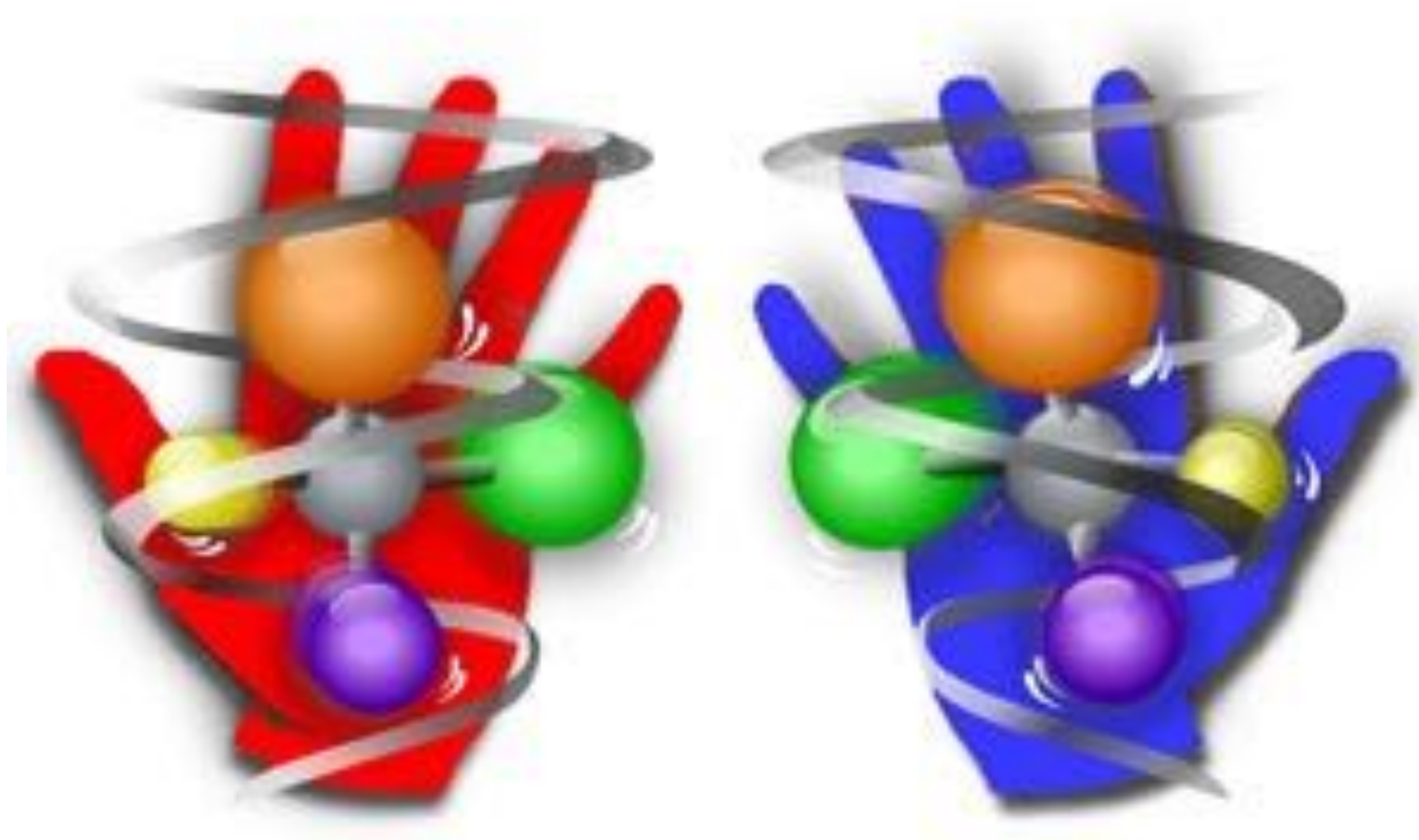
Global chiral symmetry, extra dimensions, and topology



Chiral symmetry

★ Approximate global chiral symmetry protects vector-like gauge theories (QED, QCD) from O(1/a) fine-tuning of fermion mass

★ Approximate global chiral symmetry gives proper pseudoscalar spectrum and weak interactions

★ Local chiral symmetry required for SM, GUTS, various BSM models

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This talk:

★ What we know about global chiral symmetry on the lattice

The Euclidian fermion action:

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^4}\, \overline{\Psi}_{-\mathbf{p}} \tilde{D}(\mathbf{p}) \Psi(\mathbf{p})$$

Nielsen-Ninomiya — can't have all four properties:

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_μ; $\Leftarrow$ regulated, local
2. $D(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$; $\Leftarrow$ Dirac @ long wavelength
3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_\mu = 0$; $\Leftarrow$ No doubling of flavors
4. $\{\Gamma, \tilde{D}(\mathbf{p})\} = 0$. $\Leftarrow$ respects a chiral symmetry

Γ is the γ_5 matrix... defined only in even spacetime dimensions

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e.g., Wilson fermions:

Lattice covariant derivatives

$$\mathcal{L} = \bar{\psi}\left(\not{D} + m + aD^2\right)\psi$$

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★Requires O(1/a) fine tuning to get to symmetric point
★Also breaks non-anomalous symmetries, eg $SU(2)_L \times SU(2)_R$

The minimalist way to explicitly break anomalous global chiral symmetries:

★ domain wall fermions

★ overlap fermions

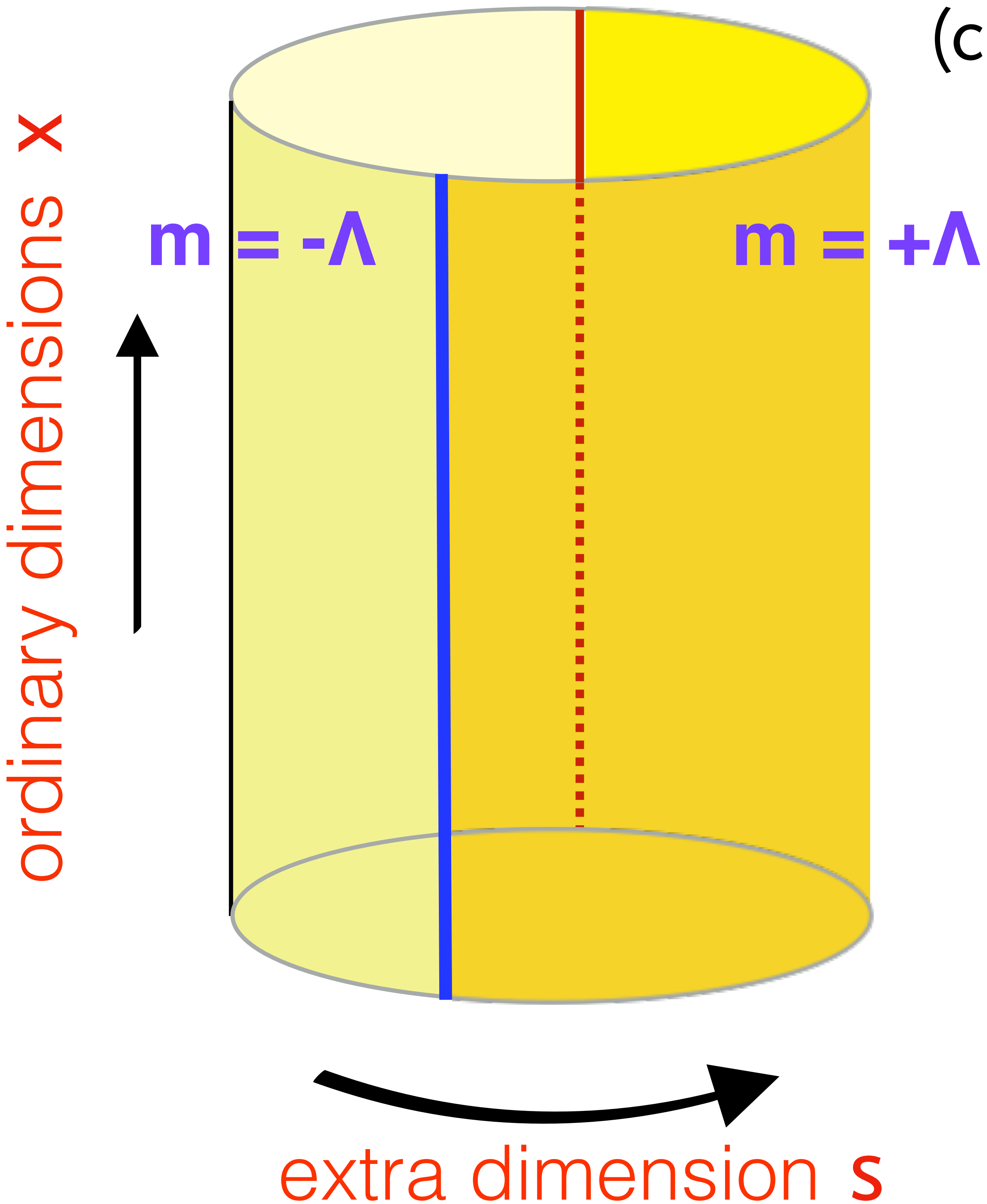
‣The only relevant symmetry breaking is proportional to continuum anomalies

‣Topology allows one to avoid any fine tuning.

‣A plausible starting point for the formulation of chiral gauge theories.

Domain Wall Fermions:

Consider a Dirac fermion in 2+1 (continuous) dimensions



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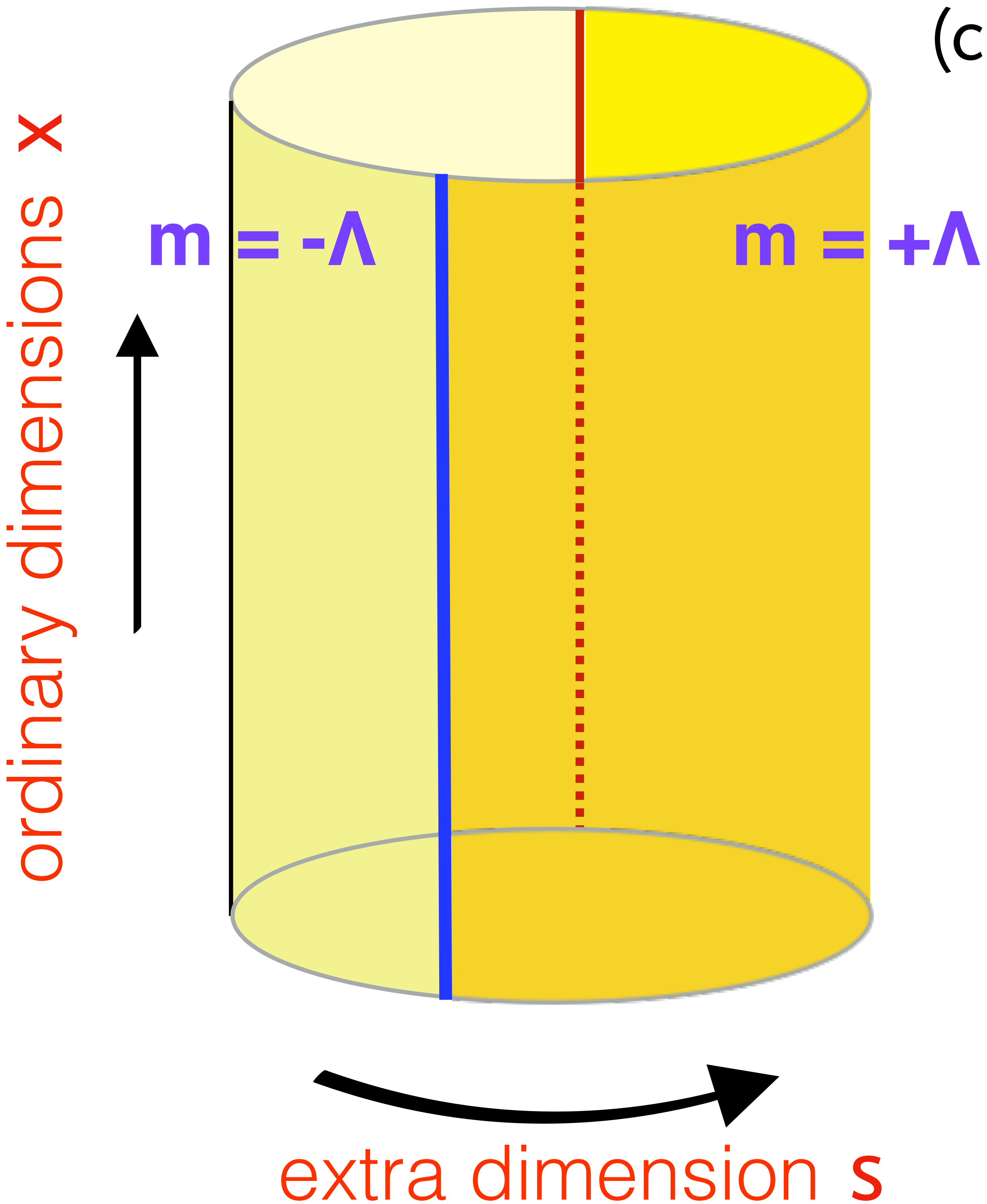

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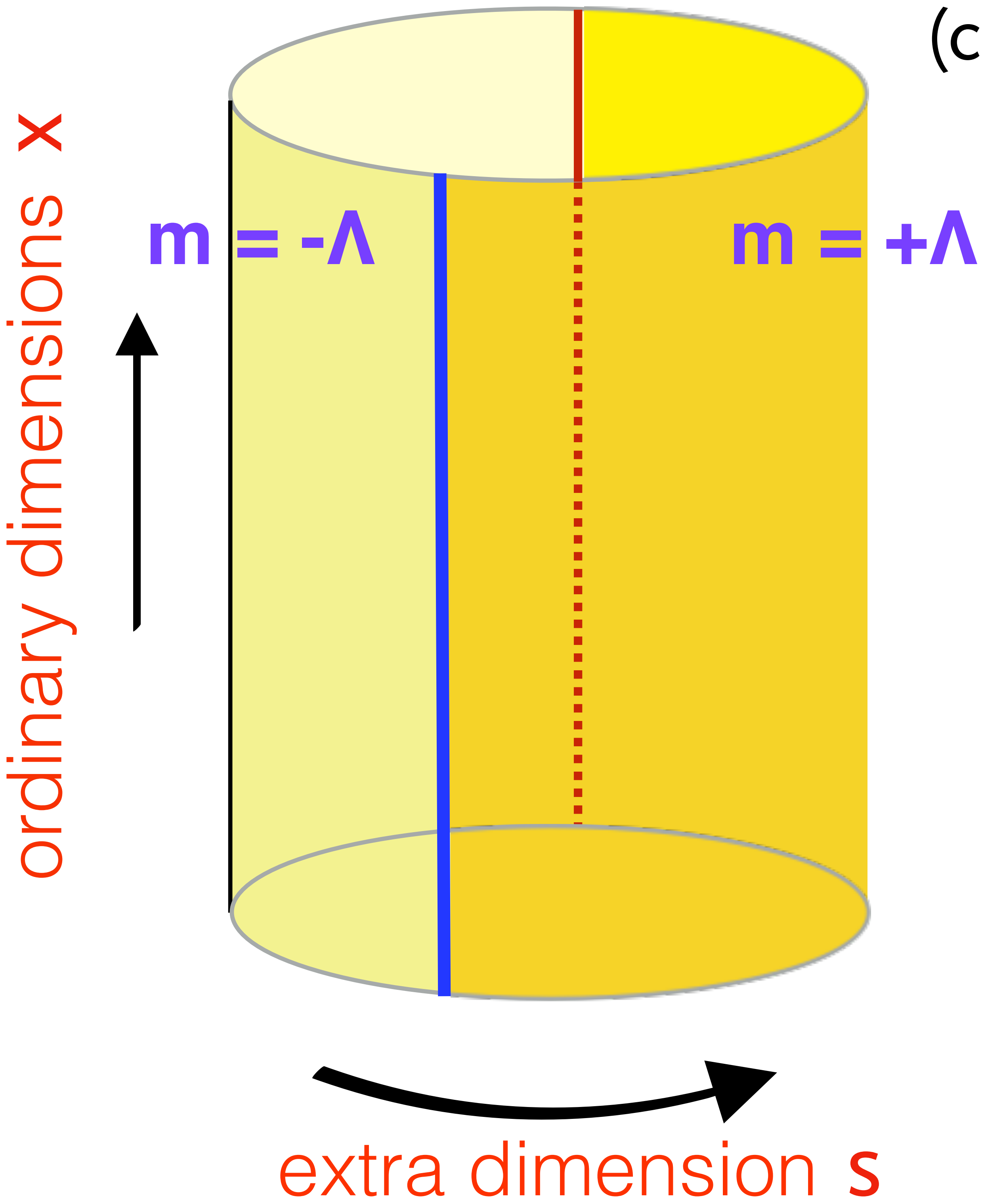

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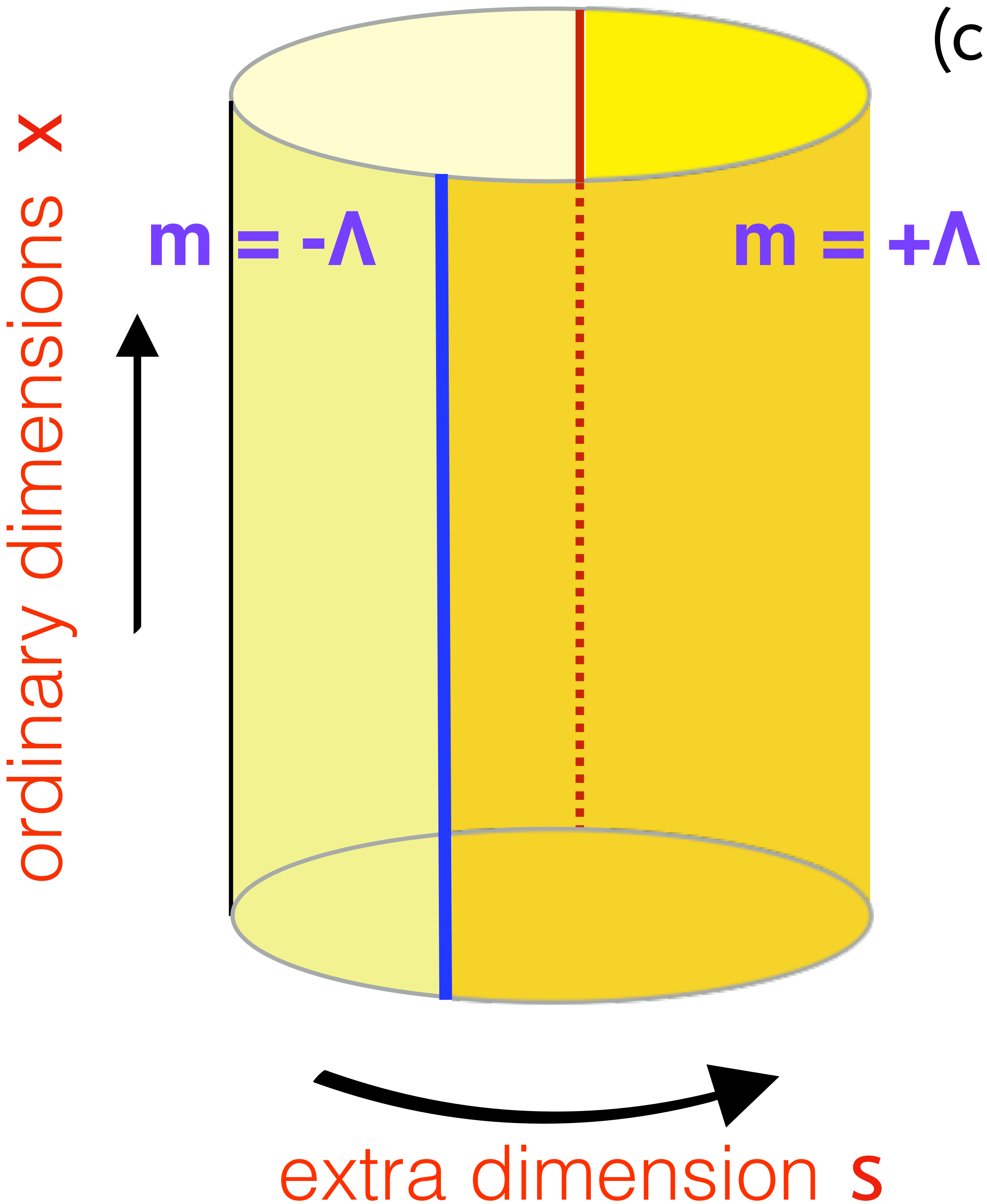


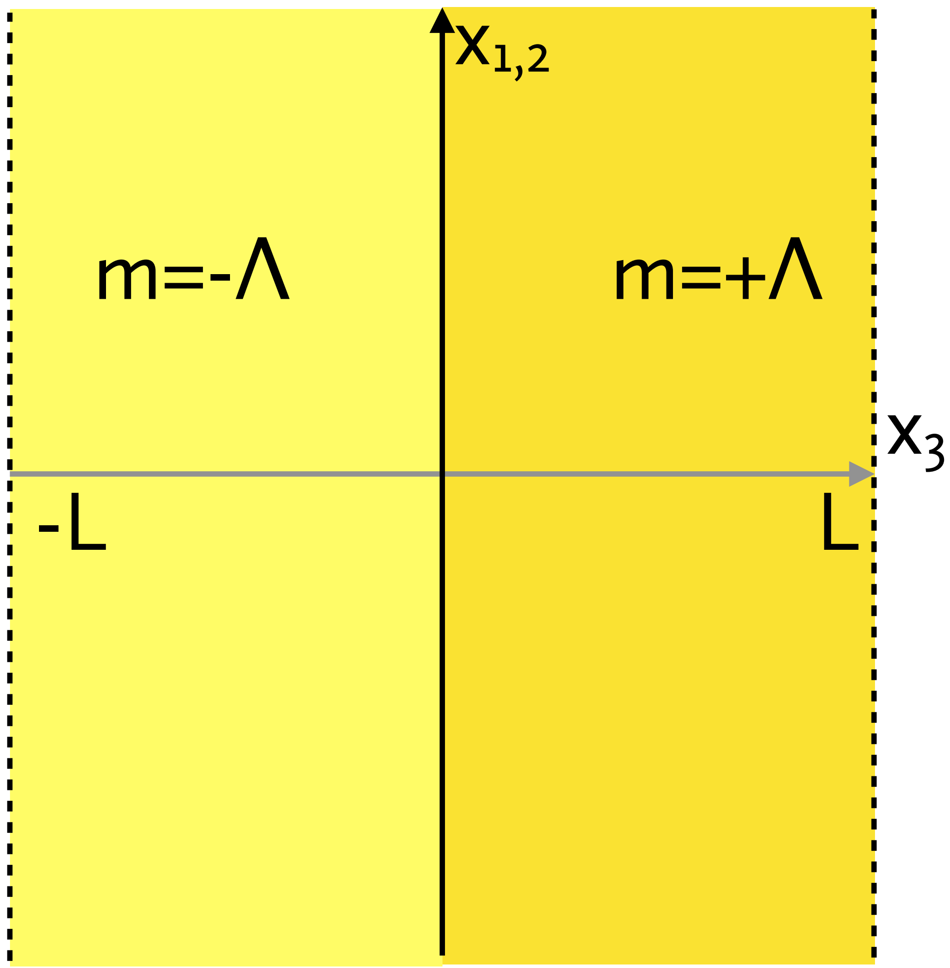
★ One spatial direction with coordinate x_3 is compactified with periodic boundary conditions

★ The fermion mass = $+\Lambda$ on one side, $-\Lambda$ on the other

★ Assume 1+1 dimensional gauge fields independent of x_3: $\{A_1(x_1,x_2), A_2(x_1,x_2)\}$

The Dirac equation is separable; expand field as

$$\Psi(\mathbf{x}, x_3) = \sum_{n=0}^{\infty} \sum_{i=1,2} \left[P_- \psi_{ni}(\mathbf{x}) b_{ni}(x_3) + P_+ \psi_{ni}(\mathbf{x}) f_n(x_3)\right]$$



- $P_\pm = (1 \pm \gamma_3)/2$
- b,f = functions of x_3
- ψ = 2-component Grassmann spinor, function of x_1, x_2
- *i* denotes even or odd in x_3 ▸ $-x_3$

Choose b,f to be solutions to a particular eigenfunction equation

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b,f are eigenfunctions of a "supersymmetric hamiltonian"

$$Q f_{ni} = \mu_n b_{ni}\ , \quad Q^\dagger b_{ni} = \mu_n b_{ni}\ , \qquad Q = (\partial_3 - \Lambda\epsilon(x_3))$$

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find exact zero modes localized on the boundaries

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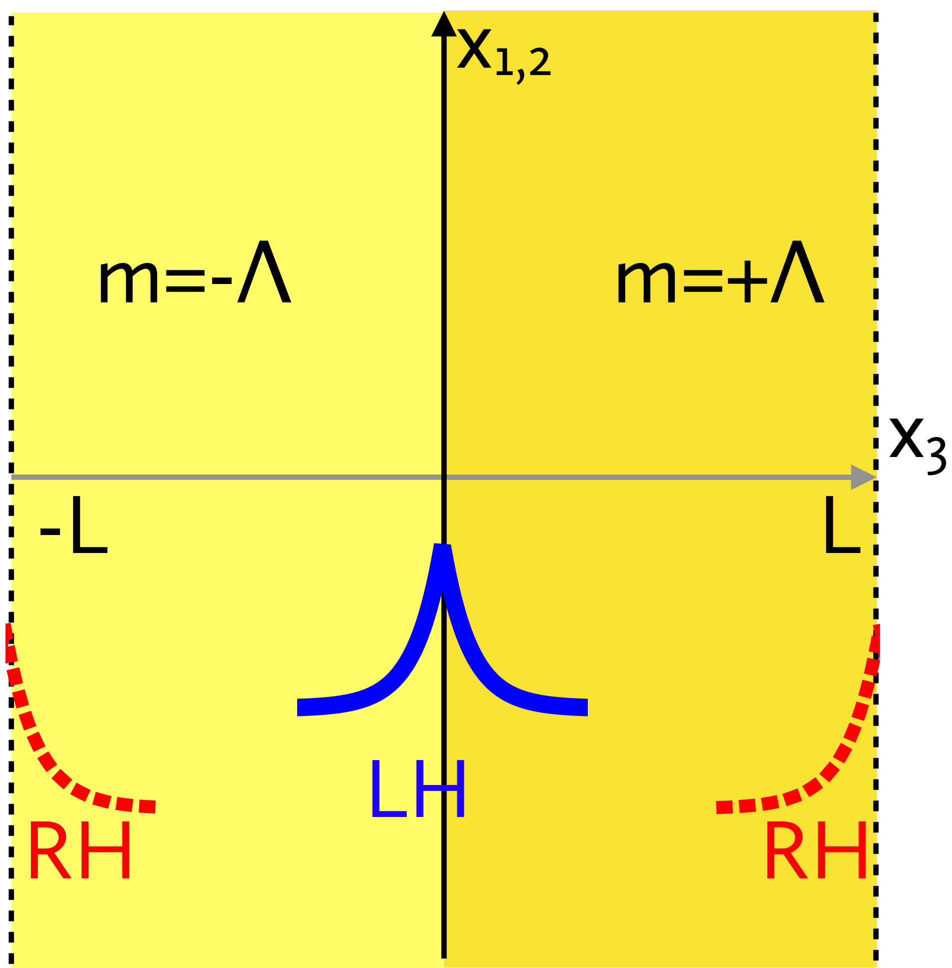
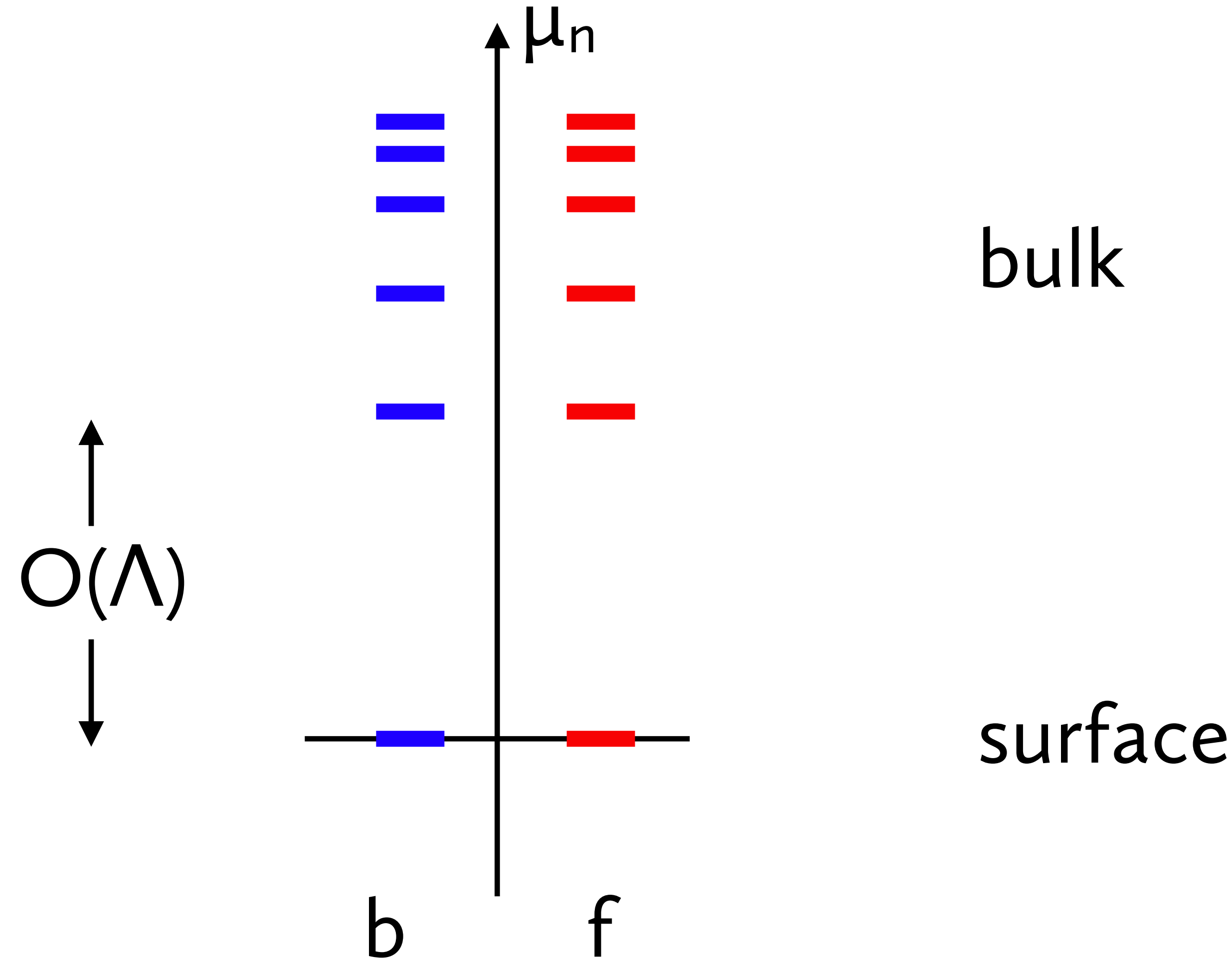




Now plug this expansion into the 3d action:

$$\Psi(\mathbf{x}, x_3) = \sum_{n=0}^{\infty} \sum_{i=1,2} \left[P_- \psi_{ni}(\mathbf{x}) b_{ni}(x_3) + P_+ \psi_{ni}(\mathbf{x}) f_n(x_3)\right]$$

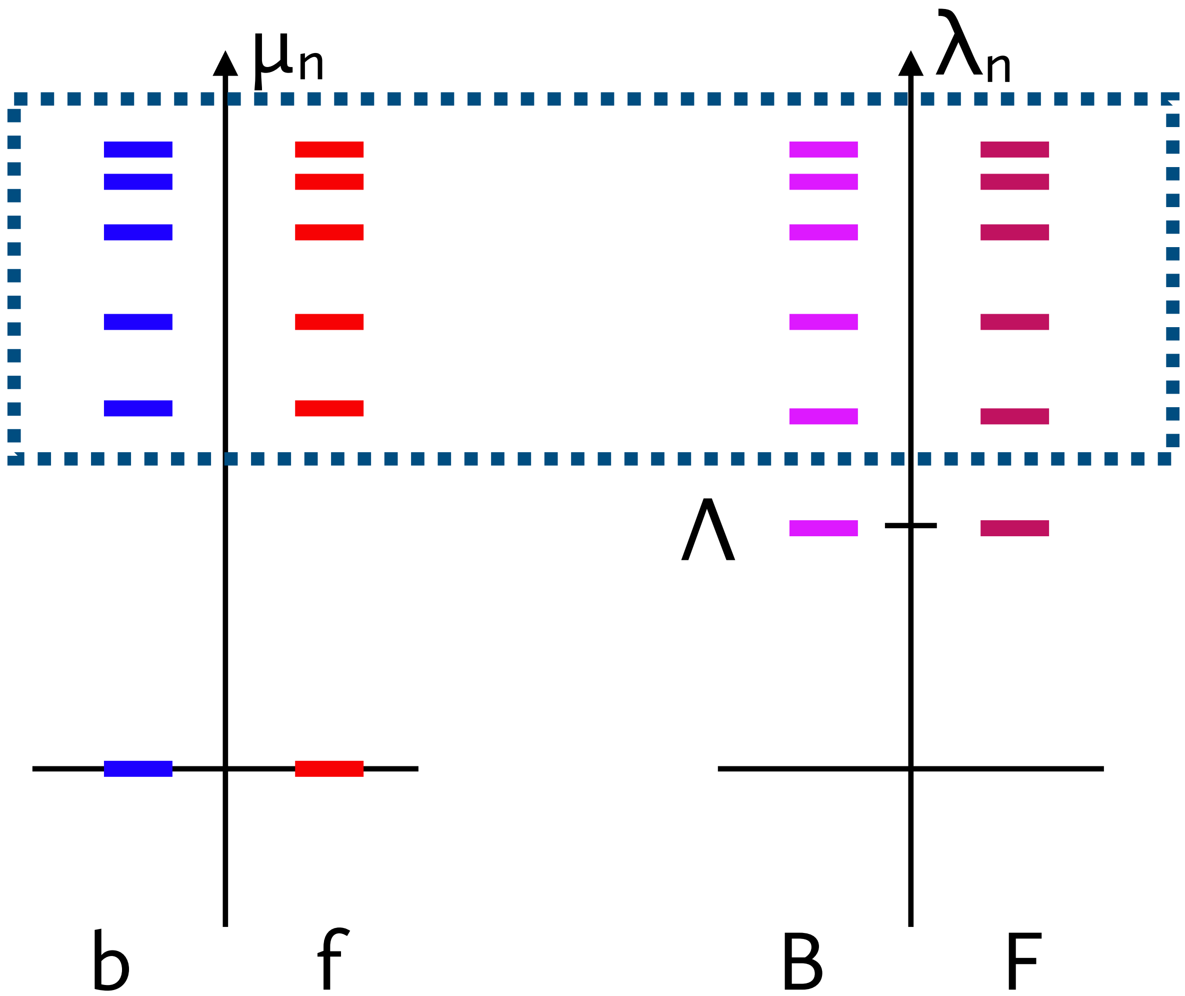
$$\begin{aligned} S &= \int d^3x \, \bar{\Psi} \left(\not{D}_3 - \Lambda \epsilon(s) \right) \Psi \\ &= \int d^2x \sum_{n,i} \bar{\psi}_{n,i} \left(\not{D}_2 + \mu_n \right) \psi_{n,i} \end{aligned}$$

- Looks like an infinite tower of states
- All have mass O(Λ) except two localized massless surface modes
- The two surface modes are chiral
- The overlap of their wave functions in x_3 is proportional to e^{-2L}

Robust? Any perturbation can only induce exponentially small mass for surface modes, proportional to e^{-2L}

Can eliminate tower of massive bulk modes by introducing Pauli-Villars field with constant mass:

$$S = \int d^3x\, \bar\Psi\left(\not{D}_3 - \Lambda\epsilon(s)\right)\Psi + \bar\Phi\left(\not{D}_3 - \Lambda\right)\Phi$$
$$= \int d^2x \sum_{n,i} \bar\psi_{n,i}\left(\not{D}_2 + \mu_n\right)\psi_{n,i} + \bar\phi_{n,i}\left(\not{D}_2 + \lambda_n\right)\phi_{n,i}$$



Exactly cancel

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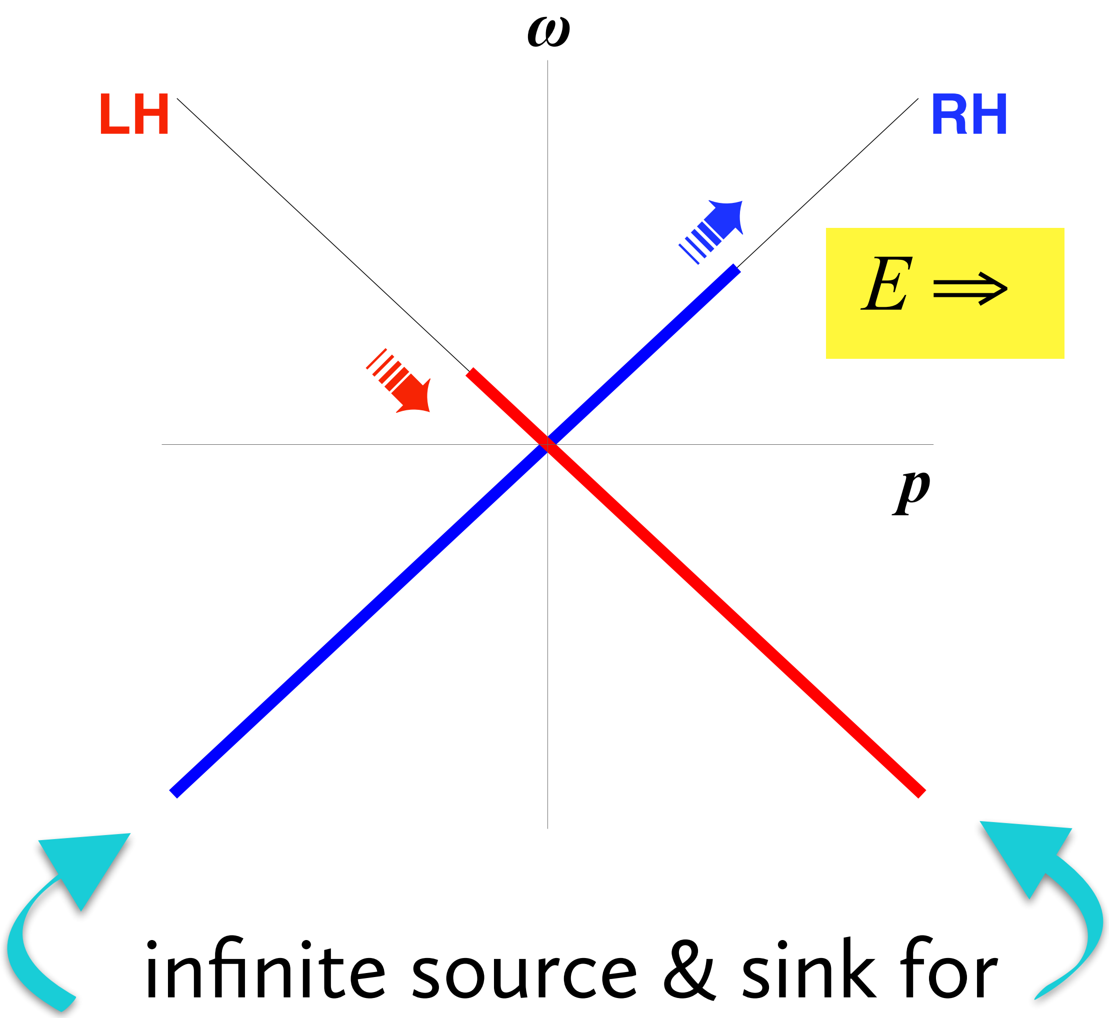
Massless Dirac fermions in an electric field E, 1+1 dim



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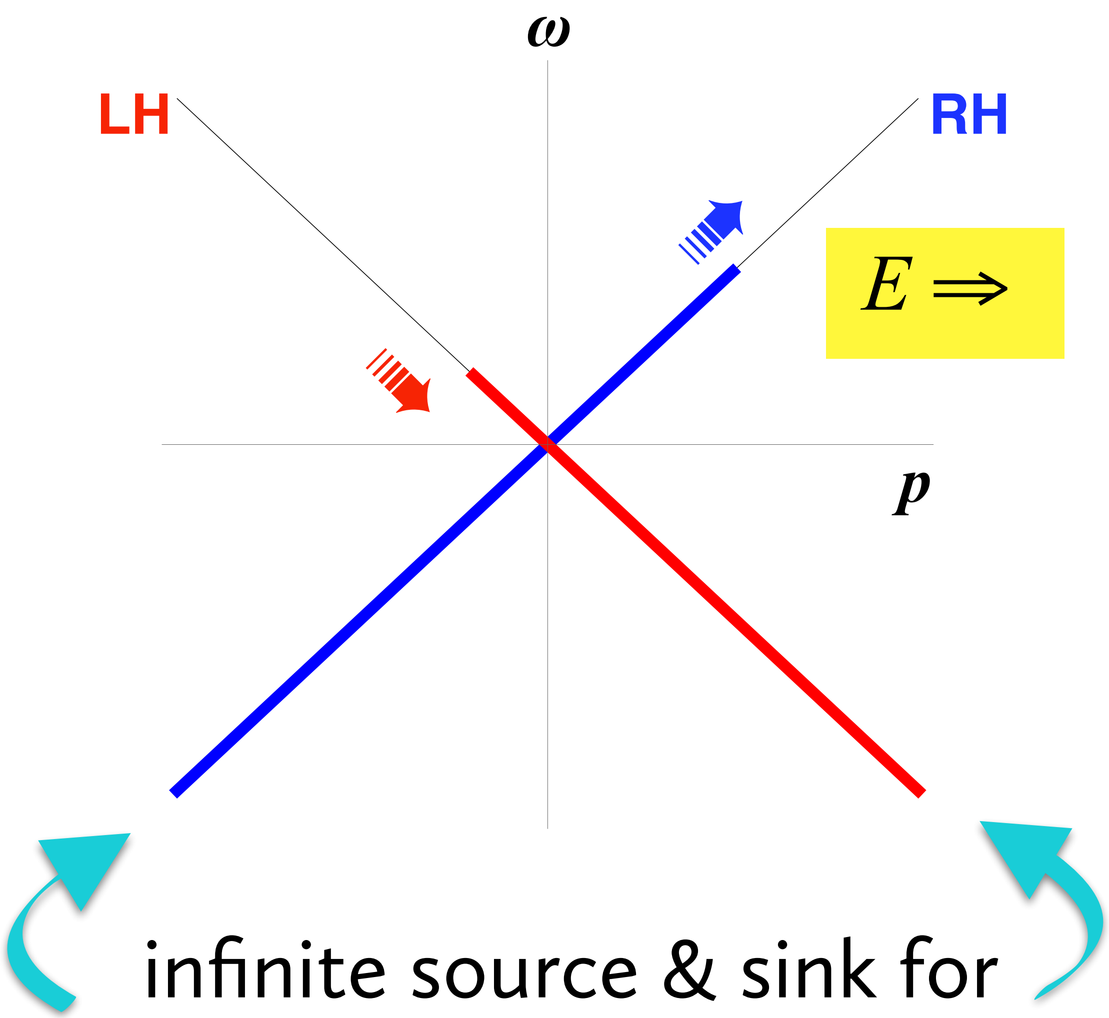


infinite source & sink for fermions

$$\frac{d(n_R - n_L)}{dt} = \frac{qE}{\pi}$$

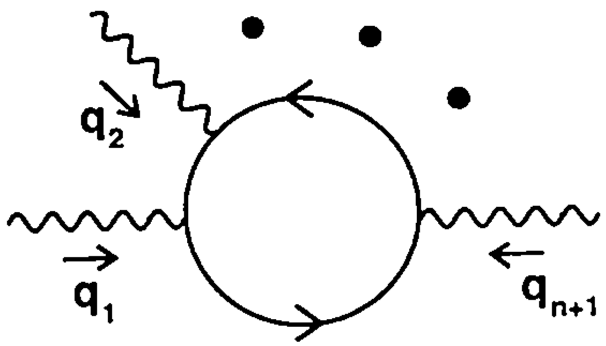
d=1+1 anomaly

Callan and Harvey (1985): compute 3d (5d) Chern Simons operator...
bulk modes don't decouple entirely!

$$\mathcal{L}_{CS} = \left(\frac{m(x_3)}{|m(x_3)|} + \frac{\Lambda}{|\Lambda|}\right)\mathcal{O}_{CS}$$

$$\mathcal{O}_{CS}^{d=3} = -\frac{e^2}{8\pi}\epsilon_{abc}A_a\partial_b A_c$$

$$\mathcal{O}_{CS}^{d=5} = -\frac{e^3}{48\pi^2}\epsilon_{abcde}A_a\partial_b A_c\partial_d A_e$$



graph in n-dimensions

Differentiate wrt A_3 (A_5) to discover a current between the domain walls:

$$J_3 = -\frac{e^2}{2\pi}\left(\epsilon(x_3)+1\right)\epsilon_{ij}\partial_i A_j$$

nonzero divergence at $x_3 = 0$ and $x_3 = \pm L$ in presence of E field

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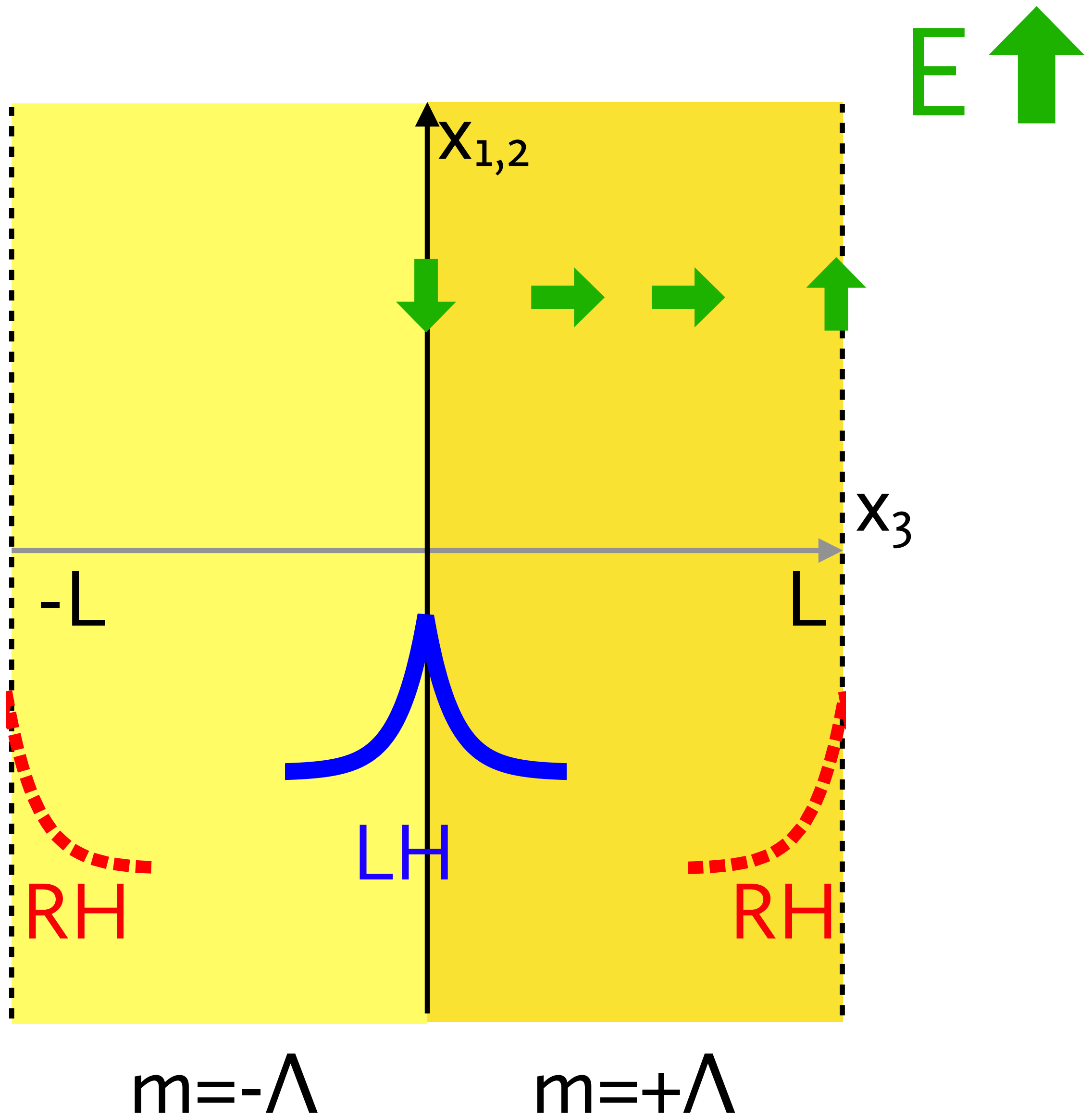

- Charge and chirality violation at each surface consistent with 2d anomaly
- Total charge is conserved in 3d
- Charge is transported by "Hall current" in the bulk despite gap
- Behavior possible because mass term in 3d violates (i) 2d chirality, (ii) time reversal, (iii) parity
- Note: no current on left side… can formulate on $0 \le x_3 \le L$

Set up seems ideally suited for realizing global chiral symmetry on the lattice, with Λ ▸ 1/a:

- 3d (5d) theory explicitly breaks chiral symmetry, so no violation of Nielsen-Ninomiya theorem
- Result is robust (eg, changes to the mass function has little effect)
- drawback: perfect chiral symmetry requires infinite L

So: put it on the lattice with “naive” derivatives and you find…

2^{d+1} doublers ! 😩

So add a Wilson term? $a\bar{\Psi}D^2\Psi$

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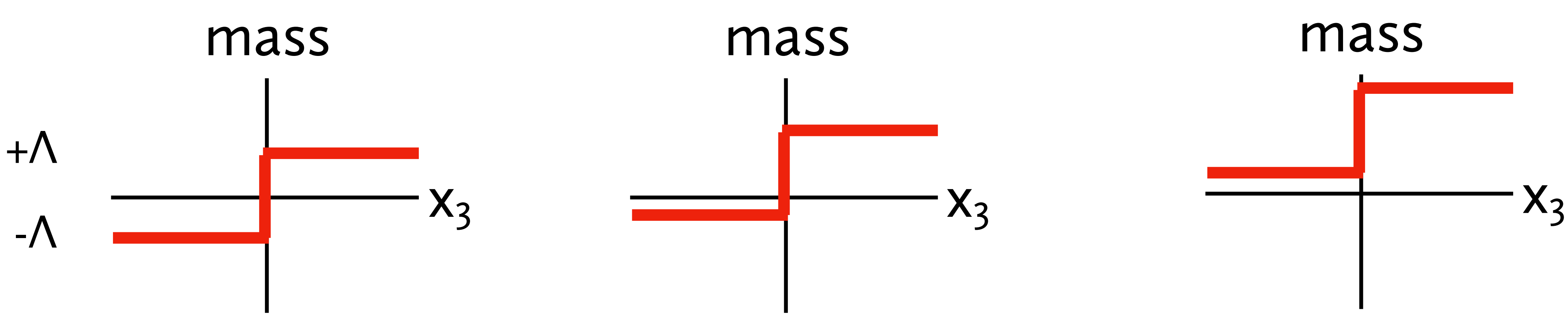
Why not, chiral symmetry is already broken maximally!

Looks like a “momentum dependent mass shift” scaling like k^2

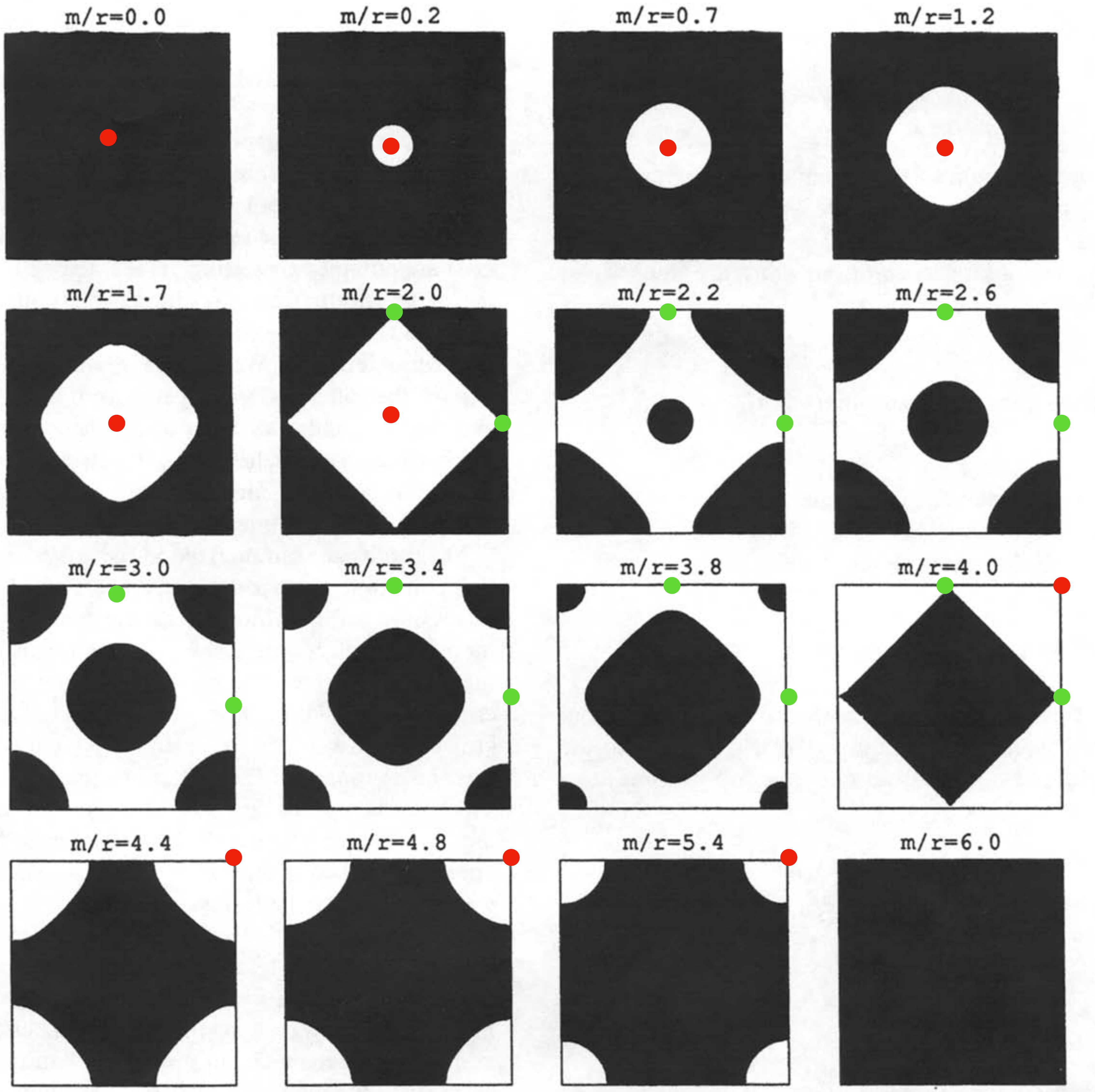


small k^2 — zeromodes ✓

medium k^2 — zeromodes ✓

large k^2 — no zeromodes ✗

K. Jansen, M. Schmaltz Physics Letters B 296 (1992) 374–378



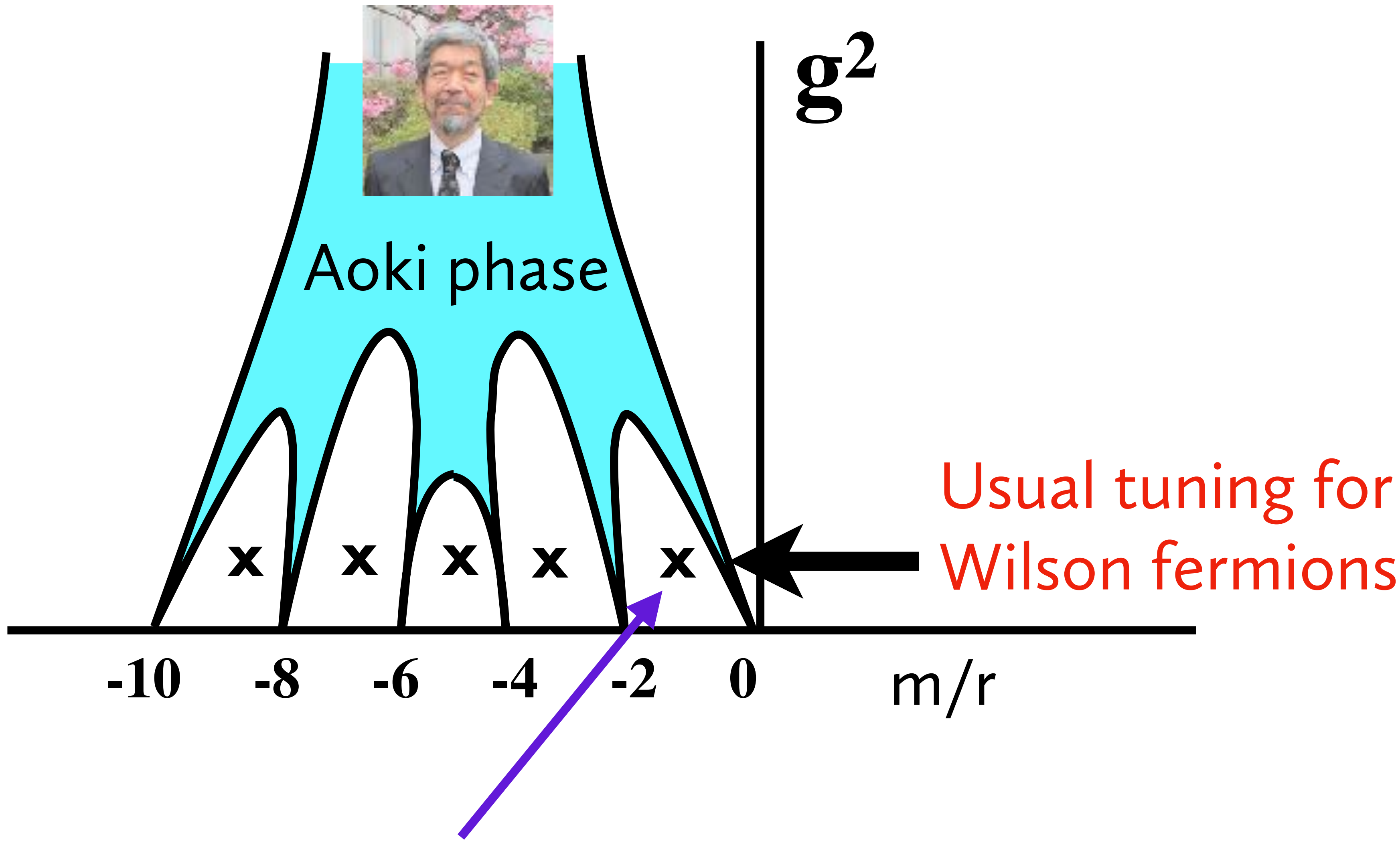
m = mass

r = Wilson operator coefficient

Poles in propagator:

- None for m/r <0
- 1 RH for 0<m/r<2
- 2 LH for 2<m/r<4
- 1 RH for 4<m/r<6
- None for m/r>6

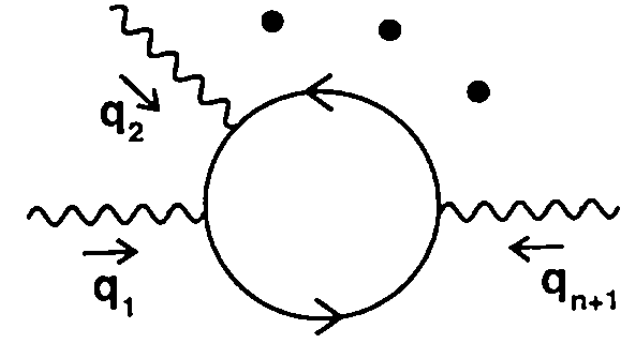
Analogue in 5d: parameter space for Wilson action:



Where to sit for one chiral DWF...you want the bulk to be gapped!

Mystery! How can coefficient of CS operator adjust discontinuously with m/r?? Result of 1-loop Feynman diagram! Should be intensity to exact lattice action!

M. Golterman, K. Jansen, DBK
Physics Letters B 301 (1993) 219–223



- Assume a generic fermion propagator S(p)

- Use the Ward identity for the photon vertex:

$$\Lambda_\mu(p,p) = -\mathrm{i}\,\frac{\partial}{\partial p_\mu} S^{-1}(p)\,,$$

- Extract the coefficient of the CS operator:

$$c_n = \frac{(-\mathrm{i})^n \epsilon_{\mu_1 \;\; \mu_{2n+1}}}{(n+1)(2n+1)!} \int \frac{\mathrm{d}^{2n+1}p}{(2\pi)^{2n+1}} \mathrm{Tr}\{[S(p)\,\partial_{\mu_1} S(p)^{-1}] \ldots [S(p)\,\partial_{\mu_{2n+1}} S(p)^{-1}]\}\,,$$

- Write S(p) as number x unitary matrix

$$S^{-1}(p) = a(p) + \mathrm{i}\boldsymbol{b}(p)\cdot\gamma = N(p)\,[\cos|\boldsymbol{\theta}(p)| + \mathrm{i}\hat{\boldsymbol{\theta}}(p)\cdot\gamma \sin|\boldsymbol{\theta}(p)|\,] \equiv N(p)\,V(p)\,,$$

- Find c_n ∝ winding number torus ‣ sphere.

Conclusion: chiral zeromodes are topologically protected

Topology resides in the dispersion relation of the bulk fermions

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Basis for topological insulators, discussed by CM theorists

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OK, but what to do with extra dimension?

Can work at large fine L...would like infinite L

Would like a better analytical understanding of chirality

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Would like a better analytical understanding of chirality

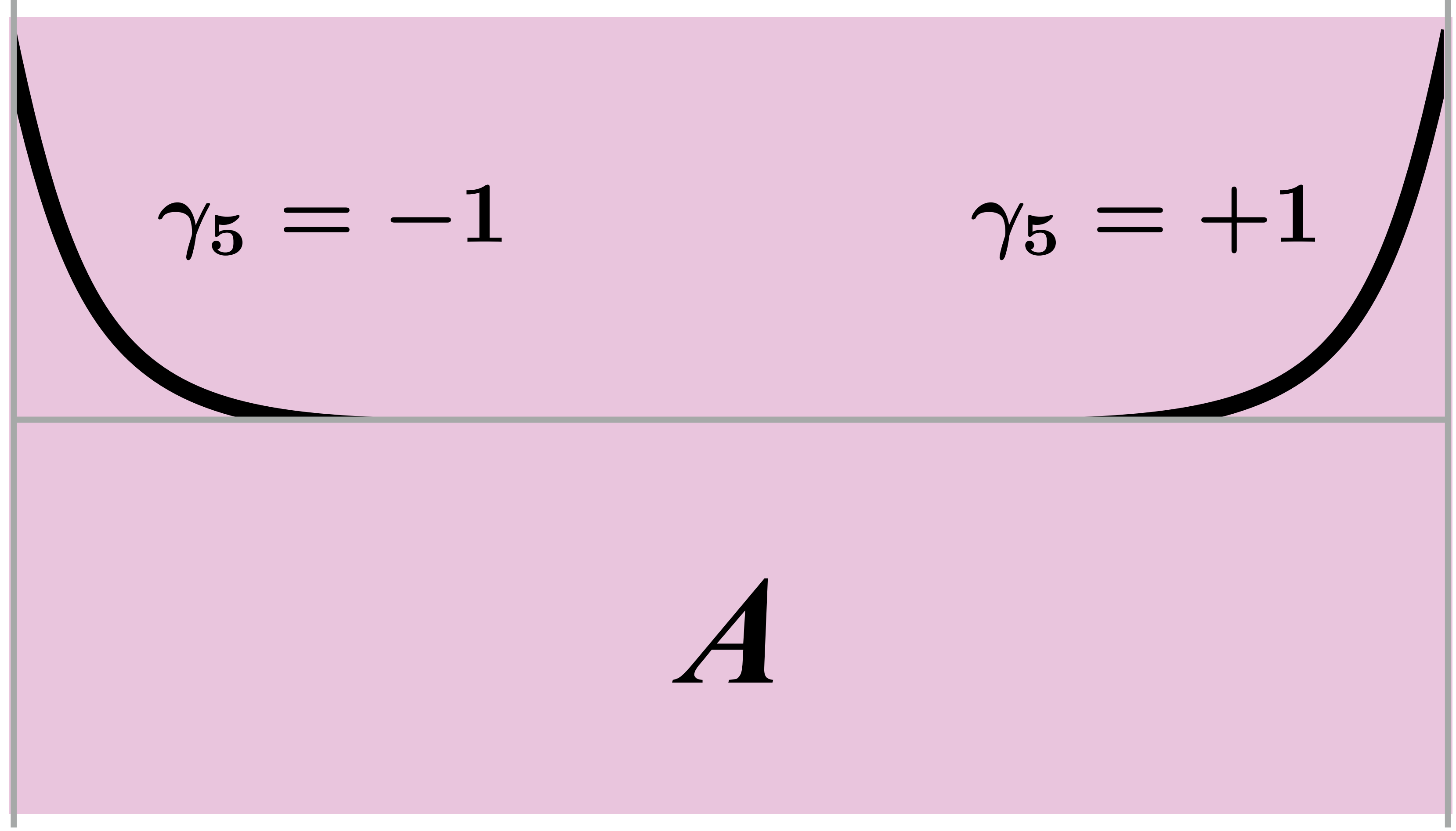
- overlap fermions (Neuberger & Narayanan)
- Ginsparg-Wilson equation

Neuberger & Narayanan:

- integrate out the bulk modes and derive the effective 2d (4d) theory of the surface modes

x

Result is: $\mathcal{L} = \bar{\psi} D \psi$



→ "time"

T = "transfer matrix"

- ε is the sign function
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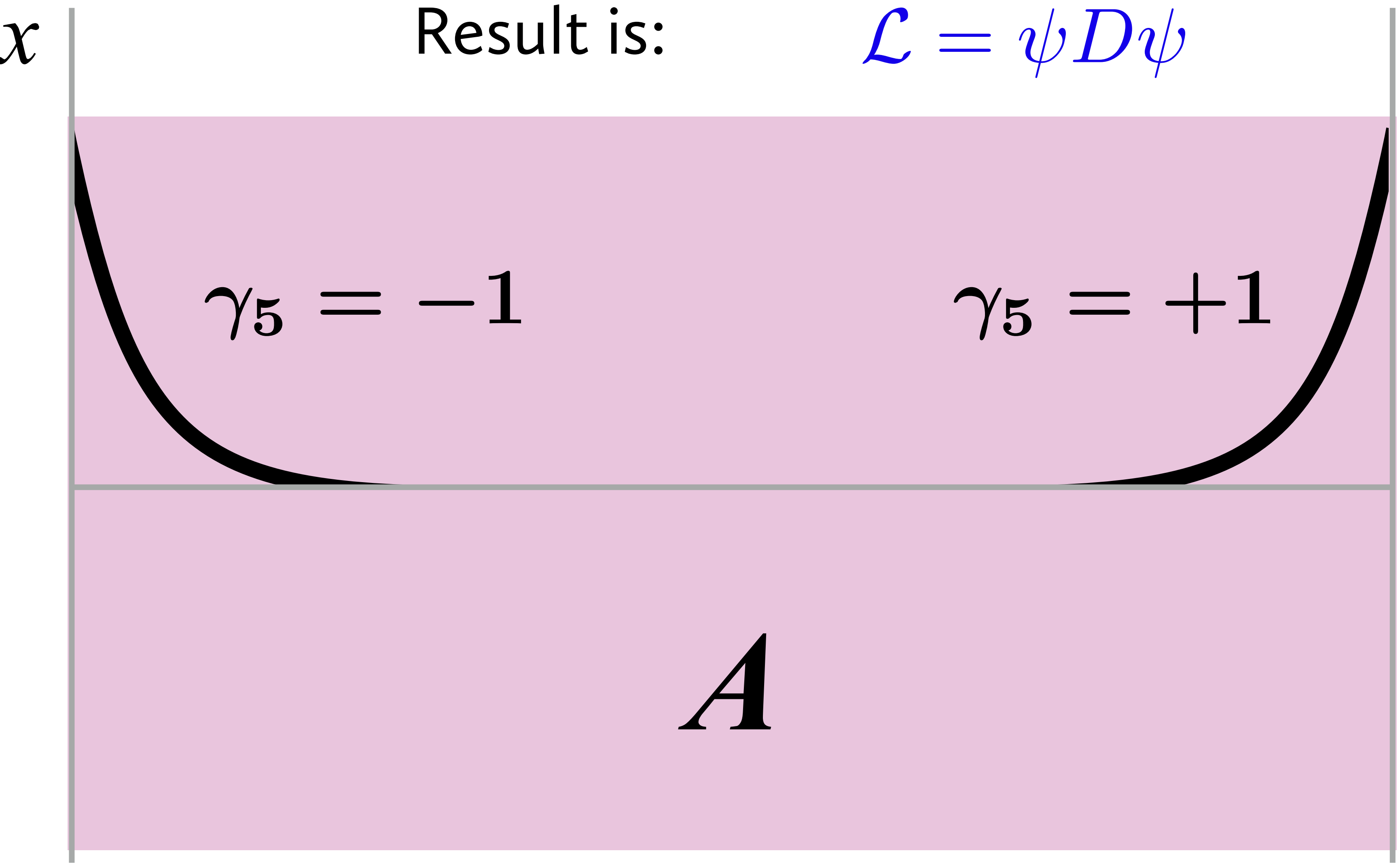

→ "time"

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$$D = \lim_{L_5 \to \infty} 1 + \gamma_5 \frac{1 - T^L}{1 + T^L} = 1 + \gamma_5 \epsilon(H)$$

- ε is the sign function
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$$D = 1 + \gamma_5 \epsilon(H) = 1 + \gamma_5 \frac{H}{\sqrt{H^2}} \equiv 1 + V$$

$$H \sim \gamma_5(\not{D} + M + D^2) = H^\dagger$$

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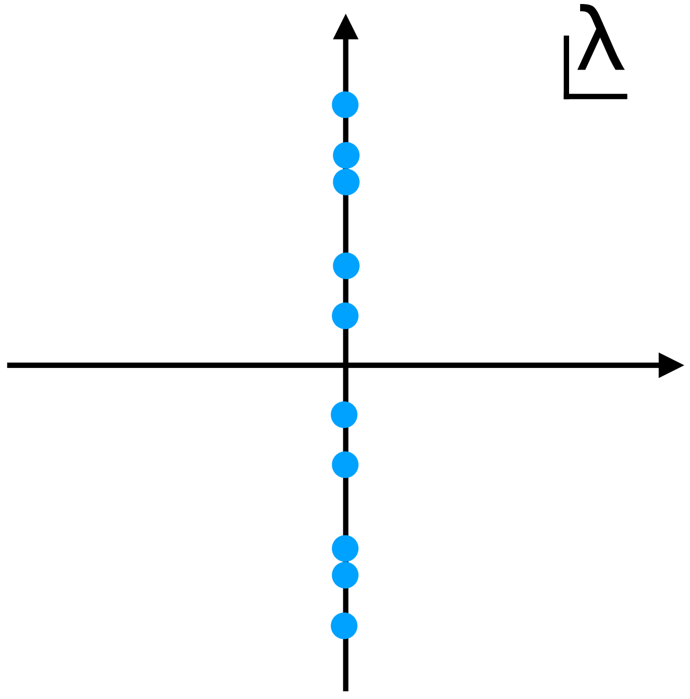

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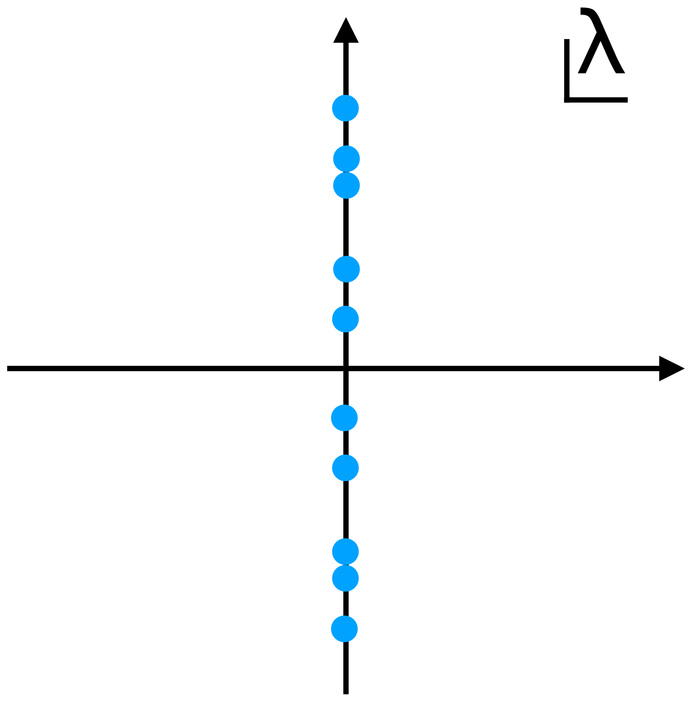
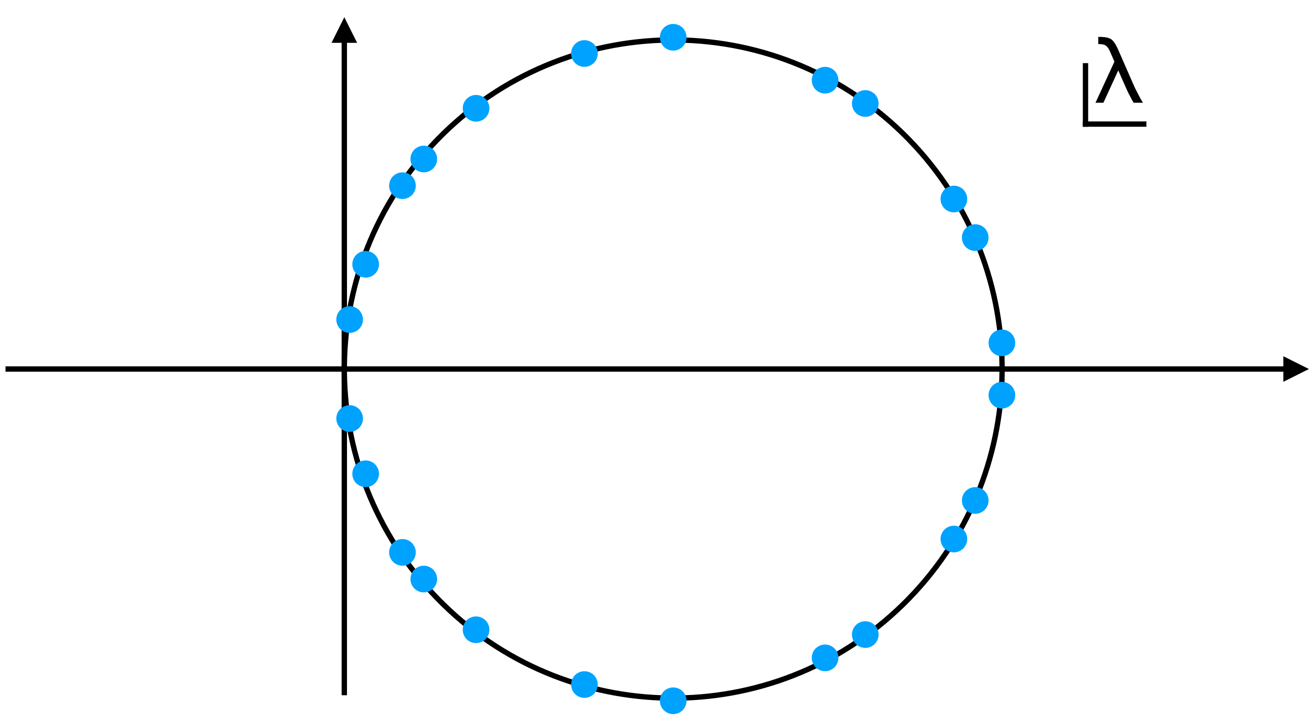


Eigenvalues of overlap op D

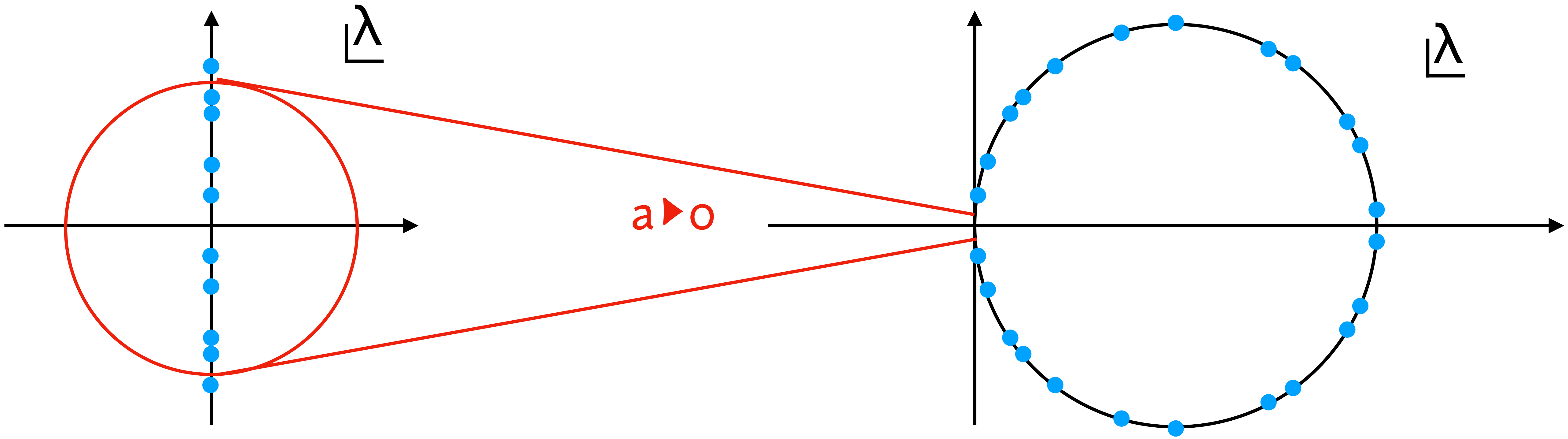


$$D = 1 + \gamma_5 \epsilon(H) = 1 + \gamma_5 \frac{H}{\sqrt{H^2}} \equiv 1 + V$$

$$H \sim \gamma_5(\not{D} + M + D^2) = H^\dagger$$

$$V^\dagger V = 1$$

Continuum eigenvalues of $\not{D}$ | Eigenvalues of overlap op D



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The overlap operator obeys the Ginsparg-Wilson equation (Neuberger, 1997; Ginsparg & Wilson, 1982)

$$\{\gamma_5, D^{-1}\}_{xy} = a\gamma_5\delta(x-y)$$

- modification is to the chiral behavior of the propagator at zero separation only
- Only effects Green functions with currents at zero separation (anomaly!)

Does this preserve enough chiral symmetry to ensure multiplicative mass renormalization? Yes! (Luscher, 1998)

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Ginsparg-Wilson:

$$\{\gamma_5, D\} = aD\gamma_5 D$$

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Luscher: If D obeys GW equation, then

action $\bar\psi D\psi$ is invariant under:

$$\delta\psi = \gamma_5\left(1-\frac{a}{2}D\right)\psi\ ,\quad \delta\bar\psi = \bar\psi\left(1-\frac{a}{2}D\right)\gamma_5$$

this generates a continuous $U(1)_A$ which is violated by mass term ▸ multiplicative renormalization (a great feature of chiral symmetry in the continuum)...

This is NOT a symmetry of the path integral measure:

$$\delta[D\psi][D\bar\psi] = a\mathrm{Tr}\,\gamma_5 D = a\,\mathrm{Tr}\,\epsilon(H)$$

Index of D ... an integer on the lattice!

$$\mathrm{Tr}\,\gamma_5 D = 2(n_+ - n_-) \to 2\nu$$

integer of D index on lattice becomes gauge winding number in the continuum

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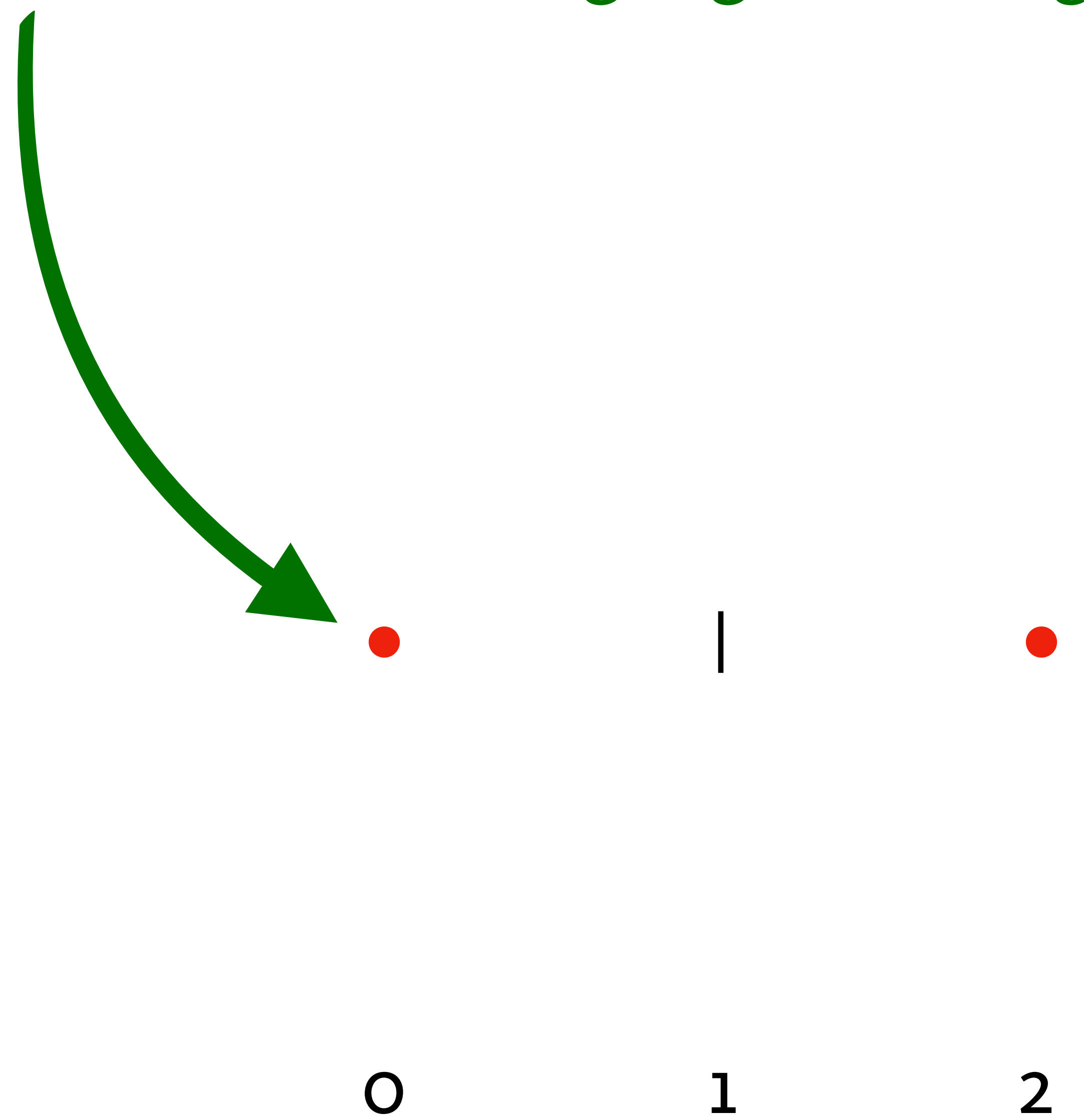

An instanton will put a + chirality mode at λ=0, and a - chirality mode at λ=2

$n_\pm$ count modes at λ=0

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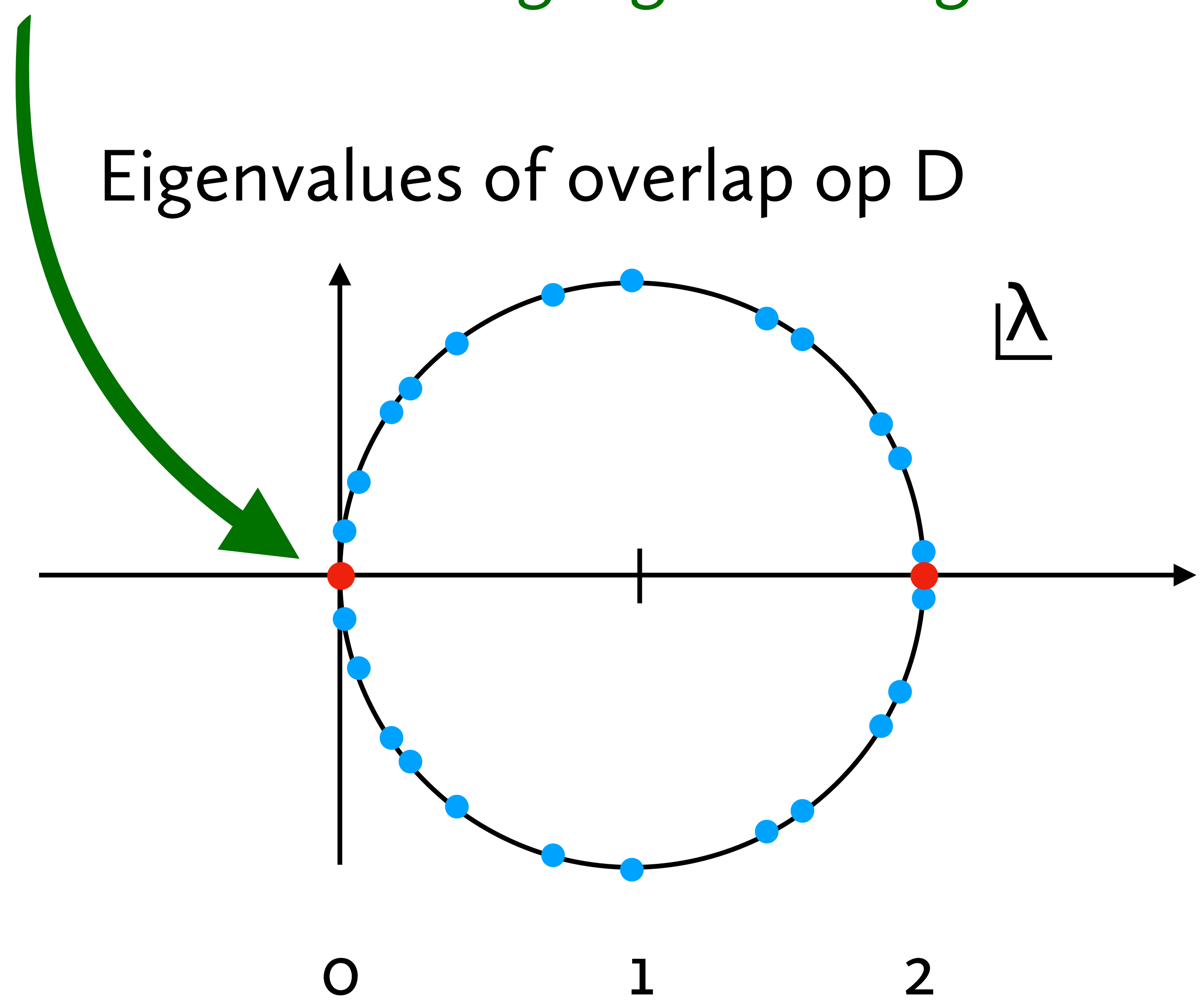

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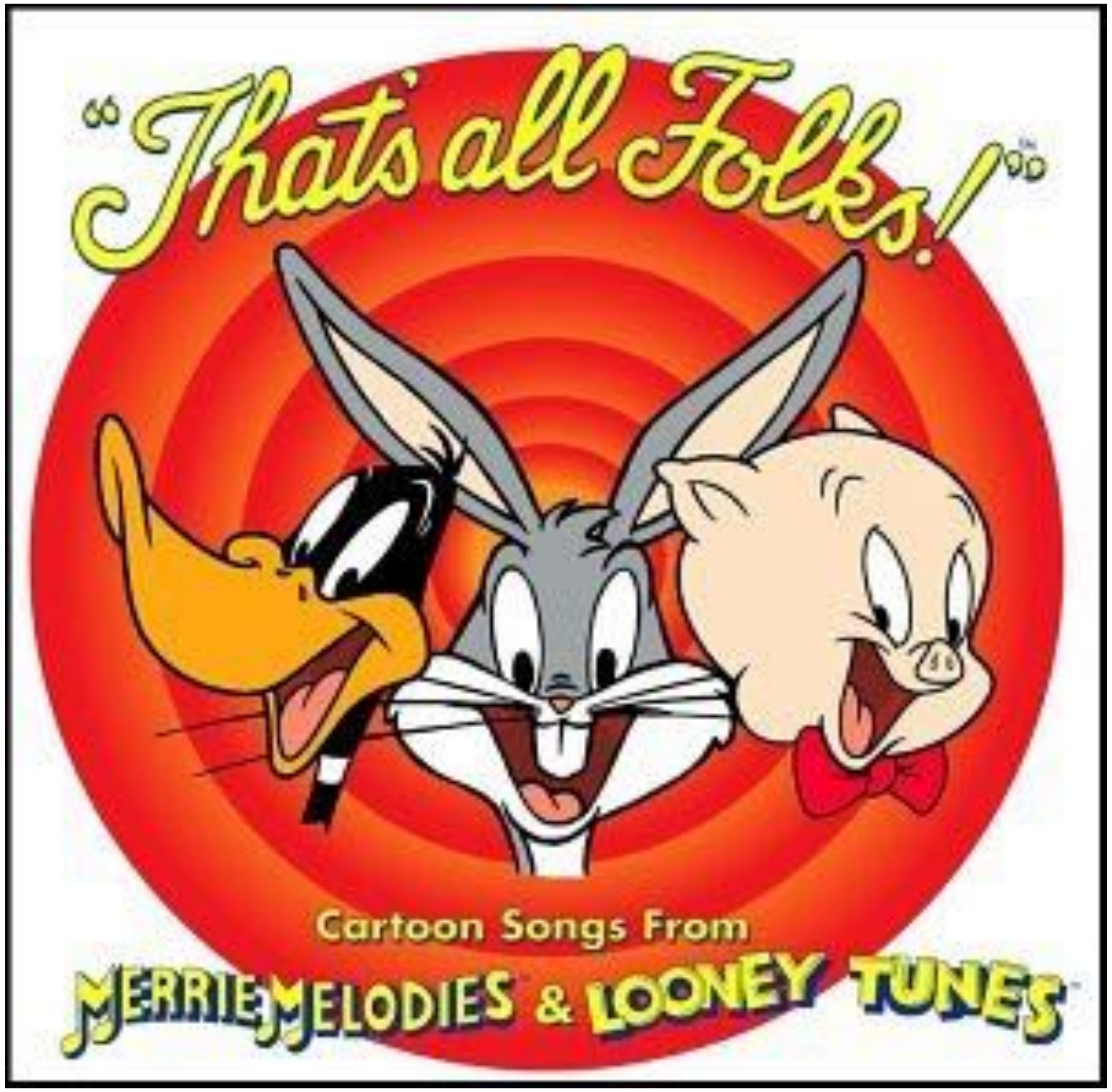
Loose end:
How in the world did Ginsparg & Wilson derive their equation in 1982?

- Start with chiral theory in the continuum
- Rewrite theory in terms of averages of fermions over cells to create a lattice theory
- Determine chiral transformation of D in latticed theory…had to have same properties as starting point: good classical $U(1)_A$ plus anomaly

…but they could't find a solution to their equation, and their paper was forgotten for 15 years…


"That's all Folks!"
Cartoon Songs From
MERRIE MELODIES & LOONEY TUNES