Quadrature domains in higher dimensions

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A quick recap of planar quadrature domains

A domain $\Omega \subset \mathbb{C}$ is said to satisfy a quadrature identity with respect to a test class of functions A if there exist points $q_1, \ldots, q_p$ in Ω and complex numbers c_{jk} where $1 \leq j \leq p$ and $0 \leq k \leq m_j - 1$ with $m_j \geq 1$ and $c_{m_j - 1} \neq 0$ such that

$$\int_\Omega f(z) = \sum_{j=1}^{p} \sum_{k=0}^{m_j - 1} c_{jk} f^{(k)}(q_j)$$

for every $f \in A$. We then say that Ω is a quadrature domain for the test class A. The points $q_1, \ldots, q_p$ are the nodes.

Aharanov and Shapiro studied quadrature domains sytematically in 1976. Gustafssön, Sakai, Putinar, Bell among many others have studied planar quadrature domains extensively.

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Aharanov and Shapiro studied quadrature domains sytematically in 1976. Gustafssön, Sakai, Putinar, Bell among many others have studied planar quadrature domains extensively.

- If $\Omega \subset \mathbb{C}$ is a domain of finite area whose complement has non-empty interior. If $a \in \Omega$ be a point such that for every function u harmonic in Ω,

$$u(a) = \frac{1}{\text{Area}(\Omega)} \int_\Omega u d\sigma,$$

then Ω is a disc centred at a.

- The necessary and sufficient condition that a simply connected domain Ω satisfies a quadrature identity for all $f \in L^1(\Omega) \cap \mathcal{O}(\Omega)$ is that some conformal map of the unit disc $B(0,1)$ on Ω be a rational function with all poles outside $\overline{B(0,1)}$.

- Let $\Omega \subset \mathbb{C}$ be a quadrature domain. Then there exists $P \in \mathbb{R}[X,Y]$, non-constant and irreducible over $\mathbb{C}$, such that the boundary $\partial\Omega$ is contained in the zero variety $\{z \in \mathbb{C} : P(z) = 0\}$.

- Let Ω be a bounded domain in $\mathbb{C}$ whose boundary consists of finitely many bounded analytic curves. Then there are quadrature domains arbitrarily close to Ω and conformally equivalent to Ω.

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Quadrature domains in higher dimensions

Let us begin by considering the natural generalization of the definition of quadrature domains to higher dimensions.

Definition

By a quadrature domain we will mean a bounded domain $D \subset \mathbb{C}^n$, $n \geq 1$ with finitely many distinct points $q_1, \ldots, q_m \in D$, positive integers $r_1, \ldots, r_p$ and complex constants $c_{j\alpha}$ such that

$$\int_D f(z) = \sum_{j=1}^{m} \sum_{|\alpha|=0}^{r_j-1} c_{j\alpha} f^{(\alpha)}(q_j)$$

for every f in the test class $H^2(D)$, the Hilbert space of square integrable holomorphic functions on D.

We shall first obtain an alternate characterization of the definition of a quadrature domain using the Bergman Kernel.

Quadrature domains and the Bergman span

We have

$$\langle f, 1\rangle_D = \int_D f(z) \quad = \sum_{j=1}^{m} \sum_{|\alpha|=0}^{r_j-1} c_{j\alpha} f^{(\alpha)}(q_j) = \left\langle f, \sum_{j=1}^{p} \sum_{|\alpha|=0}^{n_j-1} \overline{c}_{j\alpha} K_D^{(\alpha)}(\cdot, q_j) \right\rangle_D$$

for all $f \in H^2(D)$. Therefore,

$$1 = \sum_{j=1}^{p} \sum_{|\alpha|=0}^{n_j-1} \overline{c}_{j\alpha} K_D^{(\alpha)}(z, q_j)$$

for all $z \in D$.

The *Bergman span* $\mathcal{K}_D$ associated to a domain D is the complex linear span of all functions of z of the form $K_D^{(\alpha)}(z, a)$ where a varies over D and α varies over all possible multi-indices with $|\alpha| \geq 0$.

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A domain $D \subset \mathbb{C}^n$ is a quadrature domain if the constant function $h(z) \equiv 1$ belongs to the Bergman span associated to D.

Examples of quadrature domains in $\mathbb{C}^n$

1. The unit Ball in $\mathbb{C}^n$.
2. The Polydisk in $\mathbb{C}^n$.
3. Complete circular domains in $\mathbb{C}^n$.
4. Complete $(p_1, p_2, \ldots, p_n)$-circular domain in $\mathbb{C}^n$.
5. Let $D_1 \subset \mathbb{C}^{n_1}$ and $D_2 \subset \mathbb{C}^{n_2}$ be domains. Then $D_1 \times D_2$ is a quadrature domain in $\mathbb{C}^{n_1+n_2}$ if and only if D_1 and D_2 are quadrature domains in $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ respectively.

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Density of quadrature domains

Gustafssön in 1982 proved that quadrature domains are dense in the family of bounded smoothly bounded domains in $\mathbb{C}$. A natural question in this regard is whether a density theorem can be proved in higher dimensions.

Theorem (- , K Verma 2015)

Let $D \subset \mathbb{C}^{n-1}$, $n \geq 2$, be a smoothly bounded pseudoconvex domain satisfying Condition R and $\Omega \subset \mathbb{C}$ a smoothly bounded domain. Then there exist quadrature domains arbitrarily close to $D \times \Omega$.

A smoothly bounded domain $D \subset \mathbb{C}^n$, $n \geq 1$ is said to satisfy *Condition R* if

$$P_D(C^\infty(\overline{D})) \subset C^\infty(\overline{D}).$$

Sketch of the proof

Theorem (Key tool)

Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded domains and $f : D_1 \to D_2$ a biholomorphic mapping. Then D_2 is a quadrature domain if and only if the complex Jacobian $u = det[\partial f_i/\partial z_j] \in \mathcal{K}_{D_1}$.

Then we must consider the map $f = ('z, g('z, z_n)) : D \times \Omega \to \mathbb{C}^n$ such that $u = \frac{\partial g}{\partial z_n} \in \mathcal{K}_{D\times\Omega}$. Then $D_2 = f(D \times \Omega)$ will be a quadrature domain if f is biholomorphic.

Theorem (Bell's density lemma)

Let D be a smooth bounded domain in $\mathbb{C}^n$ satisfying Condition R. Then the Bergman span $\mathcal{K}_D$ is dense in $A^\infty(D)$.

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Theorem (Bell's density lemma)

Let D be a smooth bounded domain in $\mathbb{C}^n$ satisfying Condition R. Then the Bergman span $\mathcal{K}_D$ is dense in $A^\infty(D)$.

By the recent work of Debraj and Mei-Chi Shaw, it can be seen that the product domain $D \times \Omega$ satisfies Condition R.

Proposition

For a given $u \in \mathcal{K}_{D\times\Omega}$, there exists $v \in \mathcal{K}_{D\times\Omega}$ such that $\frac{\partial g}{\partial z_n} = v$ admits a single valued holomorphic solution g. Consequently, with this choice of g, we get a single valued holomorphic mapping $f : D \times \Omega \to \mathbb{C}^n$.

If $u \in \mathcal{K}_{D\times\Omega}$ is close to the constant function $h(z) \equiv 1$ in $A^\infty(D \times \Omega)$, the image $f(D \times \Omega)$ is a quadrature domain that is close to the $D \times \Omega$.

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Bell's conjecture

Bell conjectured that every one point quadrature domain with degree one is biholomorphic to a complete circular domain. The family of one point one degree quadrature domains is a large class of domains containing complete circular domains, complete p-circular domains, domains with an invariant action by subgroups of $U(n)$ whose invariant entire functions are constants.

A positive answer to this question would have given tools to transfer results from complete circular domain to one point quadrature domain with degree one. However, the following is a counter-example to the above conjecture

Theorem (-, Jaikrishnan J 2018)

There exists a $(2,3)$-circular domain containing 0 which is not biholomorphic to any complete circular domain.

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Outline of the proof

Theorem (Kaup)

Let $D_1, D_2 \subset \mathbb{C}^n$ be two bounded p-circular domains that are biholomorphic. Then we can find a biholomorphism $f : D_1 \to D_2$ that fixes 0.

Theorem (Ning–Zhou)

Let $D_1, D_2 \subset \mathbb{C}^n$ be a $p = (p_1, \ldots, p_n)$ and $p' = (p'_1, \ldots, p'_n)$-circular domain respectively that contain 0. Let $f : D_1 \to D_2$ be a biholomorphism that fixes 0. Then writing $f = (f_1, \ldots, f_n)$, we have

1. *f is a polynomial mapping.*
2. *$deg(f_i) \leq \max\{|\delta| : \delta \in \mathbb{N}^n, \delta p = \gamma p, |\gamma| = |\beta|, \beta p' = p'_i\}$.*

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The counterexample

Let D be a complete $(2,3)$-circular domain in $\mathbb{C}^2$ that is *not* circular. For instance, we might take

$$D := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 + |z^3 + w^2|^2 < 3.\}$$

To see that D as defined above is not circular, observe that $(-1,1) \in D$ but $(-i,i) \notin D$.

Suppose D were biholomorphic to a complete circular domain Ω. Then by Kaup's result we can find a biholomorphism $f : \Omega \to D$ that fixes 0. The result of Ning–Zhou now implies that f has to be a linear. But linear mappings take circular domains to circular domains and D is not circular by construction.

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