3 Dynamics at or through a phase transition

Take a piece of material in contact with an external reservoir. The material will be characterized by certain observables, energy density, magnetization density, *etc.* The external environment will be characterized by some parameters, like the temperature, magnetic field, pressure, *etc.* In principle, one is able to tune the latter and study the variation of the former. Note that we are using a **canonical setting** in the sense that the system under study is not isolated but open.

Sharp changes in the behavior of macroscopic systems at critical points (or lines) in parameter space have been observed experimentally. These correspond to **equilibrium phase transitions**, a non-trivial collective phenomenon appearing in the thermodynamic limit. We will assume that the main features of, and analytic approaches used to study, phase transitions are known.

Imagine now that one changes an external parameter instantaneously or with a finite rate going from one phase to another in the (equilibrium) phase diagram. The kind of internal system interactions are not changed. In the statistical physics language the first kind of procedure is called a **quench** and the second one an **annealing** and these terms belong to the metalurgy terminology. We will investigate how the system evolves by trying to accomodate to the new conditions and equilibrate with its environment. We will first focus on the dynamics at the critical point or going through phase transitions between well-known phases (in the sense that one knows the order parameter, the structure, and all thermodynamic properties on both sides of the transition). Later we will comment on cases in which one does not know all characteristics of one of the phases and sometimes one does not even know whether there is a phase transition.

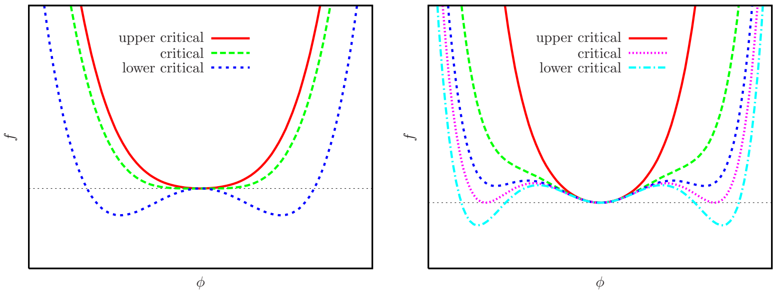

Figure 22: Left: second-order phase transition. Right: first order phase transition.

The evolution of the **free-energy landscape** (as a function of an order parameter) with the control parameter driving a phase transition is a guideline to grasp the dynamics following a quench or annealing from, typically, a disordered phase to the phase transition or into the ordered phase. See Fig. 22 for a sketch. We will discuss quenches to the phase transition and below it. In the former case, the system can get to a critical point (Fig. 22-left) in which the free-energy is metastable in the sense that its second derivative vanishes (second order phase transition cases) or to a first-order phase transition (Fig. 22-right) in which various minima are degenerate. In the latter case the initial state becomes **unstable**, that is to say a maximum, and the phase transition is of second-order (see Fig. 22-left) or **metastable**, that is to say a local minimum, and the phase transition is of first order (see Fig. 22-right) in the final externally imposed conditions.^{10} In the former case the **ordering process** occurs **throughout the material**, and not just at **nucleation sites**. Two typical examples are spinodal decomposition, *i.e.* the method whereby a mixture of two materials (with equal overall concentration) can separate into distinct regions with different material concentrations, or magnetic domain growth in ferromagnetic materials. Instead, in the latter case, the stable phase conquers the system through the **nucleation of a critical localized bubble** via thermal activation and its further growth. An example is the easy-axis magnetization reversal under an abrupt change of the sign of the applied field or phase separation across first order phase transitions.

Having described the dependence of the free-energy landscape on the external parameters we now need to choose the microscopic dynamics of the order parameter. Typically, one distinguishes two classes: one in which the order parameter is locally conserved and another one in which it is not. **Conserved** order parameter dynamics are found for example in phase separation in magnetic alloys or inmiscible liquids. Ferromagnetic domain growth is an example of the **non-conserved** case.

The kinetics of systems undergoing critical dynamics or an ordering process is an important problem for material science but also for our generic understanding of pattern formation in non-equilibrium systems and the approach to equilibrium in systems with slow dynamics that are yet not well understood. The late stage dynamics is believed to be governed by a few properties of the systems whereas material details should be irrelevant. Among these relevant properties one may expect to find the number of degenerate ground states, the nature of the conservation laws and the hardness or softness of the domain walls that is intimately related to the dimension of the order parameter. Thus, classes akin to the universality ones of critical phenomena have been identified. These systems constitute a first example of a problem with **slow dynamics**. Whether all systems with slow dynamics, in particular structural and spin glasses, undergo some kind of simple though slow growth of order is an open question.

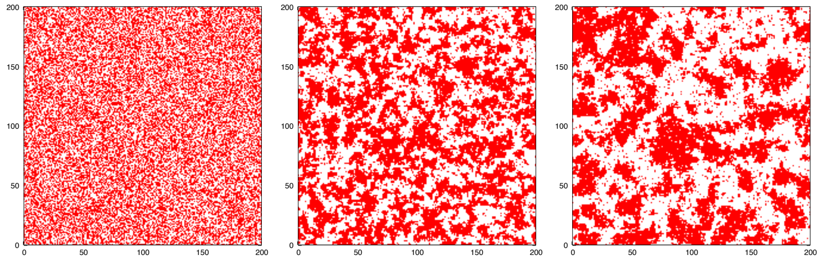

Figure 23: Monte Carlo simulations of a $2d$ Ising model. Three snapshots at $t = 1,\ 3 \times 10^5,\ 3 \times 10^6$ MCs after a quench to T_c.

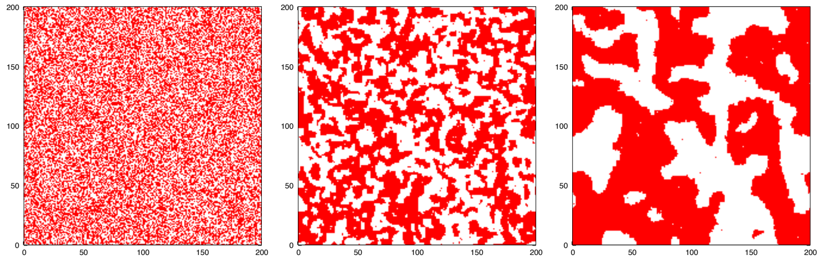

Figure 24: Monte Carlo simulations of a $2d$ Ising model. Three snapshots at $t = 1,\ 3 \times 10^5,\ 3 \times 10^6$ MCs after a quench to $0.5\ T_c$. Thermal fluctuations within the domains are visible.

3.1 Snapshots

Take a magnetic system, such as the ubiquitous Ising model with ferromagnetic uniform interactions, and quench it to its Curie point or into the low temperature phase starting from a random initial condition. Classically, the spins do not have an intrinsic dynamics; it is defined via a stochastic rule of Glauber, Metropolis, Langevin (when soft-spins are used) or similar type with or without locally conserved magnetization. For the purpose of the following discussion it is sufficient to focus on non-conserved local microscopic dynamics. Three snapshots taken after times 1, 3×10^5 and 3×10^6 MCs in a critical and two sub-critical quenches at different T below T_c are shown in Figs. 23, 24, and 25, respectively.

Time-dependent macroscopic observables are then expressed in terms of the values of the spins at each time-step. For instance, the magnetization density and its two-

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[10] Strictly speaking metastable states with infinite life-time exist only in the mean-field limit.

time self correlation function are defined as

$$m(t) \equiv N^{-1} \sum_{i=1}^{N} \langle \, s_i(t) \, \rangle \; , \qquad C(t,t') \equiv N^{-1} \sum_{i=1}^{N} \langle \, s_i(t) s_i(t') \, \rangle \; , \tag{3.1}$$

where the angular brackets indicate an average over many independent runs (i.e. random numbers) starting from identical initial conditions and/or averages over different initial configurations.

In **critical quenches**, patches with equilibrium critical fluctuations grow in time but their linear extent never reaches the equilibrium correlation length that diverges. Clusters of neighbouring spins pointing the same direction of many sizes are visible in the figures and the structure is quite intricate with clusters within clusters and so on and so forth. The interfaces look pretty rough too.

In **quenches into the ordered phase through a second order phase transition** the ferromagnetic interactions tend to align the neighbouring spins in parallel direction and in the course of time domains of the two ordered phases form and grow, see Fig. 26. At any finite time the configuration is such that the two types of domains co-exist. If one examines the configurations in more detail one reckons that there are some spins reversed within the domains. These 'errors' are due to thermal fluctuations and are responsible of the fact that the magnetization of a given configuration within the domains is smaller than one and close to the equilibrium value at the working temperature (apart from fluctuations due to the finite size of the domains). The total magnetization, computed over the full system, is zero (up to fluctuating time-dependent corrections that scale with the square root of the inverse system size). The thermal averaged spin, $\langle s_i(t) \rangle$ vanishes for all i and all finite t, see below for a more detailed discussion of the time-dependence. As time passes the typical size of the domains increases and the interfaces get flatter in a way that we will also discuss below.

Quenches across first order phase transitions will be discussed separately in Sec. .

3.2 Relaxation, equilibration and reversal times

We wish to distinguish the **relaxation time**, t_r, defined as the time needed for a given initial condition to reach equilibrium in one of the (possibly many equivalent) phases, from the **decorrelation time**, t_d, defined as the time needed for a given configuration to decorrelate from itself. To lighten the notation we do not signal out the variable that we use to study these typical times (as we did with the velocity and position in the examples of Sect. 2.4). We further define the **reversal time**, t_R, as the time needed to go from one to another of the equivalent equilibrium phases. We focus on a second-order phase transition with broken symmetry between to two equilibrium states here.

3.2.1 Quench from $T \gg T_c$ to $T > T_c$

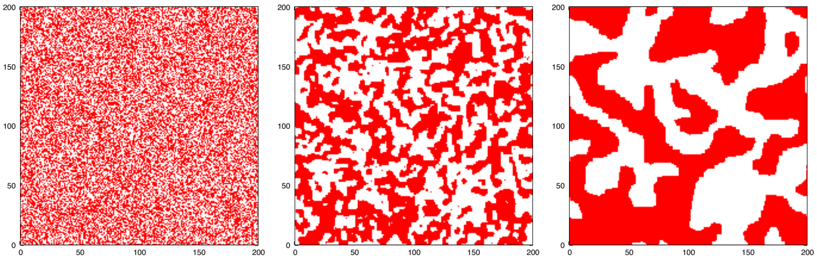

Figure 25: Monte Carlo simulations of the $2d$ Ising model. Three snapshots at $t = 1,\ 3\times10^5, 3\times10^6$ MCs after a quench to $0.01\ T_c$. There is almost perfect order within the domains ($m_{eq} \simeq 1$).

If one quenches the system to $T > T_c$ the relaxation time, t_r, needed to reach configurations sampled by the Boltzmann measure depends on the system's parameters but, most importantly, remains finite even for an infinite-size system. Once a short transient overcome, the average of a local spin approaches the limit given by the Boltzmann measure, $\langle s_i(t)\rangle \to \langle s_i\rangle_{eq} = m = 0$, for all i and all other more complex observables satisfy equilibrium laws. The relaxation time is estimated to behave as $|T-T_c|^{-\nu z_{eq}}$ close to T_c, with ν the critical exponent characterizing the divergence of the equilibrium correlation length, $\xi_{eq} \sim (T-T_c)^{-\nu}$, and z_{eq} the equilibrium exponent that links times and lengths, $R_c(t) \sim t^{1/z_{eq}}$.

The relaxation of the two-time self-correlation at $T > T_c$, when the time t' is chosen to be longer than t_r, decays exponentially

$$\lim_{t'\gg t_r} \langle s_i(t)s_i(t')\rangle \simeq e^{-(t-t')/t_d} \tag{3.2}$$

with a decorrelation time that increases with decreasing temperature and close to (but still above) T_c diverges as the power law, $t_d \sim (T-T_c)^{-\nu z_{eq}}$. The divergence of t_d is the manifestation of **critical slowing down**. The asympotic value verifies

$$\lim_{t-t'\gg t'\gg t_r} \langle s_i(t)s_i(t')\rangle = \lim_{t\gg t_r} \langle s_i(t)\rangle \lim_{t'\gg t_r} \langle s_i(t')\rangle = \langle s_i\rangle_{eq}\langle s_i\rangle_{eq} = m^2 = 0\ , \tag{3.3}$$

cfr. eq. (2.75).

Grosso modo the behavior of the spin degree of freedom is equivalent to the one derived for the position of a particle in a harmonic potential centered at zero with non vanishing positive spring constant in Sec. 2.4.2.

3.2.2 Quench from $T \gg T_c$ to $T \leq T_c$

At or below T_c, coarsening from an initial condition that is **not correlated with the equilibrium state** and with no bias field does not take the system to equilibrium

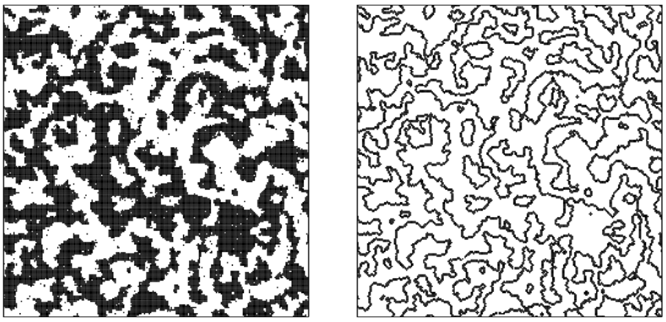

Figure 26: Snapshot of the $2d$ Ising model at a number of Monte Carlo steps after a quench from infinite to a subcritical temperature. Left: the up and down spins on the square lattice are represented with black and white sites. Right: the domain walls are shown in black.

in finite times with respect to a function of the system's linear size, L. More explicitly, if the growth law is a power law [see eq. (3.22)] one needs times of the order of $L^{z_{eq}}$ (critical) or L^{z_d} (subcrititcal) to grow a domain of the size of the system. This gives a rough idea of the time needed to take the system to one of the two equilibrium states. For any shorter time, domains of the two types exist and the system is **out of equilibrium**.

The self-correlation of such an initial state evolving at $T \leq T_c$ involves power laws or logarithms and although one cannot associate to it a decay time as one does to an exponential, one can still define a characteristic time that, quite generally, turns out to be related to the age of the system, $t_d \simeq f(t')$ [see eq. (3.21)], where t' is the first time at which the configuration of the system is measured to compute the correlation function.

In contrast, the relaxation time of an **equilibrium** magnetized configuration at temperature T vanishes since the system is already equilibrated while the decorrelation time t_d is a finite function of T.

The relaxation of the two-time self-correlation at $T < T_c$, when the time t' is chosen to be longer than t_r, that is to say, once the system has thermalized in one of the two equilibrium states, decays exponentially

$$\langle s_i(t) s_i(t') \rangle \simeq e^{-(t-t')/t_d} \tag{3.4}$$

with a decorrelation time that decreases with decreasing temperature and close to T_c (but below it) also diverges as a power law, $t_d \sim (T - T_c)^{-\nu z_{eq}}$. The asympotic value

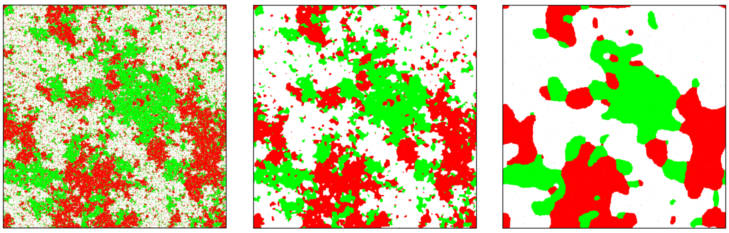

Figure 27: Snapshot of the $2d$ Potts model with $q = 3$ at a number of Monte Carlo steps after a quench from the critical to a subcritical temperature.

verifies

$$\lim_{t-t' \gg t' \gg t_r} \langle s_i(t) s_i(t') \rangle = \lim_{t \gg t_r} \langle s_i(t) \rangle \lim_{t' \gg t_r} \langle s_i(t') \rangle = \langle s_i \rangle_{eq} \langle s_i \rangle_{eq} = m^2 \geq 0 \; , \tag{3.5}$$

cfr. eqs. (2.75) and (3.3), depending on $T = T_c$ or $T > T_c$. Again, *grosso modo* the behavior of the spin degree of freedom for an ordered initial condition at $T < T_c$ is equivalent to the one derived for the position of a particle in a harmonic potential centered at a non-vanishing position with non vanishing positive spring constant in Sec. 2.4.2.

3.2.3 Summary

The lesson to learn from this comparison is that the relaxation time and the decorrelation time not only depend upon the working temperature but they also depend upon the initial condition.

In all critical or low-temperature cases we will study the relaxation time also depends on the system size L – and diverges in the infinite size limit. In short, for a random initial condition and an infinite system, $L \to \infty$, one has

$$t_r^\phi \simeq \begin{cases} \text{finite} & T > T_c \; , \\ |T - T_c|^{-\nu z_{eq}} & T \gtrsim T_c \; , \\ \infty & T \leq T_c \end{cases}$$

while for a finite system, $L < +\infty$, one finds

$$t_r^\phi \simeq \begin{cases} L^{z_{eq}} & T = T_c \; , \\ L^{z_d} & T < T_c \; . \end{cases}$$

For a random initial condition quenched to the critical or sub-critical temperature the decorrelation time grows (and diverges) with the waiting time t' but does not diverge with the system size. For a system quenched above the critical temperature this time saturates to a waiting-time independent value. A system that evolves from an ordered initial condition at sub-critical temperature has a finite decorrelation time.

Still another time scale is given by the time needed to reverse an equilibrium configuration in the low-T phase. This one is expected to be given by an Arrhenius law, with the height of the barrier being determined by the extensive free-energy barrier between the two minima, i.e. $\Delta F \simeq L^d f$, therefore,

$$\boxed{t_R^\phi \simeq e^{\beta L^d f} \qquad\qquad \text{Reversal time-scale .}} \tag{3.6}$$

The Ginzburg-Landau description allows for a pictorial interpretation of these results. The dynamics of the full system is visualized as the motion of its representative point in the Ginzburg-Landau potential. At high T the potential is harmonic in the deterministic Allen-Cahn equation, or the double-well structure in the time-dependent stochastic Ginzburg-Landau equation is completely ignored. The relaxation is similar to the one of a particle in a harmonic potential studied in Sect. 2.4.2. At low T, the initial position in the double-well potential depends on the type of initial condition $\phi(\vec{x},0) = 0$ or $\phi(\vec{x},0) \neq 0$. In the first case, the point sits on top of the central barrier and it does not detach from it in finite times with respect to a function of L. In the second case, the point starts from within one well and it simply rolls (on average) to the bottom of the well. This relaxation is similar to the one in the harmonic case. To reverse the configuration from, say, positive to negative magnetization the point needs to jump over the barrier in the double well potential and it does via thermal activation ruled by the Arrhenius law.

Note however that the phase-space of the system is actually N-dimensional while the description that is given here is projected onto one single coordinate, the one of the order-parameter. This reduction might lead to some misunderstandings and one should be very careful with it.

3.3 Growing length and dynamic scaling

In usual coarsening systems the averaged space-time correlation function

$$\boxed{L^d C(r,t) = \textstyle\sum_{ij/|\vec{r}_i - \vec{r}_j| = r} \langle s_i(t) s_j(t) \rangle}$$

allows for the identification of a growing length from, for example,

$$R_{\mathrm{a}}(T,t) \equiv \int d^d r \; r^{\mathrm{a}+1} C(r,t) / \int d^d r \; r^{\mathrm{a}} C(r,t) \tag{3.7}$$

(a is a parameter chosen to weight preferentially short or long distances; the time-dependence of $R_{\mathrm{a}}(t)$ should not depend on a.) Here and in the following $\langle \ldots \rangle$ stands for an average over different realizations of thermal histories at heat-bath temperature T and/or initial conditions. In presence of quenched disorder one adds an average over it and denotes it $[\ldots]$. The stochastic time-dependent function $N^{-1} \sum_{ij/|\vec{r}_i - \vec{r}_j| = r} s_i(t) s_j(t)$ after a quench from a random initial condition does not fluctuate in the thermodynamic limit. Therefore, the averages are not really necessary but they are usually written down. In spin-glasses and glasses this observable does not yield information on the existence of any growing length as we will discuss below.

The spherically averaged structure factor $S(k,t)$ – the Fourier transform of $C(r,t)$ – can be measured experimentally with small-angle scattering of neutrons, x-rays or light and from it $R_{\rm a}(T,t)$ can be extracted.

The ordering process is characterized by the growth of a **typical length**, $R(T,t)$. The growth regimes are summarized in the following equation and in Fig. 28:

$$\left\{ \begin{array}{lll} R_c(t) \to \xi_{eq}(T) < +\infty & T > T_c & \text{saturation,} \\ R_c(t) \to \xi_{eq}(T) \to \infty & T = T_c & \text{critical coarsening,} \\ R_c(t) \to (\xi_{eq}(T)) \gg R(T,t) \to L & T < T_c & \text{sub-critical coarsening.} \end{array} \right. \tag{3.8}$$

(Note that ξ_{eq} is defined from the connected static correlation function while R is defined from the dynamic correlation function. They measure different lengths as indicated below.) After a quench to the high temperature phase $T > T_c$ the system first grows equilibrium regions until reaching the equilibrium correlation length ξ_{eq} and next relaxes in equilibrium as explained in the previous section. The correlation length could be very short and the transient non-equilibrium regime be quite irrelevant ($T \gg T_c$). In the critical region, instead, the equilibrium correlation length is very long and it becomes important. In a critical quench the system never orders sufficiently and $R(T_c,t) < \xi_{eq}$ for all finite times. Finally, a quench into the subcritical region is characterized by two growth regimes: a first one in which the critical point dominates and the growth is as in a critical quench; a second one in which the proper sub-critical ordering is at work. The time-dependence of the growth law is different in these two regimes as we will see below. (Note that below T_c the equilibrium correlation length ξ_{eq} does not measure the size of ordered regions but the typical distance until which a fluctuation has an effect.)

In the asymptotic time domain, when $R(T,t)$ has grown much larger than any microscopic length in the system, a **dynamic scaling symmetry** sets in, similarly to the usual scaling symmetry observed in equilibrium critical phenomena. According to this hypothesis, the growth of $R(T,t)$ is the only relevant process and the whole time-dependence enters only through $R(T,t)$.

3.4 Critical coarsening

The scaling behavior of binary systems quenched to the critical point is quite well understood. It can be addressed with scaling arguments and renormalization group approaches [5] which give explicit expressions for many of the quantities of interest up to two loops order. Numerical simulations confirm the analytic results and probe exponents and scaling functions beyond the available perturbative orders. In this case the system builds correlated critical Fortuin-Kasteleyn clusters[11] with fractal

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[11] The Fortuin-Kasteleyn clusters are constructed as follows. Starting with a spin domain, one first draws all bonds linking nearest-neighbor spin on the cluster and then erases bonds with a temperature

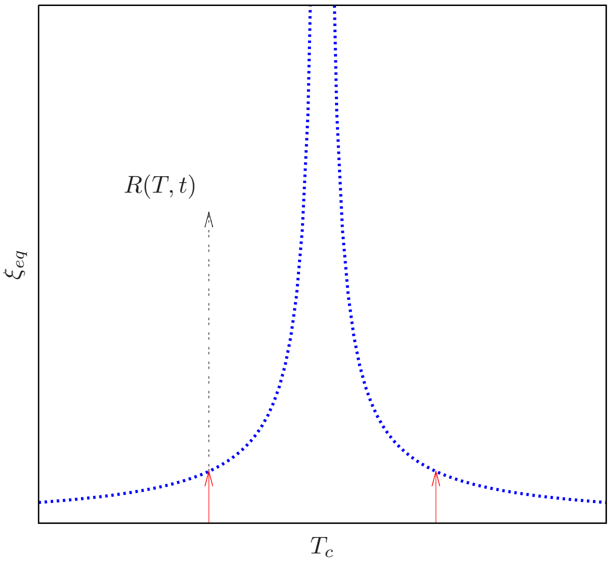

Figure 28: Sketch of the growth process in a second-order phase transition. The thick line is the equilibrium correlation length $\xi_{eq} \simeq |T-T_c|^{-\nu}$. The thin solid (red) arrows indicate the growing length R_c in the critical coarsening regime and the dashed (black) arrow the sub-critical growing length R in the coarsening regime.

dimension[12] $D_{FK} = (d+2-\eta)/2$, where η is the usual static critical exponent, in regions growing algebraically as $R_c(T_c,t) \equiv R_c(t) \sim t^{1/z_{eq}}$; henceforth we simplify the notation and avoid writing T_c within R. [As an example, for the bidimensional critical Ising class $\eta = 1/4$ and $D_{FK} = (2+2-1/4)/2 = 15/8$.]

In the asymptotic time regime the space-time correlation function has the scaling form

$$\begin{aligned} C(r,t) &= C_{eq}(r)\ C_{ag}(r,t) \\ &= C_{eq}(r)\ f\left(\frac{r}{R_c(t)}\right) \end{aligned}$$

---

dependent probability $e^{-\beta J}$. In such a way, the original bond-cluster typically diminishes in size and may even get disconnected.

[12] A possible definition of the fractal dimension is given by the box counting construction in which one counts the number of boxes of linear size ϵ that are needed to cover the set and computes $D = \lim_{\epsilon \to 0} [\ln N(\epsilon)/\ln(1/\epsilon)]$.

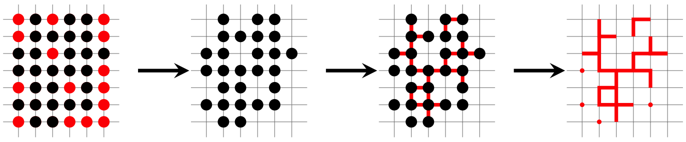

Figure 29: Sketch of the construction of a FK cluster. A domain is identified. The bonds between nearest-neighbor aligned spins (represented with black dots on the lattice sites) are erased using the FK procedure. Two FK disconnected clusters remain. The surviving bonds are highlighted on the edges of the lattice.

$$
= r^{-2(d-D_{FK})} \; f\left(\frac{r}{R_c(t)}\right) \tag{3.9}
$$

$$
\boxed{C(r,t) = r^{2-d-\eta} \; f\left(\frac{r}{R_c(t)}\right) \qquad \text{Multiplicative separation.}}
$$

The pre-factor $r^{-2(d-D_{FK})}$ takes into account that the growing domains have a **fractal nature** (hence their *density* decreases as their size grows) and the fact that the order parameter vanishes at the second order critical point. The dependence on $r/R_c(t)$ in $f(x)$ expresses the similarity of configurations at different times once lengths are measured in units of $R_c(t)$. At distances and times such that $r/R_c(t) \ll 1$ the equilibrium power-law decay, $C_{eq}(r) \simeq r^{2-d-\eta}$, should be recovered, thus $f(x) \simeq$ ct at $x \to 0$. $f(x)$ falls off rapidly for $x \gg 1$ to ensure that spins are uncorrelated at distances larger than $R_c(t)$. [More precisely, correlated as in the initial condition that, in the case of a quench from infinite temperature, means indeed uncorrelated.]

For two-time quantities, when t' is sufficiently large one has

$$
C(t,t') = C_{st}(t-t') \; f_{\rm C}\left(\frac{R_c(t)}{R_c(t')}\right) \tag{3.10}
$$

$$
\boxed{C(t,t') = R_c(t-t')^{2-d-\eta} \; f_{\rm C}\left(\frac{R_c(t)}{R_c(t')}\right) \qquad \text{Multiplicative separation.}}
$$

Here $C_{st}(t-t') \simeq R_c(t-t')^{-2(d-D_{FK})} = R_c(t-t')^{2-d-\eta}$. The scaling function $f_{\rm C}(x)$ describes the non-equilibrium behavior. It satisfies $f_c(1) = 1$ and $f_c(x \to \infty) = 0$, see the sketch in Fig. 30 (a). The former condition ensures that equilibrium is established up to the length $R_c(t)$. The latter decorrelation occurs faster than what the equilibrium relaxation tells beyond this length. In the scaling forms the equilibrium and non-equilibrium contributions enter in a **multiplicative** structure. Non-equilibrium effects are taken into account by taking ratios between the sizes of the correlated domains at the observation times t' and t in the scaling functions. Note that the reason why the equilibrium results are recovered for $t \simeq t'$ is that for very similar times one does not let the system realize that it is out of equilibrium.

In the case of non-conserved scalar order-parameter dynamics the growing length

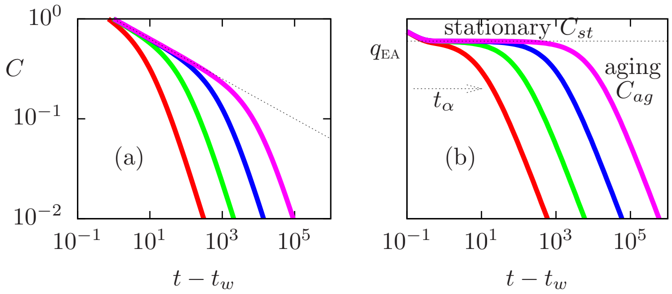

Figure 30: Sketch of the decay of the two-time correlation at T_c (a) and $T < T_c$ (b) for different values of the waiting-time, increasing from left to right.

behaves as

$$\boxed{R_c(t) \sim t^{1/z_{eq}}} \tag{3.11}$$

with z_{eq} the equilibrium dynamics exponent (note that z_{eq} is different from z_d). We will not discuss critical dynamics in detail; this problem is treated analytically with dynamic renormalization group techniques and it is very well discussed in the literature [5]. In short, the exponent z_{eq} is given by [34]

$$z_{eq} = 2 + \frac{N+2}{(N+8)^2}\left[3\ln\frac{4}{3} - \frac{1}{2}\right]\epsilon^2 + O(\epsilon^3) \tag{3.12}$$

where N is the dimension of the possibly vector field, $N = 1$ for a scalar one, and $\epsilon = 4 - d$ with d the dimension of space. Note that z_{eq} is larger than 2 for all finite N and it approaches 2 in the large N limit (at least up to this order in perturbation theory). In particular, one finds

$$z_{eq} \simeq \begin{cases} 2.0538 & d = 2 \\ 2.0134 & d = 3 \\ 2 & d = 4 \end{cases} \tag{3.13}$$

for $N = 1$. $d = 4$ is the upper critical dimension in this problem. Numerical simulations indicate $z_{eq} \simeq 2.13$ in $d = 2$. These results are valid for white noise dynamics. The effect of colored noise is to change the value of the exponent z_{eq} when it is sufficiently long-range correlated (sub-Ohmic noise with a power-law decay with an exponent smaller than a critical value that depends on the dimension of space).

The multiplicative scaling is also obtained with the dynamic RG method.

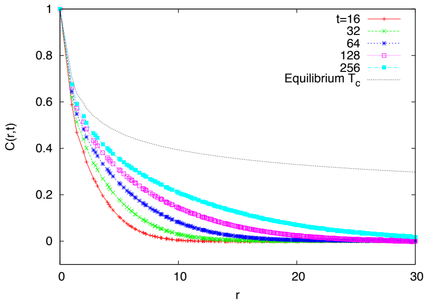

Figure 31: The space-time correlation function in the $2d$ Ising model after a quench to its critical point. With color data points the dynamic measurements. With a dashed black line the equilibrium power-law decay.

3.5 Sub-critical coarsening

3.5.1 Dynamic scaling hypothesis

The **dynamic scaling hypothesis** states that at late times and in the scaling limit

$$r \gg \xi_{eq}(g) \; , \qquad R(g,t) \gg \xi_{eq}(g) \; , \qquad r/R(g,t) \;\; \text{arbitrary} \; , \tag{3.14}$$

where r is the distance between two points in the sample, $r \equiv |\vec{x} - \vec{x}'|$, and $\xi_{eq}(g)$ is the equilibrium correlation length that depends on all parameters (T and possibly others) collected in g, there exists a **single characteristic length**, $R(g,t)$, such that the domain structure is, in statistical sense, independent of time when lengths are scaled by $R(g,t)$. Time, denoted by t, is typically measured from the instant when the critical point is crossed. In the following we ease the notation and write only the time-dependence in R. This hypothesis has been proved analytically in very simple models only, such as the one dimensional Ising chain with Glauber dynamics or the Langevin dynamics of the d-dimensional $O(N)$ model in the large N limit (see Sect. 3.7). But, as in critical coarsening, this regime sets in only after an "equilibrium-like" regime has died out. We are more precise below.

The late stage of phase-ordering in binary systems is characterized by a patchwork of large domains the interior of which is basically thermalized in one of the two equilibrium phases while their boundaries are slowly moving. This picture suggests the splitting of the degrees of freedom (spins) into two categories, providing statistically independent contributions to observables such as correlation or response functions. More precisely, a quasi-equilibrium stationary contribution arises as due to bulk spins, while boundaries account for the non-equilibrium part. Then asymptotically one has

$$\boxed{C(r,t) \simeq C_{eq}(r) + C_{ag}(r,t) \qquad \text{Additive separation.}} \tag{3.15}$$

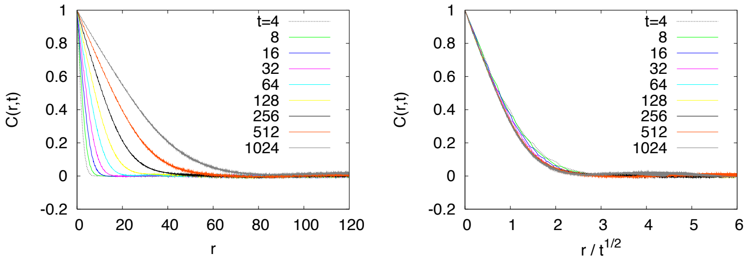

Figure 32: The equal-time correlation as a function of distance in the $2d$IM quenched below T_c. Raw (left) and scaled (right) data. These numerical simulations were performed by A. Sicilia.

The first term describes the equilibrium fluctuations in the low temperature broken symmetry pure states

$$C_{eq}(r) = (1 - \langle s_i \rangle_{eq}^2)\; g\left(\frac{r}{\xi_{eq}}\right) , \tag{3.16}$$

where $\langle s_i \rangle_{eq}$ is the equilibrium expectation value of the local spin in one of the two symmetry breaking states, $\langle s_i \rangle_{eq} = m$, and $g(x)$ is a function with the limiting values $g(0) = 1$, $\lim_{x\to\infty} g(x) = 0$. The second term takes into account the motion of the domain walls through

$$C_{ag}(r,t) = \langle s_i \rangle_{eq}^2\; f\left(\frac{r}{R(t)}\right) , \tag{3.17}$$

with $f(1) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Both C_{eq} and C_{ag} obey (separately) scaling forms with respect to the equilibrium and the non-equilibrium lengths ξ_{eq}, $R(t)$. In particular, eq. (3.17) expresses the fact that system configurations at different times are statistically similar provided that lengths are measured in units of $R(t)$, namely the very essence of dynamical scaling.

Monte Carlo simulations of the Ising model and other systems quenched below criticality and undergoing domain growth demonstrate that in the long waiting-time limit $t' \gg t_0$, the spin self-correlation $\langle s_i(t)s_i(t')\rangle$ separates into two additive terms

$$\boxed{C(t,t') \sim C_{st}(t-t') + C_{ag}(t,t') \quad \text{Additive separation}} \tag{3.18}$$

see Fig. 33, with the first one describing equilibrium thermal fluctuations within the domains,

$$C_{st}(t-t') \;=\; \begin{cases} 1 - \langle s_i \rangle_{eq}^2 = 1 - m^2 \; , & t - t' = 0 \; , \\ 0 \; , & t - t' \to \infty \; , \end{cases} \tag{3.19}$$

and the second one describing the motion of the domain walls

$$C_{ag}(t,t') \;=\; \langle s_i \rangle^2_{eq}\; f_c\left(\frac{R(t)}{R(t')}\right) = \begin{cases} \langle s_i \rangle^2_{eq} \,, & t' \to t^- \,, \\ 0 \,, & t-t' \to \infty \,. \end{cases} \qquad (3.20)$$

To ease the notation we have not written the explicit T-dependence in R that, as we will see below, is less relevant than t. Note that by adding the two contributions one recovers $C(t,t) = 1$ as expected and $C(t,t') \to 0$ when $t \gg t'$. The first term is identical to the one of a system in equilibrium in one of the two ordered states, see eq. (3.5) for its asymptotic $t - t' \gg t'$ limit; the second one is inherent to the out of equilibrium situation and existence and motion of domain walls. They vary in completely different two-time scales. The first one changes when the second one is fixed to $\langle s_i \rangle^2_{eq}$, at times such that $R(t)/R(t') \simeq 1$. The second one varies when the first one decayed to zero. The mere existence of the second term is the essence of the aging phenomenon with older systems (longer t') having a slower relaxation than younger ones (shorter t'). The scaling of the second term as the ratio between 'two lengths' is a first manifestation of **dynamic scaling**.

A decorrelation time can also be defined in this case by expandind the argument of the scaling function around $t' \simeq t$. Indeed, calling $\Delta t \equiv t - t'$ one has $R(t)/R(t') \simeq R(t'+\Delta t)/R(t') \simeq [R(t') + R'(t')\Delta t]/R(t') \simeq 1 + \Delta t/[d \ln R(t')/dt']^{-1}$ and one identifies a **t'-dependent decorrelation time**

$$\boxed{t_d \simeq [d \ln R(t')/dt']^{-1} \qquad \text{decorrelation time}} \qquad (3.21)$$

which is, in general, a growing function of t'.

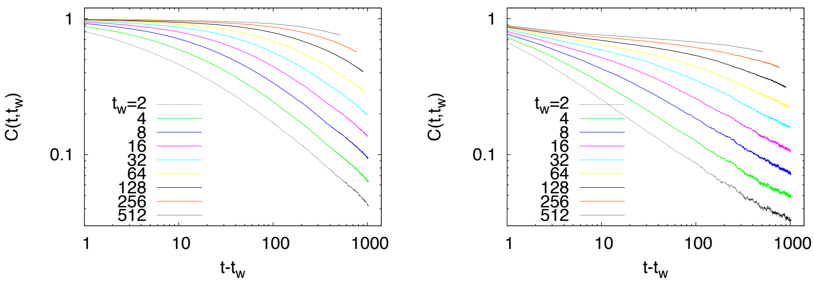

Figure 33: The two-time self-correlation in the $2d$IM with non-conserved order parameter dynamics at several waiting-times given in the key at temperature $T = 0.5$ (left) and $T = 2$ (right). Data obtained with Monte Carlo simulations. Note that the plateau is located at a lower level in the figure on the right consistently with the fact that $\langle \phi \rangle_{eq}$ decreases with increasing temperature. Data from A. Sicilia et al.

In order to fully characterise the correlation functions one then has to determine the typical growing length, R, and the scaling functions, g, f, f_c, *etc.* It turns out that the former can be determined with semi-analytic arguments and the predictions are well verified numerically – at least for clean system. The latter, instead, are harder to obtain. We will give a very brief state of the art report in Sect. 3.6.3. For a much more detailed discussion of these methods see the review articles in [4].

The time-dependent typical domain length, $R(t)$, is determined numerically by using several indirect criteria or analytically within certain approximations. The most common ways of measuring R are with numerical simulations of lattice models or the numerical integration of the continuous partial differential equation for the evolution of the order parameter. In both cases one

– Computes the 'inverse perimeter density' $R(t) = -\langle H\rangle_{eq}/[\langle H(t)\rangle - \langle H\rangle_{eq}]$ with $\langle H(t)\rangle$ the time-dependent averaged energy and $\langle H\rangle_{eq}$ the equilibrium energy both measured at the working temperature T.

– Puts the dynamic scaling hypothesis to the test and extracts R from the analysis.

3.5.2 $R(t)$ in clean one dimensional cases with non-conserved order parameter

In one dimension, a space-time graph allows one to view coarsening as the diffusion and annhilitation upon collision of point-like particles that represent the domain walls. In the Glauber Ising chain with non-conserved dynamics one finds that the typical domain length grows as $t^{1/2}$ while in the continuous case the growth is only logarithmic, $\ln t$.

3.5.3 $R(t)$ in non-conserved order parameter curvature dynamics in $d > 2$

In this case the growth law scales as $\lambda(T)t^{1/2}$ as we will show in Sec. using the time-dependent Ginzburg-Landau equation.

There are a number of ways to find the growth law

$$\boxed{R(t) = \lambda\, t^{1/z_d}} \tag{3.22}$$

with z_d the **dynamic exponent**, in **pure and isotropic** systems (see [4]). The effects of temperature enter only in the parameter λ and, for clean systems, growth is slowed down by an increasing temperature since thermal fluctuation tend to roughen the interfaces thus opposing the curvature driven mechanism. We estimate the T dependence of λ in Sect. 3.5.5.

In curvature driven Ising or Potts cases with non-conserved order parameter the domains are sharp and $z_d = 2$ with λ a weakly T-dependent coefficient. For systems with continuous variables such as rotors or XY models and the same type of dynamics, a number of computer simulations have shown that domain walls are thicker and $z_d = 4$.

3.5.4 $R(t)$ in conserved order parameter dynamics and the role of bulk diffusion

A different type of dynamics occurs in the case of phase separation (the water and oil mixture ignoring hydrodynamic interactions or a binary allow). In this case, the material is locally conserved, *i.e.* water does not transform into oil but they just separate. The main mechanism for the evolution is diffusion of material through the bulk of the opposite phase. After some discussion, it was established, as late as in the early 90s, that for scalar systems with **conserved order parameter** $z_d = 3$.

3.5.5 Crossover between critical and sub-critical coarsening

Matching critical coarsening with sub-critical one allows one to find the T-dependent prefactor λ [38]. The argument goes as follows. The out of equilibrium growth at criticality and in the ordered phase are given by

$$R(t) \quad \sim \quad \begin{cases} t^{1/z_{eq}} & \text{at} \qquad T = T_c \; , \\ (\lambda(T) t)^{1/z_d} & \text{at} \qquad T < T_c \; . \end{cases} \tag{3.23}$$

z_{eq} is the equilibrium dynamic critical exponent and z_d the out of equilibrium growth exponent. Close but below criticality one should have an interpolating expression of the kind

$$R(t) \quad \sim \quad \xi^{-a} \, t^{1/z_d} \, f\left(\frac{t}{\xi^{z_{eq}}}\right) \qquad \text{at} \qquad T = T_c - \epsilon \tag{3.24}$$

with ξ the T-dependent equilibrium correlation length, $\xi_{eq}(T) \sim (T_c - T)^{-\nu}$. The last factor tends to one, $f(x \to \infty) \to 1$, when $R(t) \gg \xi$, that is to say when the argument diverges and the system enters the sub-critical coarsening regime. It is however non-trivial when $R(t) \sim \xi$, the argument is finite and critical coarsening must be described. In particular, we determine its behavior for $x = O(1)$ by requiring that eq. (3.24) matches the subcritical growing length which is achieved by (i) recovering the correct t dependence, (ii) cancelling the ξ factor. (i) implies

$$f(x) \sim x^{-1/z_d + 1/z_{eq}} \qquad \text{for} \qquad x = O(1) \; . \tag{3.25}$$

Then eq. (3.24) becomes

$$R(t) \sim \xi^{-a + z_{eq}/z_d - 1} \, t^{1/z_{eq}} \tag{3.26}$$

and to eliminate ξ we need

$$a = z_{eq}/z_d - 1 \; . \tag{3.27}$$

Comparing now the subcritical growing length and (3.24) in the very long times limit such that $R(t) \gg \xi$ and $f(x \to \infty) \to 1$:

$$[\lambda(T)]^{1/z_d} \sim \xi^{-a} \sim (T_c - T)^{\nu(z_{eq} - z_d)/z_d} \; . \tag{3.28}$$

Note that quite generally one finds $z_{eq} > z_d$ and $\lambda(T)$ vanishes at T_c.

3.5.6 Role of weak disorder: thermal activation

The situation becomes much less clear when there is weak quenched disorder in the form of non-magnetic impurities in a magnetic sample, lattice dislocations, residual stress, *etc.* These are assumed not to modify the nature of the equilibrium states with respect to the ones of the clean system. Qualitatively, the dynamics are expected to be slower than in the pure cases since disorder pins the interfaces. In general, based on an argument due to Larkin (and in different form to Imry-Ma) one expects that in $d < 4$ the late epochs and large scale evolution is no longer curvature driven but controlled by disorder. Indeed, within a phase space view disorder generates metastable states that trap the system and thus slow down the relaxation.

A hand-waving argument to estimate the growth law in dirty systems is the following. Take a system in one equilibrium state with a domain of linear size R of the opposite equilibrium state within it. This configuration could be the one of an excited state with respect to the fully ordered one with absolute minimum free-energy. Call $\Delta F(R)$ the free-energy barrier between the excited and equilibrium states. The thermal activation argument (see Sect. 2.3) yields the activation time scale for the decay of the excited state (*i.e.* erasing the domain wall)

$$t_A \sim \tau \; e^{\Delta F(R)/(k_B T)} \; . \tag{3.29}$$

For a barrier growing as a power of R, $\Delta F(R) \sim \Upsilon(T, J) R^{\psi}$ (where J represents the disorder) one inverts (3.29) to find the linear size of the domains still existing at time t, that is to say, the growth law

$$\boxed{R(t) \sim \left(\frac{k_B T}{\Upsilon(T)} \; \ln \frac{t}{\tau} \right)^{1/\psi} \; .} \tag{3.30}$$

All smaller fluctuation would have disappeared at t while typically one would find objects of this size. The exponent ψ is expected to depend on the dimensionality of space but not on temperature. In 'normal' systems it is expected to be just $d-1$ – the surface of the domain – but in spin-glass problems, it might be smaller than $d-1$ due to the presumed fractal nature of the walls. The prefactor Υ is expected to be weakly temperature dependent.

One assumes that the same argument applies out of equilibrium to the reconformations of a portion of any domain wall or interface where R is the observation scale.

However, already for the (relatively easy) random ferromagnet there is no consensus about the actual growth law. In these problems there is a competition between the 'pure' part of the Hamiltonian, that tries to minimize the total $(d-1)$ dimensional area of the domain wall, and the 'impurity' part that makes the wall deviate from flatness and pass through the locations of lowest local energy (think of $J_{ij} = J + \delta J_{ij}$ with J and δJ_{ij} contributing to the pure and impurity parts of the

Hamiltonian, respectively). The activation argument in eq. (3.29) together with the power-law growth of barriers in $\Delta F(R) \sim \Upsilon(T,J) R^{\psi}$ imply a logarithmic growth of $R(t)$. Simulations, instead, suggest a power law with a temperature dependent exponent. Whether the latter is a pre-asymptotic result and the trully asymptotic one is hidden by the premature pinning of domain walls or it is a genuine behavior invalidating $\Delta F(R) \sim \Upsilon(T,J) R^{\psi}$ or even eq. (3.29) is still an open problem. See the discussion below for a plausible explanation of the numerical data that does not invalidate the theoretical expectations.

In the $3d$ RFIM the curvature-driven growth mechanism that leads to (3.22) is impeded by the random field roughening of the domain walls. The dependence on the parameters T and h has been estimated. In the early stages of growth, one expects the zero-field result to hold with a reduction in the amplitude $R(t) \sim (A - Bh^2)\, t^{1/2}$. The time-window over which this law is observed numerically decreases with increasing field strength. In the late time regime, where pinning is effective Villain deduced a logarithmic growth $R(t) \sim (T/h^2)\, \ln t/t_0$ by estimating the maximum barrier height encountered by the domain wall and using the Arrhenius law to derive the associated time-scale.

In the case of spin-glasses, if the mean-field picture with a large number of equilibrium states is realized in finite dimensional models, the dynamics would be one in which all these states grow in competition. If, instead, the phenomenological droplet model applies, there would be two types of domains growing and $R(t) \sim (\ln t)^{1/\psi}$ with the exponent ψ satisfying $0 \leq \psi \leq d-1$. Some refined arguments that we will not discuss here indicate that the dimension of the bulk of these domains should be compact but their surface should be rough with fractal dimension $D_s > d-1$.

3.5.7 Temperature-dependent effective exponents

The fact that numerical simulations of dirty systems tend to indicate that the growing length is a power law with a T-dependent exponent can be explained as due to the effect of a T-dependent cross-over length L_T. Indeed, if below $L_T \sim T^{\phi}$ the growth process is as in the clean limit while above L_T quenched disorder is felt and the dynamics is thermally activated:

$$R(t) \quad \sim \quad \left\{ \begin{array}{lll} t^{1/z_d} & \text{for} & R(t) \ll L_T \; , \\ (\ln t)^{1/\psi} & \text{for} & R(t) \gg L_T \; . \end{array} \right. \tag{3.31}$$

These growth-laws can be first inverted to get the time needed to grow a given length and then combined into a single expression that interpolates between the two regimes:

$$t(R) \sim e^{(R/L_T)^{\psi}} R^{z_d} \tag{3.32}$$

where the relevant T-dependent length-scale L_T has been introduced.

Now, by simply setting $t(R) \sim R^{\overline{z}(T)}$ one finds $\overline{z}(T) \sim z_d + \frac{1}{\ln R(t)} \left(\frac{R^{\psi}(t)}{L_T^{\psi}} \right)$ that replacing $R \sim t^{1/\overline{z}(T)}$ becomes $\overline{z}(T) \sim z_d + \frac{\overline{z}(T)}{\ln t} \left(\frac{t^{\psi/\overline{z}(T)}}{L_T^{\psi}} \right)$. Using now $\overline{z}(T) \simeq z_d$ in the correction term and focusing on times such that $t^{\psi/z_d}/\ln t$ is almost constant and equal to c one finds $\overline{z}(T) - z_d \simeq \mathrm{c}\ z_d/L_T^{\psi}$. Similarly, by equating $t(R) \sim \exp(R^{\overline{\psi}(T)}/T)$ one finds that $\overline{\psi}(T)$ is a decreasing function of T approaching ψ at high T.

3.5.8 Logarithmic growth in clean systems

It is possible to slow down the growth of the characteristic length by introducing competing interactions in the Hamiltonian. One such model is the $2d$ Ising model with non-conserved order parameter dynamics and an additional antiferromagnetic next nearest-neighbour interaction where the growing length is logarithmic [42]. Another possibility is to add frustration, as in the case of vertex models on a square lattice with Monte Carlo single-spin flip dynamics: in coarsening in the ordered phases is anisotropic with different growing lengths in the two perpendicular directions [43].

3.6 Time-dependent Ginzburg-Landau description

In order to treat phase-transitions and the coarsening process analytically it is preferable to introduce a coarse-grained description in terms of a continuous coarse-grained field,

$$\phi(\vec{x}, t) \equiv \frac{1}{V} \sum_{i \in V_{\vec{x}}} s_i(t) \ , \tag{3.33}$$

the fluctuating magnetization density. In a first approximation a Landau-Ginzburg free-energy functional is introduced

$$F[\phi] = \int d^d x \ \left\{ \frac{c}{2} \, [\nabla \phi(\vec{x}, t)]^2 + V[\phi(\vec{x}, t)] \right\} \ . \tag{3.34}$$

The elastic constant c is usually re-absorbed with a series of re-definitions.

With the choice of the potential one distinguishes between a second order and a first order phase transition. In the former case, the typical form is the ϕ^4 form:

$$V(\phi) = a\phi^4 + b(g)\phi^2 \ . \tag{3.35}$$

The first term in eq. (3.34) represents the energy cost to create a domain wall or the elasticity of an interface. The second term depends on a parameter, g, and changes sign from positive at $g > g_c$ to negative at $g < g_c$. Above the critical point determined by $b(g_c) = 0$ it has a single minimum at $\phi = 0$, at g_c it is flat at $\phi = 0$ and below g_c it has a double well structure with two minima, $\phi = \pm[-b(g)/(2a)]^{1/2} = \langle \phi \rangle_{eq}(g)$, that correspond to the equilibrium states in the ordered phase. Equation (3.34) is exact for a fully connected Ising model where $V(\phi)$ arises from the multiplicity of spin configurations that contribute to the same $\phi(\vec{x}) = m$. The order-parameter dependent free-energy density reads $f(m) = -Jm^2 - hm + k_B T\{(1+m)/2 \ln[(1+m)/2] + (1-m)/2 \ln[(1-m)/2]$ that close to the critical point where $m \simeq 0$ becomes $f(m) \simeq (k_B T - 2J)/2 \ m^2 - hm + k_B T/12 \ m^4$ demonstrating the passage from a harmonic form at $k_BT > k_BT_c = 2J$, to a quartic well at $T = T_c$, and finally to a double-well structure at $T < T_c$.

Exercise 14. Prove the above.

With a six-order potential one can mimic the situation in the right panel of Fig. 22. It suffices to take $V(\phi) = a + b\phi^2 + c\phi^4 + d\phi^6$. The sign of d, $d > 0$, is fixed by the condition that the potential be confining at large values of $|\phi|$. The potential has a local minimum at $\phi = 0$ for all $b > 0$. Next, we choose $c < 0$ to allow for the existence of two maxima and two minima at $\phi = \pm[(-c \pm \sqrt{c^2 - 3bd})/(3d)]^{1/2}$.

When discussing dynamics one should write down the stochastic evolution of the individual spins and compute time-dependent averaged quantities as the ones in (3.1). This is the procedure used in numerical simulations. Analytically it is more convenient to work with a field-theory and an evolution equation of Langevin-type. This is the motivation for the introduction of continuous field equations that regulate the time-evolution of the coarse-grained order parameter. Ideally these equations should be derived from the spin stochastic dynamics but in practice they are introduced phenomenologically. In the magnetic case as well as in many cases of interest, the domain wall and interface dynamics can be argued to be **overdamped** (i.e. $t \gg t_r^{\dot{\phi}}$).

Two very similar approaches are used. Assuming T is only relevant to determine the equilibrium coarse-grained field one uses the phenomenological **zero-temperature time-dependent Ginzburg-Landau** equation or **model A** in the classification of Hohenberg-Halperin deterministic equation

$$\frac{\partial \phi(\vec{x},t)}{\partial t} = -\frac{\delta F[\phi]}{\delta \phi(\vec{x},t)} \tag{3.36}$$

(the friction coefficient has been absorbed in a redefinition of time). Initial conditions are usually chosen to be random with short-range correlations

$$[\,\phi(\vec{x},0)\phi(\vec{x}',0)\,]_{ic} = \Delta\delta(\vec{x}-\vec{x}') \tag{3.37}$$

thus mimicking the high-temperature configuration ($[\ldots]_{ic}$ represent the average over its probability distribution). The numeric solution to this equation with the quartic potential and $b < 0$ shows that such a random initial condition evolves into a field configuration with patches of ordered region in which the field takes one of the two values $[-b/(2a)]^{1/2}$ separated by sharp walls. It ignores temperature fluctuations within the domains meaning that the field is fully saturated within the domains and, consequently, one has access to the aging part of the correlations only, see *e.g.* eq. (3.15). The phase transition is controlled by the parameter b in the potential.

Another, similar approach, is to add a thermal noise to the former

$$\frac{\partial \phi(\vec{x},t)}{\partial t} = -\frac{\delta F[\phi]}{\delta \phi(\vec{x},t)} + \xi(\vec{x},t)\ . \tag{3.38}$$

This is the field-theoretical extension of the Langevin equation in which the potential is replaced by the order-parameter-dependent functional free-energy in eq. (3.34) with a potential form with fixed parameters (independent of T). ξ is a noise taken to be Gaussian distributed with zero mean and correlations

$$\langle \xi(\vec{x},t)\xi(\vec{x}',t')\rangle = 2k_B T \delta^d(\vec{x}-\vec{x}')\delta(t-t') \; . \tag{3.39}$$

The friction coefficient has been absorbed in a redefinition of time. For a quartic potential a dynamic phase transition arises at a critical T_c; above T_c the system freely moves above the two minima and basically ignores the double well structure while below T_c this is important. Within the growing domains the field ϕ fluctuates about its mean also given by $[-b/(2a)]^{1/2}$ and the fluctuations are determined by T. One can describe the rapid relaxation at ties such that the domain walls do not move with this approach. This formulation is better suited to treat critical and sub-critical dynamics in the same field-theoretical framework.

These equations do not conserve the order parameter neither locally nor globally. Extensions for cases in which it is conserved exist (model B). Cases with vectorial or even tensorial order parameters can be treated similarly and are also of experimental relevance, notably for vectorial magnets or liquid crystals.

3.6.1 Short-time dynamics

Take an initial configuration $\phi(\vec{x},0)=0$ on average with small fluctuations, as in equilibrium at very high temperature, and quench the system. At very short time one can expand the non-linear potential and the Ginzburg-Landau equation (3.36), for the Fourier components, $\phi(\vec{k},t) = L^{-d/2}\int d^dx \; \phi(\vec{x},t)e^{-i\vec{k}\vec{x}}$ with $\vec{k} = 2\pi/L \; (n_1,\dots,n_d)$ and n_k integer, reads

$$\frac{\partial \phi(\vec{k},t)}{\partial t} = [-k^2 - V''(0)]\phi(\vec{k},t) + \xi(\vec{k},t) \; . \tag{3.40}$$

If $V''(0) > 0$ all modes decay exponentially and no order develops. If $V''(0) < 0$ instead modes with $-k^2 - V''(0) > 0$ are unstable and grow exponentially until a time $t^* \simeq -1/V''(0)$ when the small ϕ expansion ceases to be justified. The instability of the small wave-vector modes indicates that the system tends to order. To go beyond this analysis one needs to consider the full non-linear equation.

3.6.2 $R(t)$ in non-conserved order parameter curvature dynamics in $d > 2$

The time-dependent Ginzburg-Landau model allows us to gain some insight on the mechanism driving the domain growth and the direct computation of the averaged domain length. In clean systems temperature does not play a very important role in the domain-growth process, it just adds some thermal fluctuations within the domains, as long as it is smaller than T_c. In dirty cases instead temperature triggers thermal activation.

We focus first on the clean cases at $T = 0$ and only later we discuss thermal effects. Equation (3.36) for $T = 0$ is just a gradient descent in the energy landscape F. Two terms contribute to F: the bulk-energy term that is minimized by $\phi = \pm\phi_0$ and the elastic energy $(\nabla\phi)^2$ which is minimized by flat walls if present. As a consequence the minimization process implies that regions of constant field, $\phi(\vec{x},t) = \pm\phi_0$, grow and they separated by flatter and flatter walls.

Take a **flat domain wall** separating regions where the configuration is the one of the two equilibrium states, $\phi(\vec{x},t) = \pm\phi_0 + \delta\phi(\vec{x},t)$. Linearizing eq. (3.36) around $\pm\phi_0$ and looking for static configurations, *i.e.* $\delta\phi(\vec{x},t) = \delta\phi(\vec{x}) = \delta\phi(n)$ where n is the distance from the wall along the normal direction one finds $d^2\delta\phi(n)/dn^2 = -V''(\phi_0)\delta\phi(n)$. This equation has the solution $\delta\phi(n) \sim e^{-\sqrt{V''(\phi_0)}n}$ where n is the perpendicular distance to the wall. The order parameter approaches $\pm\phi_0$ on both sides of the wall very rapidly. This means that the free-energy of a configuration with an interface (sum of the elastic and potential terms) is concentrated in a very narrow region close to it. In consequence, the domain-wall curvature is the driving force for domain growth.

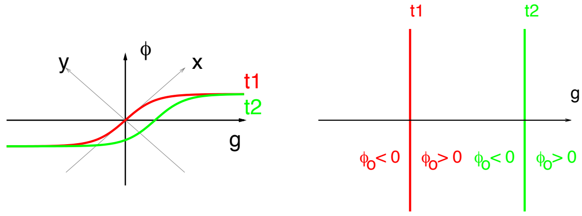

Figure 34: Left: domain wall profile. Right: view from the top. (g is n.)

Allen and Cahn showed that the local wall velocity is proportional to the local curvature working with the Ginzburg-Landau equation at $T = 0$. The proof goes as follows. Take the Ginzburg-Landau equation and transform the derivatives to apply in the direction normal to the wall:

$$\frac{\partial\phi(\vec{x},t)}{\partial t} = -\left.\frac{\partial\phi(\vec{x},t)}{\partial n}\right|_t \left.\frac{\partial n}{\partial t}\right|_\phi , \qquad \vec{\nabla}\phi(\vec{x},t) = \left.\frac{\partial\phi(\vec{x},t)}{\partial n}\right|_t \hat{n} ,$$

$$\nabla^2\phi(\vec{x},t) = \left.\frac{\partial^2\phi(\vec{x},t)}{\partial n^2}\right|_t + \left.\frac{\partial\phi(\vec{x},t)}{\partial n}\right|_t \vec{\nabla}\cdot\hat{n}$$

where the subscripts mean that the derivatives are taken at t or ϕ fixed. Using now $\frac{\partial^2\phi(\vec{x},t)}{\partial n^2}|_t = V'(\phi)$ (note that the derivative is taken at fixed t) in the GL equation one finds the Allen-Cahn result

$$v \equiv \partial_t n|_\phi = -\vec{\nabla}\cdot\hat{n} \equiv -\kappa \tag{3.41}$$

valid in all d with κ the geodesic curvature.

Equation (3.41) allows one to get an intuition about the typical growth law in such processes. Take a spherical wall in any dimension. The local curvature is constant and $\kappa = (d-1)/R$ where R is the radius of the sphere within the hull. Equation (3.41) is recast as $dR/dt = -(d-1)/R$ that implies $R^2(t) = R^2(0) - 2(d-1)t$.

A closer look at the $2d$ equation allows one to go beyond and prove, in this case, that all areas enclosed by domain walls irrespective of their being other structures within (the so-called hull-enclosed areas) tend to diminish at constant rate $dA/dt = -\lambda$. This, of course, does not mean that all domains reduce their area since a domain can gain area from the disappearance of an internal domain of the opposite sign, for instance. The proof is simple and just uses the Gauss-Bonnet theorem: $\frac{dA}{dt} = \oint \vec{v} \wedge d\vec{\ell} = \oint v d\ell$. The local wall-velocity, $\vec{v}$, is proportional to the local geodesic curvature, κ, and the Gauss-Bonnet theorem implies $\oint \kappa d\ell = 2\pi$ for a planar $2d$ manifold with no holes. Therefore, the hull-enclosed area decreases with constant velocity for any geometry.

Therefore the local velocity points in the direction of the local centre of curvature. The effect is to reduce the wall roughness by rendering them smoother.

3.6.3 Scaling functions for subcritical coarsening

Even though the qualitative behavior of the solution to eq. (3.36) is easy to grasp, it is still too difficult to solve analytically and not much is known exactly on the scaling functions. A number of approximations have been developed but none of them is fully satisfactorily (see [4] for a critical review of this problem).

The **super-universality hypothesis** states that in cases in which temperature and quenched disorder are 'irrelevant' in the sense that they do not modify the nature of the low-temperature phase (*i.e.* it remains ferromagnetic in the case of ferromagnetic Ising models) the scaling functions are not modified. Only the growing length changes from the, say, curvature driven $t^{1/2}$ law to a logarithmic one. Numerical evidence for the validity of this hypothesis in a number of two and three dimensional models including the RBIM and the RFIM was given in [47] but more recent numerical studies [48] claim that it does not hold, at least for the latter model. This issue is not settled.

3.6.4 Breakdown of dynamic scaling

Some special cases in which dynamic scaling does not apply have also been exhibited. Their common feature is the existence of two (or more) growing lengths associated to different ordering mechanisms. An example is given by the Heisenberg model at $T \to 0$ in which the two mechanisms are related to the vectorial ordering within domains separated by couples of parallel spins that annihilate in a way that is similar to domain-wall annihilation in the Ising chain.

3.7 The large $\mathcal{N}$ approximation

We would like to study, in full detail, the Langevin equation for the scalar field ϕ with Ginzburg-Landau free-energy

$$F = \int d^d x \left[\frac{1}{2} (\vec{\nabla}\phi)^2 - \frac{g}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right] , \tag{3.42}$$

As we will soon introduce a new vector and an another set of indices, let us recall here that space has d dimensions, the vector position is $\vec{x} = (x_1, \dots, x_d)$ and the local contribution to the elastic term is proportional to

$$(\vec{\nabla}\phi)^2 = \vec{\nabla}\phi \cdot \vec{\nabla}\phi = \frac{\partial \phi}{\partial x_1}\frac{\partial \phi}{\partial x_1} + \dots \frac{\partial \phi}{\partial x_d}\frac{\partial \phi}{\partial x_d} . \tag{3.43}$$

We used the standard notation for the parameters $g > 0$ and $\lambda > 0$ in the potential, and we chose the signs to have the double-well structure. The potential part has two minima at

$$\phi_0^2 = \frac{g}{\lambda} \tag{3.44}$$

and it can be rewritten in a convenient way

$$V(\phi) = \frac{\lambda}{4}\left[\left(\phi^2 - \phi_0^2\right)^2 - \phi_0^4 \right] \tag{3.45}$$

Without loss of generality we neglect the constant term $-\phi_0^4$. We therefore work with

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \phi_0^2\right)^2 . \tag{3.46}$$

A very useful approximation is to upgrade the scalar field to a vectorial one with $\mathcal{N}$ components

$$\phi(\vec{x}, t) \to \vec{\phi}(\vec{x}, t) = (\phi_1(\vec{x}, t), \dots, \phi_{\mathcal{N}}(\vec{x}, t)) , \tag{3.47}$$

and modify the free-energy

$$F = \int d^d x \left[\frac{1}{2} (\vec{\nabla}\vec{\phi})^2 + \frac{\mathcal{N}\lambda}{4}(\phi_0^2 - \mathcal{N}^{-1}\phi^2)^2 \right] , \tag{3.48}$$

with $\phi^2 = \sum_{\alpha=1}^{\mathcal{N}} \phi_\alpha^2$ and $\phi_0^2 > 0$ and finite. Note that the double-well structure of the potential is assumed from the start. The factors $\mathcal{N}$ and $\mathcal{N}^{-1}$ are added to ensure that each contribution to the free-energy F is of order $\mathcal{N}$ (note that ϕ^2 is expected to be order $\mathcal{N}$). The elastic contribution has now a double scalar product structure, in the $\mathcal{N}$-dimensional space and in the d-dimensional space. More precisely,

$$\begin{aligned}
(\vec{\nabla}\vec{\phi})^2 &= \vec{\nabla}\phi_\alpha \cdot \vec{\nabla}\phi_\alpha = \frac{\partial \phi_\alpha}{\partial x_1}\frac{\phi_\alpha}{\partial x_1} + \dots + \frac{\partial \phi_\alpha}{\partial x_d}\frac{\phi_\alpha}{\partial x_d} \\
&= \frac{\partial \phi_1}{\partial x_1}\frac{\phi_1}{\partial x_1} + \dots \frac{\partial \phi_{\mathcal{N}}}{\partial x_1}\frac{\phi_{\mathcal{N}}}{\partial x_1} + \dots + \frac{\partial \phi_1}{\partial x_d}\frac{\phi_1}{\partial x_d} + \dots + \frac{\partial \phi_{\mathcal{N}}}{\partial x_d}\frac{\phi_{\mathcal{N}}}{\partial x_d} .
\end{aligned} \tag{3.49}$$

This problem can be studied statically within the canonical formalism and a finite critical temperature $T_c(d, \phi_0)$ is found.

3.7.1 Statics

If the volume V is kept finite the system equilibrates in a finite time t_{eq} and the order parameter probability distribution reaches the Gibbs state

$$P_{eq}[\vec{\phi}(\vec{k})] = \frac{1}{Z} \exp\left(-\frac{1}{2TV}\sum_{\vec{k}}(k^2 + \xi_{eq}^{-2})\vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k})\right) \tag{3.50}$$

where ξ_{eq} is the correlation length

$$\xi_{eq}^{-2} = -g + \frac{\lambda}{N}\langle \vec{\phi}^2(\vec{x})\rangle_{eq} \tag{3.51}$$

with $\langle \cdots \rangle_{eq}$ standing for the average taken with (3.50).

In order to analyze the properties of $P_{eq}[\vec{\phi}(\vec{k})]$ it is necessary to extract from (3.51) the dependence of ξ_{eq}^{-2} on T and V. Evaluating the average, the above equation yields

$$\xi_{eq}^{-2} = -g + \frac{\lambda}{V}\sum_{\vec{k}} \frac{T}{k^2 + \xi_{eq}^{-2}}. \tag{3.52}$$

The solution of this equation is well known[45] and here we summarize the main features, as presented in [46]. Separating the $\vec{k} = 0$ term under the sum, for very large volume we may rewrite

$$\xi_{eq}^{-2} = -g + \lambda T B(\xi_{eq}^{-2}) + \frac{\lambda T}{V\xi_{eq}^{-2}} \tag{3.53}$$

where

$$B(\xi_{eq}^{-2}) = \lim_{V\to\infty} \frac{1}{V}\sum_{\vec{k}} \frac{1}{k^2 + \xi_{eq}^{-2}} = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2 + \xi_{eq}^{-2}} \tag{3.54}$$

regularizing the integral by introducing the high momentum cutoff Λ. The function $B(x)$ is a non negative monotonically decreasing function with the maximum value at $x = 0$

$$B(0) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2} = (4\pi)^{-\frac{d}{2}} \frac{2}{d-2}\Lambda^{d-2} \quad . \tag{3.55}$$

By graphical analysis one can easily show that (3.53) admits a finite solution for all T. However, there exists the critical value of the temperature T_c defined by

$$-g + \lambda T_c B(0) = 0 \tag{3.56}$$

such that for $T > T_c$ the solution is independent of the volume, while for $T \leq T_c$ it depends on the volume. Using

$$B(x) = (4\pi)^{-\frac{d}{2}} x^{\frac{d}{2}-1} e^{\frac{x}{\Lambda^2}} \Gamma\left(1 - \frac{d}{2}, \frac{x}{\Lambda^2}\right) \tag{3.57}$$

where $\Gamma(1-\frac{d}{2}, \frac{x}{\Lambda^2})$ is the incomplete gamma function, for $0 < \frac{T-T_c}{T_c} \ll 1$ one finds $\xi_{eq} \sim (\frac{T-T_c}{T_c})^{-\nu}$ where $\nu = 1/2$ for $d > 4$ and $\nu = 1/(d-2)$ for $d < 4$, with logarithmic corrections for $d = 4$. At T_c one has $\xi_{eq} \sim V^{\overline{\lambda}}$ with $\overline{\lambda} = 1/4$ for $d > 4$ and $\overline{\lambda} = 1/d$ for $d < 4$, again with logarithmic corrections for $d = 4$. Finally, below T_c one finds $\xi^2_{eq} = \frac{M^2 V}{T}$ where $M^2 = \phi_0^2 \left(\frac{T_c - T}{T_c}\right)$ and $\phi_0^2 = g/\lambda$.

Let us now see what are the implications for the equilibrium state. As (3.50) shows the individual Fourier components are independent random variables, Gaussianly distributed with zero average. The variance is given by

$$\frac{1}{\mathcal{N}} \langle \vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k}) \rangle_{eq} = V C_{eq}(\vec{k}) \tag{3.58}$$

where

$$C_{eq}(\vec{k}) = \frac{T}{k^2 + \xi_{eq}^{-2}} \tag{3.59}$$

is the equilibrium structure factor. For $T > T_c$, all $\vec{k}$ modes behave in the same way, with the variance growing linearly with the volume. For $T \leq T_c$, instead, ξ_{eq}^{-2} is negligible with respect to k^2 except at $\vec{k} = 0$, yielding

$$C_{eq}(\vec{k}) = \begin{cases} \frac{T_c}{k^2}(1 - \delta_{\vec{k},0}) + cV^{2\overline{\lambda}} \delta_{\vec{k},0} & \text{for } T = T_c \quad , \\ \frac{T}{k^2}(1 - \delta_{\vec{k},0}) + M^2 V \delta_{\vec{k},0} & \text{for } T < T_c \quad , \end{cases} \tag{3.60}$$

where c is a constant. This produces a volume dependence in the variance of the $\vec{k} = 0$ mode growing faster than linear. Therefore, for $T \leq T_c$ the $\vec{k} = 0$ mode behaves differently from all the other modes with $\vec{k} \neq 0$. For $T < T_c$ the probability distribution (3.50) takes the form

$$P_{eq}[\vec{\phi}(\vec{k})] = \frac{1}{Z} e^{-\frac{\vec{\phi}^2(0)}{2M^2V^2}} e^{-\frac{1}{2TV} \sum_{\vec{k}} k^2 \vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k})} \quad . \tag{3.61}$$

Therefore, crossing T_c there is a transition from the usual disordered high temperature phase to a low temperature phase characterized by a macroscopic variance in the distribution of the $\vec{k} = 0$ mode. The distinction between this phase and the mixture of pure states, obtained below T_c when $\mathcal{N}$ is kept finite can be discussed but we will not discuss it here.

3.7.2 Dynamics

The (over-damped) Langevin equation then becomes

$$\partial_t \phi_\alpha(\vec{x},t) = \nabla^2 \phi_\alpha(\vec{x},t) - \phi_\alpha(\vec{x},t)\,[\phi_0^2 - \mathcal{N}^{-1}\phi^2(\vec{x},t)] + \xi_\alpha(\vec{x},t)\;, \tag{3.62}$$

where the friction coefficient has been absorbed in a redefinition of time, the constant λ has been conveniently set to one, and the initial condition is taken from a Gaussian distribution with zero-mean and correlations

$$[\phi_\alpha(\vec{x},0)\phi_\beta(\vec{x}',0)]_{ic} = \Delta\delta^d(\vec{x}-\vec{x}')\delta_{\alpha\beta}\;. \tag{3.63}$$

We take the noise to be delta correlated in space and time. It will turn out to be convenient (to avoid short distance divergencies) to introduce a short-distance cut-off:

$$\langle \xi_\alpha(\vec{x},t)\xi_\beta(\vec{x}',t')\rangle = 2k_BT\delta_{\alpha\beta}e^{-|\vec{x}-\vec{x}'|^2\Lambda^2/4}\;\delta(t-t')\;, \tag{3.64}$$

$$\langle \xi_\alpha(\vec{k},t)\xi_\beta(\vec{k}',t')\rangle = 2k_BT(2\pi)^d\delta_{\alpha\beta}e^{-k^2/\Lambda^2}\;\delta(\vec{k}+\vec{k}')\;\delta(t-t')\;. \tag{3.65}$$

In the limit $\mathcal{N}\to\infty$ while keeping the dimension of real space fixed to d, the factor that couples the different components in the cubic term in the right-hand-side can be replaced by

$$\mathcal{N}^{-1}\phi^2(\vec{x},t) \to \mathcal{N}^{-1}\;\langle[\,\phi^2(\vec{x},t)\,]_{ic}\rangle \equiv \tilde{a}(t) \tag{3.66}$$

since $\mathcal{N}^{-1}\phi^2(\vec{x},t)$ does not fluctuate, it is equal to its average over the initial conditions and temperature, and it is therefore not expected to depend on the spatial position if the initial conditions are chosen from a distribution that is statistically translational invariant. For the scalar field theory the replacement (3.66) is just the **Hartree approximation**. The dynamic equation is now **linear** in the field $\phi_\alpha(\vec{x},t)$ that we rename $\phi(\vec{x},t)$ (and it is now order 1):

$$\partial_t\phi(\vec{x},t) = [\nabla^2 + a(t)]\phi(\vec{x},t) + \xi(\vec{x},t)\;, \tag{3.67}$$

where the time-dependent harmonic constant

$$a(t) = -\left[\phi_0^2 - \;\langle[\phi^2(\vec{x},t)]_{ic}\rangle\right] = -\left[\phi_0^2 - \tilde{a}(t)\right] \tag{3.68}$$

has to be determined self-consistently. The factor $1/\mathcal{N}$ disappeared since we are now working with a single component of the $\mathcal{N}$-vector $\vec{\phi}$.

Consistently with the decoupling performed above, the dynamics is **isotropic** in the $\mathcal{N}$-dimensional space implying that all α components have the same self-correlation and that they are not correlated between themselves:

$$C_{\alpha\beta}(\vec{x},t;\vec{x}',t') = \delta_{\alpha\beta}C(\vec{x},t;\vec{x}',t')\;. \tag{3.69}$$

Equation (3.67) can be Fourier transformed

$$\partial_t\phi(\vec{k},t) = [-k^2 + a(t)]\phi(\vec{k},t) + \xi(\vec{k},t)\;, \tag{3.70}$$

and it takes now the form of almost independent oscillators under different time-dependent harmonic potentials coupled only through the self-consistent condition on $a(t)$. The stability properties of the oscillators depend on the sign of the prefactor $-k^2 + a(t)$ in the RHS. The solution is

$$\phi(\vec{k},t) = \phi(\vec{k},0)\ e^{-k^2 t + \int_0^t dt'\ a(t')} + \int_0^t dt'\ e^{-k^2(t-t') + \int_{t'}^t dt'' a(t'')}\ \xi(\vec{k},t') \tag{3.71}$$

and the equation on $a(t)$ reads:

$$a(t) = -\phi_0^2 + \Delta\ e^{2\int_0^t dt' a(t')} \left(\frac{2\pi}{4t}\right)^{d/2} + k_B T \int_0^t dt' \left(\frac{2\pi}{4(t-t')}\right)^{d/2} e^{\int_{t'}^t dt'' a(t'')} \tag{3.72}$$

where one used $[\phi^2(\vec{x},t)]_{ic} = [\phi^2(\vec{0},t)]_{ic}$ and a delta-correlated Gaussian distribution of initial conditions with strength Δ. The self-consistency equation is not singular at $t = 0$ since there is an underlying cut-off in the integration over k corresponding to the inverse of the lattice spacing, this implies that times should be translated as $t \to t + 1/\Lambda^2$ with $\Lambda = 1/a$ the lattice spacing. This comment can be made rigorous by including a short-distance cut-off $a = 1/\Lambda$ in the noise-noise correlations:

$$\langle \xi_\alpha(\vec{x},t)\xi_\beta(\vec{x}',t')\rangle = 2k_B T \delta_{\alpha\beta} e^{-|\vec{x}-\vec{x}'|^2\Delta^2/4}\ \delta(t-t')\ , \tag{3.73}$$

$$\langle \xi_\alpha(\vec{k},t)\xi_\beta(\vec{k}',t')\rangle = 2k_B T (2\pi)^d \delta_{\alpha\beta} e^{-k^2/\Delta^2}\ \delta(\vec{k}+\vec{k}')\ \delta(t-t')\ . \tag{3.74}$$

Without giving all the details of the calculation, eq. (3.72) can be solved at all temperatures [46]. One finds that there exists a finite $T_c(d)$, the same that is found with the equilibrium analysis of the static free-energy, and

Upper-critical quench

$$a(t) \to -\xi_{eq}^{-2} < 0 \tag{3.75}$$

with ξ_{eq} the equilibrium correlation length, and the 'mass' (in field theoretical terms) or the harmonic constant saturates to a finite value: $-k^2 + a(t) \to -k^2 - \xi^{-2}$.

Critical quench

The time-dependent contribution to the harmonic term vanishes asymptotically

$$a(t) \to -\frac{w}{2t} \quad \text{with} \quad w = 0 \ \text{ for } \ d > 4 \ \text{ and } \ w = (d-4)/2 \ \text{ for } \ d < 4\ . \tag{3.76}$$

The dynamics is trivial for $d \geq 4$ but there is critical coarsening in $d < 4$. z_{eq} equals 2 in agreement with the result from the ϵ expansion once evaluated at $\mathcal{N} \to \infty$. The averaged field is

$$\langle \phi(\vec{k},t)\rangle \simeq \phi(\vec{k},0)\ e^{-k^2 t}\ t^{(4-d)/4} \tag{3.77}$$

Zero-temperature sub-critical coarsening

The zero-T equation admits a simple solution. In the long times limit in which the system tends to decrease its elastic and potential energies $[\,\phi^2(\vec{x},t)\,]_{ic}$ must converge to $\phi_0^2 \neq 0$ **below criticality** and this imposes $2\int_0^t dt'\, a(t') \simeq \frac{d}{2}\ \ln(t/t_0)$ with $t_0 = \pi/2\ (\Delta/\phi_0^2)^{2/d}$ at large times, *i.e.*

$$a(t) \simeq \frac{d}{4t} \qquad \text{for} \qquad t \gg t_0 \tag{3.78}$$

and the time-dependent contribution to the spring constant vanishes asymptotically.

Knowing the long-time behavior of $a(t)$ implies that each mode $[\phi(\vec{k},t)]_{ic}$ with $\vec{k} \neq 0$ vanishes exponentially both in critical and sub-critical quenches but the $\vec{k} = 0$ mode grows as $t^{d/4}$. The growth of the $\vec{k} = 0$ reflects the domain growth process whereby all modulations tend to disappear and the configuration gets more and more uniform as time passes. The averaged field is

$$\langle \phi(\vec{k},t)\rangle \simeq \phi(\vec{k},0)e^{-k^2 t}\ t^{d/4} \tag{3.79}$$

We focus now on two interesting cases: quenches to T_c and $0 = T < T_c$ (in this way the equilibrium relaxation is set to one in the correlation). We study the two space and two time correlation.

$$C(\vec{x},\vec{x}';t,t') \equiv [\,\langle \phi(\vec{x},t)\phi(\vec{x}',t')\rangle\,]_{ic} \tag{3.80}$$

This general correlation becomes the space-time one for $t = t'$ and the two-time one for $\vec{x} = \vec{x}'$.

Critical quench

The asymptotic behavior of the space-time correlation function is

$$[\,\langle \phi(\vec{x},t)\phi(\vec{x}',t')\rangle\,]_{ic} = \phi_0^2\, t'^{1-d/2} f(t/t')\ \exp\left[-\frac{(\vec{x}-\vec{x}')^2}{4(t+t')}\right]\ , \tag{3.81}$$

for a quench to T_c. We focus on $d < 4$. Note that ϕ_0 is still present

In critical quenches the two-time dependent pre-factor is of the form expected from dynamic scaling of $C(r,t)$ or $C(t,t')$ as discussed in eqs. (3.9) and (3.10) above.

The correlation decays to zero due to the prefactor that goes as $t^{(2-d)/2}$ and vanishes in all $d > 2$. The aging curves have an envelope that approaches zero as a power law. $d = 2$ is the lower critical dimension in this problem.

Sub-critical quench

The asymptotic behavior of the space-time correlation function after a quench to $T = 0$ is

$$[\,\phi(\vec{x},t)\phi(\vec{x}',t')\,]_{ic} = \phi_0^2 \left[\frac{4tt'}{(t+t')^2}\right]^{d/4}\ \exp\left[-\frac{(\vec{x}-\vec{x}')^2}{4(t+t')}\right]\ , \tag{3.82}$$

for $t \geq t'$ for a quench to $T < T_c$.

- The expression above has to be complemented with an additive contribution for quenches to finite temperature, $0 < T < T_c$. We are showing here the ageing contribution only.
- In sub-critical quenches, for any finite and fixed $(\vec{x}-\vec{x}')$, in the long times limit the exponential factor approaches one and one obtains a function of t'/t only.
- Due to the exponential factor, for fixed but very large time t and t' the correlation falls off to zero over a distance $|\vec{x}-\vec{x}'| \propto \sqrt{t+t'}$. This means that, at time t, the typical size of the equilibrated regions is $R(t) \propto t^{1/2}$. This holds for critical and sub-critical quenches as well and it is a peculiar property of the large $\mathcal{N}$ $O(\mathcal{N})$ model that has $z_{eq} = z_d$.
- For fixed $|\vec{x}-\vec{x}'|$, the correlation always falls to zero over a time separation $t-t'$ *which is larger than* t'. This means that the time it takes to the system to decorrelate from its configuration at time t' is of the order of t' itself, $t_d \simeq t'$. The age of the system is the characteristic time-scale for the dynamical evolution: the older is the system, the slower is its dynamics. After a time of the order of the age of the system any point $\vec{x}$ will be swept by different domain walls and the correlation will be lost.
- In a sub-critical quench, for any finite and fixed $(\vec{x}-\vec{x}')$, in the long t' and t limit such that $t'/t \to 1$ the time dependence disappears and the correlation between two points converges to ϕ_0^2. This means that, typically, if one looks at a finite spatial region on a finite time-scale this region will be in one of the two states $\pm\phi_0$, i.e. within a domain.

Note that we have obtained the field and then computed correlations from the time-dependent configuration. We have not needed to compute the linear response. We will see later that in other more complex glassy systems one cannot follow this simple route and one needs to know how the linear response behave. We refer to the reviews in [56] for detailed accounts on the behavior of the linear response in critical dynamics.

3.8 The $2d$ XY model

The classical xy, rotator or $O(2)$ model is a special case of the $O(N)$ model. This model is quite special in $d=2$. Although there is no exact solution, contrary to the Ising case, several very convincing arguments and approximate calculations allowed one to reach a good understanding of its static behaviour. The model has a Kosterlitz-Thouless transition at T_{KT} that separates a disordered high-T from a critical low-T phase, both of them with vanishing order parameter, $m = 0$.[13] This phase transition is not accompanied by symmetry breaking. Topological defects proliferate in the disordered phase and they bind in pairs in the one with quasi long-range order and correlation functions that decay algebraically. Physical realisations are two-dimensional

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[13] N. D. Mermin and H. Wagner, *Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models*, Phys. Rev. Lett. **17**, 1133 - 1136 (1966).

planar ferromagnets, superconducting films, Josephson-junction arrays, especially tailored nematic liquid crystals and toy models for two-dimensional turbulence. The $2d$ xy model is the paradigm in this class of *topological phase transitions*.

For any translational invariant two-body interaction $J_{ij} = J(i-j)$ and a site-dependent external field h_i, the Hamiltonian is

$$\begin{aligned} H &= -\sum_{i\neq j} J_{ij}\vec{s}_i\cdot\vec{s}_j - \sum_i \vec{h}_i\cdot\vec{s}_i & (3.83)\\ &= -\sum_{i\neq j} J_{ij}\cos(\theta_i-\theta_j) - \sum_i h_i\cos(\theta_i-\theta_{h_i}) & (3.84)\end{aligned}$$

The model is fully solvable in the spin-wave approximation in which the field is supposed to vary smoothly in space and, hence, vortices are neglected. Indeed, at low temperature one can assume that the spins vary little from site to site the cos function can be expanded to second order and, in the absence of external fields, the Hamiltonian becomes

$$H \simeq -\sum_{i\neq j} J_{ij}\left[1-\frac{1}{2}(\theta_i-\theta_j)^2\right] . \qquad (3.85)$$

In the continuum limit the two-component spin is parametrized as

$$\vec{s}(\vec{x}) = \cos\theta(\vec{x})\ e_x + \sin\theta(\vec{x})\ e_y \qquad (3.86)$$

where the modulus $|\vec{s}(\vec{x})|$ has been fixed to one at all space points $\vec{x}$ and the angle $\theta \in (-\pi,\pi]$ is measured between the local spin and a chosen fixed axis. Taking now an homogeneous case, $J_{ij} = J$ for all ij pairs, the functional Ginzburg-Landau free-energy is proposed to be

$$F = \frac{J}{2}\int d^2x\ (\vec{\nabla}\theta(\vec{x}))^2 \qquad (3.87)$$

where an irrelevant additive constant has been neglected. With the Fourier transform $\theta(\vec{x}) = (2\pi)^{-2}\int d^dk e^{i\vec{x}\cdot\vec{k}}\theta(\vec{k})$ one has

$$F = \frac{J}{2}\int \frac{d^2k}{(2\pi)^2}\ k^2|\theta(\vec{k})|^2 \qquad (3.88)$$

and the modes are now manifestly decoupled. Within this approximation, the angle correlation functions in equilibrium are

$$\langle\left(\theta(\vec{r})-\theta(\vec{0})\right)^2\rangle = \frac{k_BT}{\pi J}\ln\frac{r}{a} \qquad (3.89)$$

leading to

$$C(r) = \langle\vec{s}(\vec{r})\cdot\vec{s}(\vec{0})\rangle = \left(\frac{a}{r}\right)^{k_BT/\pi J} = \left(\frac{a}{r}\right)^{\eta(T)} \qquad (3.90)$$

a power-law decay typical of a critical point. Spin-waves are non-local and extensive excitations.

The high-temperature series analysis of the partition function shows that the correlation function decays exponentially in this limit, $C(r) \simeq e^{-r/\xi_{eq}}$, with a correlation length $\xi_{eq}(T) \propto a/\ln(k_BT/J)$ that tends to zero only at $T \to \infty$ and diverges at $T \to 0$.

The different decays found at low and high T imply that there should be a phase transition in between. When the effect of vortices is included this finite temperature phase transition is found. Vortices are local and intensive. They cannot be eliminated by simple perturbations but they annihilate when a vortex and an anti-vortex encounter.

The over-damped Langevin equation acting on the angle θ is

$$\partial_t \theta(\vec{x}) = \nabla^2 \theta(\vec{x}, t) + \xi(\vec{x}, t) \tag{3.91}$$

where a white-noise scalar noise is proposed to act additively on the angle θ and the friction coefficient has been set to one. This equation can be readily solved in Fourier space

$$\theta(\vec{k}, t) = \theta(\vec{k}, 0) e^{-k^2 t} + \int_0^t dt' \; e^{-k^2(t-t')} \; \xi(\vec{k}, t') \; . \tag{3.92}$$

The noise-noise correlation is usually taken to be delta-correlated in space and time. As for the large $\mathcal{N}$ field it is convenient to use a microscopic short-distance cut-off, a, inherited from the lattice spacing that translates into a large k cut-off Λ:

$$\langle \xi(\vec{k}, t) \xi(\vec{k}', t') \rangle = (2\pi)^d 2 k_B T e^{-k^2/\Lambda^2} \delta(\vec{k} + \vec{k}') \delta(t - t') \; . \tag{3.93}$$

The global correlation and linear response, $C(t, t') = V^{-1} \int d^2x \; \langle \vec{s}(\vec{x}, t) \cdot \vec{s}(\vec{x}, t') \rangle$ and $R(t, t') = V^{-1} \int d^2r \; \left. \frac{\delta \langle \vec{s}(\vec{x}, t) \rangle}{\delta \vec{h}(\vec{x}, t')} \right|_{\vec{h}=0}$ take the following scaling forms in the limit $t - t' \gg \Lambda^{-2}$:

$$C(t, t') \sim \frac{1}{(t - t')^{\eta(T)/2}} \; \Phi_C \left(\frac{R_c(t)}{R_c(t')} \right) \tag{3.94}$$

$$R(t, t') \sim \frac{1}{4\pi \rho(T) (t - t')^{1 + \eta(T)/2}} \; \Phi_R \left(\frac{R_c(t)}{R_c(t')} \right) \tag{3.95}$$

with Φ_C and Φ_R two scaling functions and $R_c(t)$ the growing correlation length (that should not be confused with the linear response). The first remarkable property of these functions is that they are both decomposed in the product of a function of the time-difference $t - t'$ and a function of the ratio $\lambda \equiv R_c(t')/R_c(t)$, like in the general critical coarsening case. When $t - t' \ll [d \ln R_c(t')/dt']^{-1}$, the argument of the scaling functions gets close to one, $\lambda \sim 1$, and the decay is stationary

$$C(t, t') \sim (t - t')^{-\eta(T)/2} \; , \quad R(t, t') \sim (t - t')^{-1 - \eta(T)/2}$$

and the FDR equals one. This limit defines a quasi-equilibrium regime. When the time difference increases and λ becomes larger than one the relaxation enters an aging regime in which the decay of the correlation and response depends on the waiting-time t'. The behavior in the aging regime depends on the initial conditions as discussed below.

Uniform initial conditions.

The uniform initial condition contains no free vortices and none are generated by thermal fluctuations at any $T < T_{\rm KT}$. The evolution is well captured by the simple spin-wave approximation and after a simple calculation one finds

$$\Phi_C\left(\frac{R_c(t)}{R_c(t')}\right) = \left[\frac{(1+\lambda)}{4\lambda}\right]^{\eta(T)/4} , \qquad R_c(t) = t^{1/2} . \tag{3.96}$$

Beyond the crossover time $t-t' \sim t'$, when $C(2t',t') \sim t'^{-\eta(T)/2}$ and λ becomes smaller than one, the correlation and response decay to zero as power laws of the waiting-time t'. There is no clear-cut separation of time-scales characterised by the correlation reaching a constant value independently of the waiting-times but only a t' dependent pseudo-plateau where the behavior of the two-time correlation changes. This is to be confronted to the behavior of ferromagnetic coarsening systems quenched to the low-temperature phase for which the crossover occurs at $C(2t',t') = m_{eq}^2$. Above this plateau, the relaxation corresponds to the equilibrium fluctuations of short wave-length while below the plateau the decorrelation is due to the domain-wall motion that manifests into a scaling in terms of $\lambda = t'/t$ only. In the 2d XY case the order parameter vanishes and there is no plateau at any finite value of C.

In the aging regime the fluctuation – dissipation ratio is larger than one. This *a priori* surprising result can be understood when interpreted in terms of the effective – temperature. The completely order configuration is the equilibrium state at zero temperature. The evolution of this initial state at finite temperature can be thought of as representing a sudden inverse quench of the system from $T = 0$ to $T > 0$. If the FDR is related to a remembrance of the temperature of the initial condition, in this case this is lower than the working temperature T and thus, the effective temperature also turns out to be lower than T.

Random initial conditions.

When random initial conditions with only short-ranged spatial correlations are considered, free vortices and antivortices are present initially. The relaxation occurs via the annihilation of vortex-antivortex pairs and this coarsening process is much slower than the relaxation of spin-waves. The simple Gaussian theory is no more suited to describe this dynamics and a full analytic treatment is too hard to implement. With scaling and numeric analysis the dynamic correlation length has been estimated to be [4]

$$\boxed{R_c(t) \sim (t/\ln t)^{1/2} \ .}$$

The numerical simulations of Berthier, Holdsworth and Sellitto have proven that the two-time correlation and response are correctly described by the scaling form (3.94) and (3.95) with this length scale and the full decay looks like the one shown in the sketch above. The FDR is rather different from the one following the evolution of a uniform initial condition. The non-equilibrium susceptibility is now smaller than the equilibrium value, and in terms of the effective temperature this means that the fluctuations of the wave-lengths longer than $R_c(t)$ occur at a $T_{\rm EFF} > T$ and hence keep a memory of the initial temperature $T = \infty$. The effective temperature will be discussed later.

3.9 Annealing: crossover from critical to subcritical coarsening

There has been recent interest in understanding how a finite rate cooling affects the defect density found right after the quench. A scaling form involving equilibrium critical exponents was proposed by Zurek following work by Kibble. The interest is triggered by the similarity with the problem of quantum quenches in atomic gases, for instance. An interplay between critical coarsening (the dynamics that occurs close in the critical region) that is usually ignored (!) and sub-critical coarsening (once the critical region is left) is the mechanism determining the density of defects right after the end of the cooling procedure.

The usual protocol is such that the control parameter is tuned linearly in time with a convention such that at time zero the system crossed the critical point. In this way, at negative times the system is in the disordered phase while at positive times it enters the ordered one. In short,

$$T(t) = T_c \left(1 - \frac{t}{\tau_Q}\right) \;, \tag{3.97}$$

with τ_Q the annealing time.

We assume that, for positive time t, the dynamic growing length that, for infinite rapid quenches, is a function of time and the control parameter, $R(t, T)$, becomes, for very slow quenches a function of time in the form $R(t, T(t))$. Then it is natural to propose a new scaling behaviour for the growing length

$$\begin{aligned} R(t, \epsilon(t)) &\sim \epsilon^{-\nu}(t)\ f[t\epsilon^{z_{eq}\nu}(t)] \qquad\qquad \epsilon(t) = |T(t) - T_c|/T_c \\ &\sim \xi_{eq}(T(t))\ f\left[\frac{t}{\xi_{eq}^{z_{eq}}(T(t))}\right] \end{aligned} \tag{3.98}$$

with the limiting values

$$f(x) \to \left\{ \begin{array}{lll} \mathrm{ct} & x \ll -1 & \text{Equilibrium at high } T \\ x^{1/z_d} & x \gg 1 & \text{Coarsening at low } T \end{array} \right.$$

t is measured from the instant when the critical point is crossed and $x \in (-1, 1)$ is the critical region.

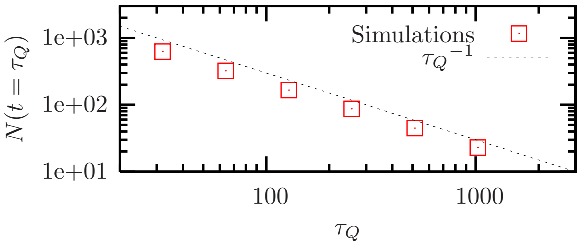

Figure 35: The number of interfaces after an annealing (quench at finite rate) as defined in (3.97) through the critical point, measured at time $t = \tau_Q$, *i.e.* at $T = 0$, in a 2dIM.

Dynamic scaling allows one to relate the growing length to the number of defects, $n_d(t, \tau_Q) \simeq R^{-d}(t, \tau_Q)$ and one finds

$$n_d(t, \tau_Q) \simeq \tau_Q^{d\nu(z_{eq}-z_d)/z_d} \; t^{-d[1+\nu(z_{eq}-z_d)]/z_d} \tag{3.99}$$

that is to say a function of both t and τ_Q. In particular, for $t \simeq \tau_Q$ one has

$$N_d(t \simeq \tau_Q, \tau_Q) \simeq n_d(t \simeq \tau_Q, \tau_Q) L^2 \simeq \tau_Q^{-1} \tag{3.100}$$

which is very different from the behaviour found assuming there is no more dynamics after the system falls out of equilibrium in the disordered phase ($N_{KZ} \simeq \tau_Q^{-\nu/(1+\nu z_{eq})} \simeq \tau_Q^{-0.31}$). In particular, $N_d \ll N_{KZ}$.

A careful analysis of this problem can be found in Biroli, LFC, Sicilia (2010) and the extension to the Kosterlitz-Thouless transition in Jelic, LFC (2012).

3.10 Effect of external agents or non-Hamiltonian dynamics

As mentioned in the Introductory section, external forces can contribute to maintain a given system out of equilibrium.

In the context of coarsening problems, such forces can be shearing ones, that can, for instance, impose a given length in one direction and let the system coarsen in the orthogonal one. Note that a shear is a force that does not derive from a potential.

Another external effect of interest is the one induced by a gravitational force that may act differently on different species in the problem provoking phase separation in the vertical direction.

Intriguing coarsening effects, in the form of phase separation, also exist in other areas of science. Ecology is one of them. Phase separation exists in models à la Lotka-Volterra of different species in interaction in a given space. Interactions can be of predator-prey type and the effect of the environment is taken into account via some reaction rates. Sociology is another one where, for instance, models with discrete variables taking three values $(\pm, 0)$, the so-called Blume-Capel model, is used to mimic neighbour segregation in towns.

3.11 Summary

In the table below we summarize the results describe above.

In short, critical and sub-critical coarsening occurs in models with conventional second order phase transitions (or for systems with first order phase transitions when one quenches well below the region of metastability). Close to the critical point the dynamics is characterized by **critical slowing down** with the relaxation time diverging as a power law of the distance to criticality. Growth of order is characterized by a growing length that depends on time as a power law at criticality and with a different power below the transition (in the absence of disorder). The dynamic mechanisms are well understood but quantitative results are hard to obtain since the equation to solve are highly non-linear and there is no small parameter to expand around.

In structural glasses the slowing down is not of power law type so such a simple coarsening description seems to be excluded for these systems.

For spin-glasses this modeling has been pushed by Bray, Moore, Fisher and Huse. It is not clear whether it is correct as no clearcut experimental evidence for the coarsening type of scaling has been presented yet.

3.12 Nucleation and growth

In a **first-order** phase transition the equilibrium state of the system changes abruptly. Right at the transition the free-energies of the two states involved are identical and the transition is driven by lowering the free-energy as the new phase forms, see Fig. 22. The original phase remains meta-stable close to the transition. The nucleation of a sufficiently large bubble of the truly stable phase into the metastable one needs to be thermally activated to trigger the growth process [3]. The rate of the process can be very low or very fast depending on the height of the free-energy barrier between the two states and the ambient temperature.

Two types of nucleation are usually distinguished: homogeneous (occurring at the bulk of the meta-stable phase) and heterogeneous (driven by impurities or at the surface). The more intuitive examples of the former, on which we focus here, are the condensation of liquid droplets from vapor and the crystallization of a solid from the melt.

The **classical theory of nucleation** applies to cases in which the identification of the nucleus is easy. It is based on a number of assumptions that we now list. First, one associates a number of particles to the nucleus (although in some interesting cases this is not possible and a different approach is needed). Second, one assumes that there is no memory for the evolution of the number densities of clusters of a given

	g_c	$g < g_c$
Order param.	0	$\neq 0$
Growing length	$R_c(t) \simeq$ $t^{1/z_{eq}}$ clean $\frac{t^{\frac{1}{2}}}{\ln^{\frac{1}{2}} \frac{t}{t_0}}$ $2d$ xy ? disordered	$R(t) \simeq$ $t^{1/2}$ sc. NCOP $t^{1/3}$ sc. COP $\left(\ln \frac{t}{t_0}\right)^{\frac{1}{\psi}}$ dis.
$V \simeq R^{D_F^V}(t)$	$D_F^V < d$	$D_F^V = d$
$S \simeq R^{D_F^S}(t)$	$D_F^S < d-1$	$D_F^S = d-1$
$C(r,t)$	$r^{2-d-\eta}\ f\left(\frac{r}{R_c(t)}\right)$	$C_{st}(r) +\ C_{ag}\left(\frac{r}{R_c(t)}\right)$
$C(t,t')$	$R_c^{2-d-\eta}(t-t')\ g\left(\frac{R_c(t')}{R_c(t)}\right)$	$C_{st}(t-t') + C_{ag}\left(\frac{R_c(t')}{R_c(t)}\right)$

Table 1: This table summarizes the behavior of growing structures and correlation functions in critical and sub-critical quenches. V and S are the volume and surface of the equilibrium growing structures (FK clusters and geometric domains in critical and sub-critical quenches respectively). D_F^V and D_F^S are their fractal dimension. Interesting information is also contained in the behavior of the linear response function but we will discuss it later.

size in time (concretely, a Markov master equation is used). Third, one assumes that clusters grow or shrink by attachment or loss of a single particle, that is to say, coalescence and fission of clusters are neglected. Thus, the length-scale over which the slow part of the dynamics takes place is the one of the critical droplet size, the first one to nucleate. Fourth, the transition rates satisfy detail balance and are independent of the droplet form. They just depend on the free-energy of the droplet with two terms: a contribution proportional to the droplet volume and the chemical potential difference between the stable and the metastable states, Δf, and a contribution proportional to the bubble surface that is equal to the surface area times the surface tension, σ, that is assumed to be the one of coexisting states in equilibrium - that is to say the energy of a flat domain wall induced by twisted boundary conditions. Fift, the bubble is taken to be spherical and thus dependent of a single parameter, the radius. Thus

$$\Delta F[R] = \sigma \ \Omega_{d-1} \ R^{d-1} - |\Delta f| \ \Omega_d \ R^d \tag{3.101}$$

for $d > 1$. Ω_d is the volume of the unit sphere in d dimensions. For small radii the surface term dominates and it is preferable to make the droplet disappear. In contrast, for large radii the bulk term dominates and the growth of the bubble is favored by a decreasing free-energy. Thus the free-energy difference has a maximum at

$$\boxed{R^* = \frac{(d-1) \ \Omega_{d-1} \ \sigma}{d \ \Omega_d \ |\Delta f|} \propto \sigma |\Delta f|^{-1}} \tag{3.102}$$

and the system has to thermally surmount the barrier $\Delta F^* \equiv \Delta F[R^*]$. The Kramers escape theory, see Sect. 2.3, implies that the nucleation rate or the average number of nucleations per unit of volume and time is suppressed by the Arrhenius factor

$$\boxed{r_A = t_A^{-1} \sim e^{-\beta \Delta F^*} \qquad \text{with} \qquad \Delta F^* = \frac{(d-1)^{d-1}}{d^d} \ \frac{\Omega_{d-1}^d}{\Omega_d^{d-1}} \ \frac{\sigma^d}{|\Delta f|^{d-1}}} \tag{3.103}$$

As expected, ΔF^* increases with increasing σ and/or $|\Delta f|^{-1}$ and r^{-1} vanishes for $T \to 0$ when thermal agitation is switched off. The implicit assumption is that the time to create randomly the critical droplet is much longer than the time involved in the subsequent growth. The relaxation of the entire system is thus expected to be given by the inverse probability of escape from the metastable well. The determination of the pre-factor [that is ignored in eq. (3.103)] is a hard task.

3.13 Elastic manifold dynamics

An interface is a frontier separating two regions of space with two phases. It could be the border between water and oil in a liquid mixture, the border between regions with positive and negative magnetization in a magnet, the limit of a fluid invading