Finiteness theorems for the space of holomorphic mappings

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Notation

- Let X and Y be complex manifolds. We will use the following notations:
  - $\mathcal{O}(X, Y)$ denotes the space of holomorphic mappings from X into Y.
  - $\mathcal{O}_*(X, Y)$ denotes the space of non-constant holomorphic mappings from X into Y.
  - $\mathcal{O}_{\mathrm{dom}}(X, Y)$ denotes the space of dominant holomorphic mappings from X into Y, i.e., those mappings for which the range has a non-empty interior.
- All function spaces will given the compact-open topology.

Some classical finiteness/non-existence theorems

1. Our objective is to study scenarios under which the cardinality of $\mathcal{O}_*(X, Y)$ or $\mathcal{O}_{\text{dom}}(X, Y)$ is finite.
2. **Liouville's Theorem:** If D is a bounded planar domain then $\mathcal{O}_*(\mathbb{C}, D) = \emptyset$.
3. **Little Picard Theorem:** $\mathcal{O}_*(\mathbb{C}, \mathbb{C} \setminus \{0, 1\}) = \emptyset$.
4. **Theorem of de Franchis:** If R and S are compact Riemann surfaces both of genus at least 2 then $\mathcal{O}_*(R, S)$ is a finite set.
5. $\mathcal{O}_*(\mathbb{C} \setminus \{0, 1\}, \mathbb{C} \setminus \{0, 1\})$ is a finite set.

Imayoshi's Theorem

Result (Imayoshi, 1982)

Let R be a Riemann surface of finite type and let S be a Riemann surface of finite type (g, n) with $2g - 2 + n > 0$. Then $O_*(R, S)$ is a finite set.

A *Riemann surface of finite type* (g, n) is a Riemann surface that is biholomorphic to a Riemann surface obtained by removing n points from a compact Riemann surface of genus g.

Our main result

Theorem (Divakaran and Jaikrishnan, 2017, IJM)

Let $X := X_1 \times \cdots \times X_n$ be a product of hyperbolic Riemann surfaces of finite type and let $Y = \Omega/\Gamma$ be an m-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and Γ is fixed-point-free discrete subgroup of $Aut(\Omega)$.

1. *If N is a tautly embedded complex submanifold of Y then $\mathcal{O}_{dom}(X, N)$ is a finite set.*
2. *If Y is geometrically finite and Ω is complete hyperbolic then $\mathcal{O}_{dom}(X, Y)$ is a finite set.*
3. *In addition to the conditions in (2), if the essential boundary dimension of Ω is zero, then $\mathcal{O}_*(X, Y)$ is a finite set.*

Tautness and normal families

- If M and N are two Kobayashi hyperbolic complex manifolds, then by the distance decreasing property of the Kobayashi distance, the space $\mathcal{O}(M, N)$ is an equicontinuous family.

Definition

A subset $\mathcal{F} \subset C(M, N)$ is said to be a *normal family* if every sequence $\{f_n\} \subset \mathcal{F}$ has either a subsequence that converges uniformly on compacts to a function in $C(M, N)$ or has a compactly divergent subsequence.

A complex manifold N is said to be *taut* if for every complex manifold M the set $\mathcal{O}(M, N)$ is a normal family.

Let N be a complex manifold and let Y be a complex submanifold. We say that Y is *tautly embedded* in N if every sequence of holomorphic mappings $\{f_n : M \to Y\}$, where M is any complex manifold, admits a subsequence that converges uniformly on compacts to a holomorphic map $f : M \to N$.

- Complete hyperbolic manifolds are taut.

Meaning of technical terms

Instead of giving of precise definitions, we will try to explain, loosely, the meanings of the various technical terms.

1. Every topological manifold can be assigned a *space of ends* which is roughly the various connected components after excising a suitably large connected compact set.
2. A geometrically finite complex manifold is one that has only finitely many ends each of which satisfies a certain technical condition.
3. Riemann surfaces of finite type are geometrically finite.
4. The essential boundary dimension of a bounded domain in $\mathbb{C}^n$ is roughly the maximal dimension of analytic sets sitting in ∂D. The unit ball in $\mathbb{C}^n$ and more generally strictly pseudoconvex domains have essential boundary dimension 0 whereas the polydisk has essential boundary dimension $n - 1$.

A rigidity result

We will now give a sketch of our proof in the special case that X and Y are compact hyperbolic Riemann surfaces. We need a rigidity result.

Theorem

Let $X := \mathbb{D}/G$ and $Y := \mathbb{D}/\Gamma$ be compact hyperbolic Riemann surfaces. If non-constant holomorphic mappings $\phi, \psi : X \to Y$ induce the same homomorphisms from G to Γ then $\phi = \psi$.

What is the induced homomorphism?

Let $\widetilde{x} \in \mathbb{D}$ and $g \in G$. Let $\widetilde{\phi}$ be a lift. This lifting induces a homomorphism $\chi : G \longrightarrow \Gamma$ which can be described in two ways:

1. There is an unique element $h \in \Gamma$ that takes $\widetilde{\phi}(\widetilde{x})$ to $\widetilde{\phi}(g((\widetilde{x}))$. The choice of h is independent of the choice of $\widetilde{x}$ and we get a homomorphism χ of groups. We have the following relationship

$$\widetilde{\phi} \circ g = \chi(g) \circ \widetilde{\phi}.$$

2. Let $x \in X$ be the image of $\widetilde{x}$ under the quotient map and let $y := \phi(x)$. Let $\Phi : G \longrightarrow \pi_1(X, x)$ and $\Lambda : \pi_1(Y, y) \longrightarrow \Gamma$ be the natural isomorphisms. Then $\Lambda \circ \phi_* \circ \Phi$, where $\phi_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$ is the induced map on fundamental groups, is the required homomorphism.

Proof sketch

- Let $\{f_k\} \subseteq \mathcal{O}(X, Y)$ be a sequence of distinct non-constant holomorphic mappings. We may assume that the sequence $\{f_k\}$ converges in the compact-open topology to a map $f : X \to Y$.
- We show that for suitably large k, we can find lifts $\widetilde{f_k}$ and $\widetilde{f}$ that induce the same homomorphism on G.
- We first choose k suitably large so that $z_k := f_k(x)$ and $y := f(x)$ belong to an evenly covered coordinate ball in Y, say U.
- Choose $\widetilde{f}$ and $\widetilde{f_k}$ to be the lifts of f and f_k, respectively, such that $\widetilde{f}(\widetilde{x}), \widetilde{f_k}(\widetilde{x}) \in \widetilde{U}$.
- Let χ and χ_k be the homomorphism induced by $\widetilde{f}$ and $\widetilde{f_k}$, respectively.
- Each $g \in G$ can be represented by a closed loop based at x, say γ. Then $f \circ \gamma$ and $f_k \circ \gamma$ are loops in Y based at y and z_k, respectively.

Proof sketch...

- Let $\sigma := f \circ \gamma$ and $\sigma_k := \overline{\delta}_k * (f_k \circ \gamma) * \delta_k$ be two loops based at the point y, where δ_k is a curve lying in U that connects y to z_k.
- Now $\sigma_k \to \sigma$ uniformly. Therefore σ and σ_k are equivalent in $\pi_1(Y, y)$ for suitably large k.
- Let $\widetilde{\sigma}$ and $\widetilde{\sigma}_k$ be the lifts of σ and σ_k, respectively, that start at $\widetilde{f}(\widetilde{x})$. As σ and σ_k represent the same element in $\pi_1(Y, y)$, the endpoints of $\widetilde{\sigma}$ and $\widetilde{\sigma}_k$ must be the same and equal to $\chi(g)\left(\widetilde{f}\,(\widetilde{x})\right)$.

Conclusion of proof sketch

Since the quotient map is a homeomorphism from $\widetilde{U}$ to U and δ_k lies entirely in U, a lift of δ_k starting at $\widetilde{f}(\widetilde{x})$ ends in $\widetilde{U}$. Similarly, a lift of $\overline{\delta}_k$ that ends in $\chi(g)\left(\widetilde{U}\right)$ has to begin in $\chi(g)\left(\widetilde{U}\right)$. Thus a lift of $f_k \circ \gamma$ starting in $\widetilde{U}$ (at $\delta_k(1)$) has to end in $\chi(g)\left(\widetilde{U}\right)$. Thus, $\chi_k(g)\left(\widetilde{U}\right) \cap \chi(g)\left(\widetilde{U}\right) \neq \emptyset$. Since U is an evenly covered neighborhood, it follows that $\chi_k(g) = \chi(g)$.

THANK YOU