Complex geometry of Teichmüller domains

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Overview

1. Complex Geometry of Convex Domains
   - Geometry of intrinsic metrics
   - Automorphisms of Convex Domains

2. Complex Geometry of Teichmüller Space
   - Preliminaries

The Kobayashi and Caratheodory metrics

On any bounded domain Ω in $\mathbb{C}^N$, the *infinitesimal Kobayashi metric* is defined by the following norm for a tangent vector v at a point $p \in \Omega$:

$$K_\Omega(p, v) = \inf_{h:\Delta\to\Omega} \frac{|v|}{|h'(0)|} \tag{1}$$

where the infimum is over all holomorphic maps $h : \Delta \to \Omega$ such that $h(0) = p$ and $h'(0)$ is a multiple of v.

The *Kobayashi metric* d_Ω^K on Ω is then the distance defined in the usual way: one first defines lengths of piecewise C^1 curves in Ω using the above norm and then takes the infimum of lengths of curves joining two given points to get the distance between them.

The Kobayashi and Caratheodory metrics

The *Carathéodory metric* d_Ω^C on Ω is defined by

$$d_\Omega^C(p,q) = \sup\{d_\Delta(f(p), f(q))\},$$

where d_Δ denotes the Poincaré metric on Δ and the supremum is over all holomorphic maps from Ω to Δ. Note that d_Δ is the distance function associated to the Kähler metric $4\frac{dz \otimes \overline{dz}}{(1-|z|^2)^2}$ on Δ.

The following is immediate from the definitions: For any $p, q \in \Omega$ we have

$$d_\Omega^C(p,q) \leq d_\Omega^K(p,q) \tag{2}$$

and we have the contracting property

$$d_{\Omega_2}^K(f(p), f(q)) \leq d_{\Omega_1}^K(p,q) \text{ and } d_{\Omega_2}^C(f(p), f(q)) \leq d_{\Omega_1}^C(p,q) \tag{3}$$

for any holomorphic map $f : \Omega_1 \to \Omega_2$.

Complex Geodesics

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. A *complex geodesic* in Ω is a holomorphic and isometric embedding $\tau : (\mathbb{H}, d^K_{\mathbb{H}}) \to (\Omega, d^K_{\Omega})$. As noted above, the Kobayashi $d^K_{\mathbb{H}}$ on $\mathbb{H}$ is the distance function of the hyperbolic metric, and is isometric to the Poincaré disk (Δ, d_Δ).

It is a remarkable fact, due to Lempert and generalized by Royden–Wong, that complex geodesics exist in abundance when Ω is a bounded convex domain.

Complex Geodesics in Convex Domains

Theorem (Lempert, Royden–Wong)

Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain.

(i) For any $p, q \in \Omega$ there exists a complex geodesic $\tau : \mathbb{H} \to \Omega$ with $\tau(0) = p$ and $\tau(z) = q$ for some $z \in \mathbb{H}$.

(ii) For any $p \in \Omega$ and $v \in \mathbb{C}^n$ there exists a complex geodesic $\tau : \mathbb{H} \to \Omega$ with $\tau(0) = p$ and $\tau'(0) = tv$ for some $t > 0$.

This was proved for domains with $\mathcal{C}^2$-smooth strongly convex domains by Lempert and generalized to arbitrary convex domains by Royden and Wong.

Complex Geodesics in Convex Domains

The second major result proved by Lempert is the existence of *holomorphic retracts*:

Theorem (Lempert, Royden–Wong)

Let $\tau : \mathbb{H} \to \Omega$ be a complex geodesic. Then there exists a holomorphic map $\Phi : \Omega \to \mathbb{H}$ satisfying

$$\Phi \circ \tau \; = \; id : \mathbb{H} \to \mathbb{H}$$

These theorems immediately imply the following

Corollary

If Ω is a bounded convex domain, then

$$d_\Omega^K = d_\Omega^C .$$

A Theorem of S. Frankel

The following result answered a question of S.-T. Yau:

Theorem (Frankel)

Let $\Omega \subset \mathbb{C}^n$ a bounded convex domain. If there is a discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ acting freely on Ω such that Ω/Γ is compact, then Ω is biholomorphic to a bounded symmetric domain.

It is unknown if this results if the assumption of compactness of Ω/Γ is relaxed to finiteness of volume (with respect to some intrinsic metric).

Let X be a compact Riemann surface of genus $g \geq 2$. Let K_X denote the canonical bundle of X.

A *holomorphic quadratic differential* on X is a holomorphic section of $K_X \otimes K_X$. Let $\mathcal{Q}(X)$ be the space of holomorphic quadratic differentials on X.

A *Beltrami differential* on X is a L^∞-section of $K_X^{-1} \otimes \overline{K}_X$. We denote the space of Beltrami differentials on X by $\mathcal{BD}(X)$. In a holomorphic chart $U \subset X$, an element of $\mathcal{BD}(X)$ has the form

$$\mu \frac{d\bar{z}}{dz}$$

where $\mu \in L^\infty(U)$ is called a *Beltrami coefficient*.

Preliminaries

By the Uniformization Theorem, the universal cover of the surface X is the upper half plane $\mathbb{H}$, and we have

$$X = \mathbb{H}/\Gamma$$

for Γ a *Fuchsian group*, namely a discrete subgroup of $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$.

A *quasiconformal map* between two Riemann surfaces X and Y is a homeomorphism $f : X \to Y$ with weak partial derivatives (in the sense of distributions) that are locally square-integrable, such that the Beltrami coefficient

$$\mu = \frac{f_{\bar{z}}}{f_z} \tag{4}$$

satisfies $\|\mu\|_\infty < 1$.

Preliminaries

From this perspective, one can define Teichmüller space by considering quasiconformal maps from a fixed basepoint, up to homotopy (or isotopy):

$\mathcal{T}_g = \{(Y, k) \ : \ Y$ is a Riemann surface, $k : X \to Y$ is quasiconformal map $\}$

where $(Y, k) \sim (Z, h)$ if and only if there is a *conformal* homeomorphism $c : Y \to Z$ such that the composition

$$f = h^{-1} \circ c \circ k : X \to X$$

is a quasiconformal map that lifts to a map $\tilde{f} : \mathbb{H} \to \mathbb{H}$ that extends to the identity map on $\mathbb{R}$. Note that the last condition is equivalent to the map f being homotopic to the identity map.

The Teichmüller metric

The *Teichmüller distance* between two marked surfaces X and Y is defined by

$$d_{\mathcal{T}}(X, Y) = \frac{1}{2} \inf_{f} \ln K(f) \tag{5}$$

where the infimum is taken over quasiconformal homeomorphisms preserving the marking and

$$K(f) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

, where μ is the *quasiconformal dilatation* of f.

The infimum is, in fact, attained by a *Teichmüller map* $\Psi : X \to Y$.

The Bers Embedding Theorem

We endow Teichmüller space with the topology by $d_{\mathcal{T}}$.

Theorem (L. Bers)

$\mathcal{T}_g$ can be embedded as a bounded contractible domain in $\mathbb{C}^{3g-3}$.

We regard Teichmüller as a complex manifold with the induced complex structure.

Royden's Theorem

The *mapping class group* MCG(S) is the group of self-homeomorphisms of S up to homotopy. It is immediate from the definitions that each element of MCG(S) is a holomorphic isometry of $(\mathcal{T}_g, d_{\mathcal{T}})$.
One has the following fundamental results of H. Royden:

Theorem

(i) The Teichmüller metric is equal to the Kobayashi metric on $\mathcal{T}_g$.
(ii) $\mathrm{Aut}(\mathcal{T}_g) = \mathrm{Isom}(\mathcal{T}_g, d_{\mathcal{T}}) = \mathrm{MCG}(S)$.

Complex Geodesics in $\mathcal{T}$

Theorem (Earle–Kra–Krushkal)

(i) For any $p, q \in \mathcal{T}$ *there exists a complex geodesic* $\tau : \mathbb{H} \to \mathcal{T}$ *with* $\tau(0) = p$ *and* $\tau(z) = q$ *for some* $z \in \mathbb{H}$.

(ii) For any $p \in \mathcal{T}$ *and* $v \in \mathbb{C}^n$ *there exists a complex geodesic* $\tau : \mathbb{H} \to \mathcal{T}$ *with* $\tau(0) = p$ *and* $\tau'(0) = tv$ *for some* $t > 0$.

It was unknown if, as in the convex domain case, there are holomorphic retracts for every complex geodesic.

Siu's Conjecture

Conjecture: The Teichmüller space of a closed surface of genus $g \geq 2$ cannot be biholomorphic to any convex domain in $\mathbb{C}^{3g-3}$.

It is known that there is a discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathcal{T}_g)$ acting freely on $\mathcal{T}_g$ such that the quotient $\mathcal{T}_g/\Gamma$ has finite Kobayashi volume. However $\mathcal{T}_g/\Gamma$ is noncompact and a generalization of Frankel's result to the case of convex domains with finite-volume quotients is not known.

Markovic's Theorem

By the work of Lempert, the existence of a convex embedding of $\mathcal{T}_g$ would imply that the Kobayashi and Carathéodory metrics coincide. In fact, it was proved by I. Kra that these two metrics agree on Teichmüller disks associated to Abelian quadrat differentials. However, this is not true for all complex geodesics:

Theorem (v. Markovic)

If $g \geq 2$, $d_{\mathcal{T}}^K \neq d_{\mathcal{T}}^C$.

In particular, $\mathcal{T}_g$ cannot be biholomorphic to a bounded convex domain, confirming Siu's Conjecture .

Local Convexity

It turns out that a Teichmüller domain cannot even be *locally* strictly convex.

We say that a domain $\Omega \subset \mathbb{C}^n$ is *locally convex* at $p \in \partial\Omega$ if $\Omega \cap B(p, r)$ is convex for some $r > 0$, where $B(p, r)$ denotes the Euclidean ball with center p and radius r. Moreover, it is locally *strictly* convex if $\Omega \cap B(p, r)$ is strictly convex.

Theorem (S. Gupta - H. S.)

The Teichmüller space of a closed surface of genus $g \geq 2$ cannot be biholomorphic to a domain in $\mathbb{C}^{3g-3}$ with a locally strictly convex boundary point.

This has some similarities to Rosay's localization of Wong's result.

Local Convexity

It is an elementary and well-known fact that any bounded domain with C^2-smooth boundary has a locally strictly convex boundary point. Hence one has:

Corollary

The Teichmüller space of a closed surface of genus $g \geq 2$ cannot be biholomorphic to a domain in $\mathbb{C}^{3g-3}$ with C^2-smooth boundary.

The strategy of the proof is inspired by K.-T. Kim's proof of the following result:

Theorem (Kim)

For $g \geq 2$, the Bers embedding of $\mathcal{T}_g$ is not convex.

As in Kim's proof, our proof involves two distinct components. Let $\Omega \subset \mathbb{C}^{3g-3}$ be a Teichmüller domain with a locally strictly convex boundary point $p \in \partial\Omega$.

Step 1. The goal is to show that any point p as above is an orbit accumulation point for $\mathrm{Aut}(\Omega) = \mathrm{MCG}(S)$.

A sketch of the proof

In Kim's work, one has the corresponding result for the Bers domain: *every* point of the Bers boundary is an orbit accumulation point. This follows from McMullen's result that cusps are dense in the Bers boundary, since powers of a Dehn twist converge to such cusps. Note that the orbit accumulation point property may not be preserved under biholomorphisms of domains as biholomorphism may not extend in an absolutely continuous manner (or indeed, even as a well-defined function!) to closures of domains.

A sketch of the proof

Step 2: This step consists of *rescaling* at a "smooth" orbit accumulation point to obtain an unbounded convex domain Ω_∞ in the limit with the following properties: (i) Ω_∞ is biholomorphic to Ω and (ii) $\mathrm{Aut}(\Omega_\infty)$ contains a one-parameter subgroup. These lead to a contradiction: Royden's theorem and (i) together imply that $\mathrm{Aut}(\Omega_\infty)$ is the discrete group MCG(S).

If Ω is a domain in $\mathbb{C}^N$, we say that a point $q \in \partial\Omega$ is *Alexandrov smooth* if $\partial\Omega$ is the graph, near q, of a function which has a second-order Taylor expansion at q. It is a theorem of A.D. Alexandrov that almost every point of $\partial\Omega$ is smooth in this sense if Ω is convex. In particular, if Ω is locally convex at q then we can assume that q is Alexandrov smooth, without loss of generality.

A sketch of the proof

The original rescaling argument of Pinchuk requires the C^2-smoothness of the boundary of the domain under consideration. However, in Frankel's setting or the case of the Bers domain, the boundary is far from being smooth. In his work Frankel introduced a rescaling technique along an orbit of the automorphism group accumulating at a boundary point which dispenses with the C^2-smoothness assumption.

Subsequently, Kim and Krantz ([**?**]) developed a variant of Pinchuk's method in the convex case which does not require a C^2-smooth boundary and proved that, in fact, this recovers Frankel's rescaling. In brief, they show the following: Suppose Ω is a domain, locally convex at an Alexandrov smooth point $q \in \partial\Omega$. Suppose that $p_j = \gamma_j(p) \to q$ for $\gamma_j \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$. Then there exist invertible complex affine transformations $A_j : \mathbb{C}^N \to \mathbb{C}^N$ such that the biholomorphic embeddings $A_j \circ \gamma_j : \Omega \to \mathbb{C}^N$ converge, on compact sets, to an embedding $\psi : \Omega \to \mathbb{C}^N$. Moreover, the convex domains $A_j \circ \gamma_j(\Omega)$ converge in the local Hausdorff sense (The local convexity at a boundary point is crucial to obtain a subconvergent sequence of domains.) The limiting image $\psi(\Omega)$ is

A sketch of the proof

Orbit accumulation points:
The proof of the orbit accumulation property in the first step is based on two basic results in Teichmüller theory:

First, the abundance of complex geodesics in Teichmüller space. More precisely, one knows that through every point $p \in \Omega$ and every direction $v \in T_p\mathcal{T}_g$, there exists a Teichmüller disk $\tau : \Delta \to \mathcal{T}_g$ with $\tau(0) = p$ and $\tau'(0) = tv$ for some $t > 0$.

Second, the Masur-Veech ergodicity theorem for the Teichmüller geodesic flow. Their work implies that for any Teichmüller disk, almost every radial ray gives rise to a geodesic ray in $\mathcal{T}_g$ that projects to a dense set in moduli space $\mathcal{M}_g := \mathcal{T}_g/\mathrm{MCG}(S)$. In particular, there is a sequence of points along such rays that recur to any fixed compact set in $\mathcal{M}_g$.

A sketch of the proof

Given these facts, the proof involves an elementary but delicate analysis of the boundary behaviour of holomorphic functions on the unit disk in $\mathbb{C}$.

We choose a point $p' \in \Omega$ which is close to the strictly convex point $p \in \partial\Omega$ and a complex geodesic $\tau : \Delta \to \Omega$ with $\tau(0) = p'$. Since the boundary point p is locally strictly convex, there is a pluriharmonic "barrier" function h , namely the height from a supporting hyperplane at p, whose sub-level sets nest down to the single point p. This allows us to "trap" the complex geodesic τ, by proving the existence of a positive measure set of radial directions in Δ which under the holomorphic map τ limit to boundary points arbitrarily close to p. One can then apply the Masur-Veech ergodicity result to infer the existence of an orbit points shadowing such a radial ray, which accumulate arbitrarily close to p.