The 3-point spectral Pick interpolation problem

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[1] Spectral Pick interpolation problem

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$ centered at 0. Given $n \in \mathbb{Z}_+$ the set $\Omega_n := \{A \in M_n(\mathbb{C}) : \sigma(A) \subset \mathbb{D}\}$ is the spectral unit ball of dim. n^2, where $M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices and σ denotes the spectrum of a matrix.

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In the case $n = 1$, $(*)$ was first solved by George Pick in 1916 and later by Nevanlinna in 1929, who also found a parametrization of all interpolants.

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Bercovici, Foias and Tannenbaum using a spectral version of the commutant lifting theorem, under the restriction that $\sup_{\zeta \in \mathbb{D}} \rho(F(\zeta)) < 1$, where ρ denotes the spectral radius, provided a characterization for the existence of an interpolant. This characterization involves a search for M appropriate matrices in $GL_n(\mathbb{C})$.

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Agler and Young observed that in the case $W_1, \ldots, W_M$ are all non-derogatory, then $(*)$ is equivalent to an interpolation problem from $\mathbb{D}$ to the n-dimensional symmetrized polydisc G_n, $n \geq 2$. Its relevance to $(*)$ is that, for "generic" matricial data $(W_1, \ldots, W_M)$, the problem $(*)$ descends to a region of much lower dimension with many pleasant properties. This idea has further been developed by Costara, and in Ogle's thesis.

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A matrix $A \in M_n(\mathbb{C})$ is said to be non-derogatory if it admits a cyclic vector. It is a fact that A being non-derogatory is equivalent to A being similar to the companion matrix of its characteristic polynomial.

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Recall: given a monic polynomial $p(t) = t^k + \sum_{j=1}^{k} a_j\, t^{k-j}$, where $a_j \in \mathbb{C}$, the companion matrix of p is the matrix C_p given by:

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By way of the G_n-interpolation problem, Costara and Ogle arrived independently at a necessary condition for the existence of an interpolant for the problem $(*)$ when the data $(W_1, \ldots, W_M)$ are non-derogatory.

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Bharali, later, observed that when $n \geq 3$, the necessary condition given by Costara and Ogle is not sufficient. He also established, for the case $M = 2$, a new necessary condition for the existence of an interpolant which is reminiscent of the inequality in the classical Schwarz lemma.

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Theorem(Bharali 2007). Let $F \in \mathcal{O}(\mathbb{D}, \Omega_n)$, $n \geq 2$, and let $\zeta_1, \zeta_2 \in \mathbb{D}$.

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Bharali also showed in the same article that for each $n \geq 3$, there exists a data-set for which the above condition implies that it cannot admit an interpolant whereas the condition by Costara and Ogle is inconclusive.

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The existence of maps B_A is extremely useful, since the automorphism group of Ω_n does not act transitively on Ω_n, $n \geq 2$ (whence the *classical* Schur algorithm is not even meaningful).

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$_7$Statement of the main result

Theorem 2(V. C., 2018)**.** Let $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{D}$ be distinct points and let $W_1, W_2, W_3 \in \Omega_n$, $n \geq 2$. Let $m(j, \lambda)$ denote the multiplicity of λ as a zero of the minimal polynomial of W_j, $j \in \{1, 2, 3\}$.

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$$\leq \mathcal{M}_{\mathbb{D}}\left(\zeta_{L(k)}, \zeta_{G(k)}\right),$$

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- or there exists a $\theta_0 \in \mathbb{R}$ such that

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Here, $[\cdot]$ denotes the greatest-integer function. Given $a \in \mathbb{C}$ and a function g that is holomorphic in a neighbourhood of a, $\text{ord}_a g$ will denote the order of vanishing of g at a (with the understanding that $\text{ord}_a g = 0$ if g does not vanish at a).

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Remark The above theorem provides a necessary condition that is inequivalent to the necessary conditions hitherto known for this problem. It also incorporates information about the Jordan structure of the matricial data. Infact in the same work of which the above theorem is a part we presented a class of 3-point matricial data in $\mathbb{D} \times \Omega_n$, $n \geq 4$, for which the conditions known before provide no information while Theorem 2 above implies that these data do not admit a $\mathcal{O}(\mathbb{D}, \Omega_n)$-interpolant.

Sketch of proof of the main theorem

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Now we shall use the result by Bharali stated in the begining but first we need to know the minimal polynomials for the matrices $B_k(W_{L(k)}, W_{G(k),\,k}$. This is provided by Theorem 1. Applying these two together gives us the first part of our main theorem.

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- A proper holomorphic correspondence Γ from D_1 to D_2 induces the following set-valued map: $F_\Gamma(z) := \{w \in D_2 : (z, w) \in \Gamma\} \;\; \forall z \in D_1$.

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$$\Gamma := \{(z, w) : w^n + \sum_{j=1}^{n} (-1)^j \mathscr{S}_j(\text{spec}(F(z)))w^{n-j} = 0\},$$

where $\mathscr{S}_j$ is the jth elementary symmetric polynomial in n indeterminates.