proper holomorphic map $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$, $\Psi^{-1}(S)$ is the union of three totally real planes in $\mathbb{C}^2$.

Why? Polynomial convexity remains invariant under proper holomorphic maps. In hyperbolic case of Bishop's trichotomy this method is used.

Consider $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$, proper, holomorphic
$\Phi(z,w) := (z, p(z,w)) = (z, a_3 w^3 + a_2 w^2 z + a_1 w z^2)$.

Lemma (Thomas)

Let $\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be as above. Then $\Phi^{-1}(S)$ is the union of three totally-real planes if and only if $a_2^2 = 3a_1a_3$.

Normal form: By a theorem of Harris (1985), we can take the surface S into a normal form:

$$S_t = \{(z,w) \in \mathbb{C} : w = p_t(z, \bar(z)) = z^2\overline{z} + tz\overline{z}^2 + \frac{t^2}{3}\overline{z}^3,$$

for some $t \in (0, \infty)$. Invertible $\mathbb{C}$-linear transformations are used.

Hence, any surface M can be transformed locally to
$M_t \cap \overline{B(0;r)} = \{(z,w) \in \mathbb{C}^2 : \phi(z,w) = 0\}$, where
$\phi(z,w) = p_t(z,\overline{z}) + F(z,\overline{z}) - w$ with $F(z) = o(|z|^3)$.

Compare with Bishop's normal form: $t = 1$ case is similar to that of parabolic.

Theorem

For $t \in (0, \infty)$, let $S_t = \{(z, w) \in \mathbb{C}^2 : w = p_t(z, \overline{z})\}$, where $p_t(z, \overline{z}) = z^2\overline{z} + tz\overline{z}^2 + t^2\overline{z}^3/3$. Then

i) *Any compact set of S_t is polynomially convex at the origin if*
$$t \geq \frac{\sqrt{15 - \sqrt{33}}}{2\sqrt{2}} = 1.0756066...$$

ii) *Any compact subset of S_1 is polynomially convex.*

iii) *For $t < 1$, S_t is not locally polynomially convex at the origin.*

iv) *For every $\delta > 0$ the polynomial hull of $S_t \cap \overline{\mathbb{B}(0;\delta)}$ contains an open ball centred at the origin if $t < \sqrt{3}/2$.*

v) *For every $\delta > 0$ the polynomial hull of $S_t \cap \overline{\mathbb{B}(0;\delta)}$ contains a one parameter family of analytic discs passing through the origin if $\sqrt{3}/2 \leq t < 1$.*

Theorem

For $t > 0$, Let M_t be as above. Then

i) *M_t is locally polynomially convex at the origin if*

$$t > \frac{\sqrt{15 - \sqrt{33}}}{2\sqrt{2}} = 1.0756066....$$

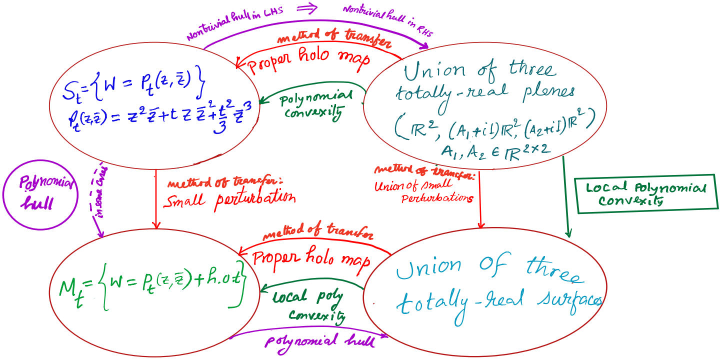
ii) *M_t is not locally polynomially convex if $t < 1$.*

iii) *For every $\delta > 0$, the polynomially convex hull of $M_t \cap \overline{\mathbb{B}(0;\delta)}$ contains an open ball centred at the origin if $t < \sqrt{3}/2$.*

iv) *For every $\delta > 0$ the polynomial hull of $M_t \cap \overline{\mathbb{B}(0;\delta)}$ contains a one parameter family of analytic discs passing through the origin if $\sqrt{3}/2 \leq t < 1$.*

Conjecture: M_t is locally polynomially convex for $t > 1$.

Idea of proofs:



A result of Wiegerinck: used as technique to get the hull

Theorem (Wiegerinck)

Suppose that 0 *is an isolated CR-singularity of* $S = \{w = F(z, \overline{z})\}$, $F(z, \overline{z})$ *is homogeneous of degree* k *and the Maslov-type index at* $0 \in \mathbb{C}^2$ *is* j, $0 < j < k$. *Assume that* $\mathrm{Re}(F(z, \overline{z})/z^{j-1})$ *is a subharmonic but nowhere harmonic function in* $\mathbb{C}$. *Then*

(i) S *is not locally polynomially convex at* $0 \in \mathbb{C}^2$.

(ii) *For evey* $r > 0$, $S \cap \overline{B(0; r)}$ *contains a* $(2j - 1)$*-parameter family of analytic discs with boundary in* S *passing around zero if and only if the curve* $\mathscr{C}(z) = \dfrac{F(z, \overline{z})}{z^k}$, $|z| = 1$, *has the following property:* (*)*If for two different points* $z_1 \neq z_2$ *on the unit circle* $\mathscr{C}(z_1) = \mathscr{C}(z_2)$, *then* z_1 *and* z_2 *divide the unit circle in two segments of length at least* $\dfrac{\pi}{k - j + 1}$.

iii) *If Property* $(*)$ *does not hold, then for evey* $r > 0$, $\widehat{S \cap \overline{B(0; r)}}$ *contains at least a 1-parameter family of analytic discs with boundary passing through the origin.*

Thank You