Moebius and Conformal Maps Between Boundaries of CAT(-1) Spaces

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Moebius group in n dimensions

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Moeb($\mathbb{R}^n \cup \{\infty\}$) $\curvearrowright \mathbb{R}^n \cup \{\infty\}$

= < reflections in hyperplanes, inversions in spheres >

= group of homeomorphisms preserving **cross-ratio**

where cross-ratio of a quadruple of distinct points defined by

$$[\xi, \xi', \eta, \eta'] := \frac{||\xi - \eta|| \, ||\xi' - \eta'||}{||\xi - \eta'|| \, ||\xi' - \eta||}$$

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= conjugate of Moeb($\mathbb{R}^n \cup \{\infty\}$) by stereographic projection $\mathbb{R}^n \cup \{\infty\} \to S^n$

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Upper half-space model: $\mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}^+$, $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$

Isom($\mathbb{H}^n$) $=$ < reflections/inversions in mirrors $\perp \partial\mathbb{H}^n$ >

The map

$$\text{Isom}(\mathbb{H}^n) \to Moeb(\mathbb{R}^{n-1} \cup \{\infty\})$$
$$f \mapsto f_{|\partial\mathbb{H}^n}$$

is an isomorphism.

Thus

$$\text{Isom}(\mathbb{H}^n) \simeq Moeb(\partial\mathbb{H}^n)$$

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Theorem

(Mostow) For $n \geq 3$, any isomorphism $\phi : \pi_1(M) \to \pi_1(N)$ between fundamental groups of closed hyperbolic n-manifolds M, N is induced by an isometry $f : M \to N$.

Sketch of proof:

Step 1. Choosing a basepoint $x_0 \in \mathbb{H}^n$, ϕ induces an equivariant quasi-isometry $f_0 : \pi_1(M) \cdot x_0 \to \pi_1(N) \cdot x_0, g \cdot x_0 \mapsto \phi(g) \cdot x_0$.

Step 2. f_0 extends to an equivariant quasi-conformal homeomorphism $F : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$.

Step 3. F equivariant and quasi-conformal implies F conformal, hence Moebius.

Step 4. F Moebius implies F extends to an equivariant isometry $f : \mathbb{H}^n \to \mathbb{H}^n$.

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Marked length spectrum rigidity

X closed negatively curved n-manifold

Each free homotopy class of closed curves contains a unique closed geodesic

Length function $l_X : \pi_1(X) \to \mathbb{R}^+$

Question: Given X, Y closed negatively curved n-manifolds, and $\phi : \pi_1(X) \to \pi_1(Y)$ an isomorphism such that $l_Y \circ \phi = l_X$, is X isometric to Y?

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(Otal) Yes, if $n = 2$.

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(Hamenstadt) Marked length spectra of X, Y are equal iff geodesic flows of X, Y are topologically conjugate.

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CAT(-1) spaces

(X, d) metric space is CAT(-1) if:

(1) X is a length space: For all $p, q \in X$, exists isometric embedding $\gamma : [0, T = d(p, q)] \to X$ with $\gamma(0) = p, \gamma(T) = q$.

(2) X satisfies CAT(-1) inequality: Geodesic triangles thinner than in $\mathbb{H}^2$, $d(s, t) \leq d_{\mathbb{H}^2}(\overline{s}, \overline{t})$.

Facts:

Unique geodesic joining any two points.

Contractible.

Examples:

X complete simply connected manifold, $K \leq -1$.

X metric tree.

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Boundary at infinity

$\partial X := \{[\gamma] : \gamma : [0, \infty) \to X$ geodesic ray$\}$, where $\gamma_1 \sim \gamma_2$ if $\{d(\gamma_1(t), \gamma_2(t)) : t \geq 0\}$ bounded.

$\gamma(\infty) := [\gamma]$.

$\forall x \in X, \xi \in \partial X, \exists!$ geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = x, \gamma(\infty) = \xi$.

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Examples:

X simply connected complete manifold, $K \leq -1$, then the map

$$\begin{aligned} T^1_x X &\to \partial X \\ v &\mapsto \gamma(\infty) \end{aligned}$$

(where $\gamma =$ unique geodesic ray with $\dot{\gamma}(0) = v$) is a homeomorphism, $X \cup \partial X \simeq \mathbb{B}^n \cup \partial \mathbb{B}^n$.

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Comparison angle at infinity: For $x \in X, \xi, \eta \in \partial X$, angle between ξ, η as viewed from x,

$$\theta_x(\xi, \eta) := \lim_{p \to \xi, q \to \eta} \theta_x(p, q)$$

$\theta_x(\xi, \eta) = 0$ iff $\xi = \eta$, $\theta_x(\xi, \eta) = \pi$ iff $x \in (\xi, \eta)$

Visual metric based at x:

$$\rho_x(\xi, \eta) = \sin\left(\frac{1}{2}\theta_x(\xi, \eta)\right)$$

Diameter one metric on ∂X compatible with topology on ∂X.

Example: For $X = \mathbb{H}^n, \partial X = S^{n-1}, \rho_0 = \frac{1}{2} \times$ chordal metric on S^{n-1}.

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Gromov inner product:
$(x|y)_z := \frac{1}{2}(d(x,z) + d(y,z) - d(x,y)), x, y, z \in X$.

For X metric tree, $(x|y)_z =$ length of common segment of $[x,z], [y,z]$.

For $\xi, \eta \in \partial X, (\xi|\eta)_x := \lim_{y \to \xi, y' \to \eta} (y|y')_x$ (y, y' converge radially).

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Example: X rooted binary tree, $\partial X = \{0,1\}^{\mathbb{N}}, \rho_x =$ standard ultrametric on Cantor set.

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Moebius and conformal maps between metric spaces

(Z, ρ) metric space, cross-ratio of quadruple of distinct points $\xi, \xi', \eta, \eta' \in Z$ defined by

$$[\xi, \xi', \eta, \eta']_\rho := \frac{\rho(\xi,\eta)\rho(\xi',\eta')}{\rho(\xi,\eta')\rho(\xi',\eta)}$$

Embedding $F : (Z_1, \rho_1) \to (Z_2, \rho_2)$ **Moebius** if it preserves cross-ratios.

Embedding $F : (Z_1, \rho_1) \to (Z_2, \rho_2)$ **conformal** if

$$dF_{\rho_1,\rho_2}(\xi) := \lim_{\eta \to \xi} \frac{\rho_2(F(\xi), F(\eta))}{\rho_1(\xi, \eta)}$$

exists for all $\xi \in Z_1$ (assuming Z_1 has no isolated points).

F Moebius implies F conformal

Moreover, "Geometric Mean-Value Theorem" holds for Moebius maps:

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$$\frac{d\rho_y}{d\rho_x}(\xi) := d\mathrm{id}_{\rho_x,\rho_y}(\xi) = \exp(B(x,y,\xi))$$

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Marked length spectrum and Moebius maps

Theorem

(Otal) X, Y closed negatively curved n-manifolds have same marked length spectrum $\Leftrightarrow F : \partial\tilde{X} \to \partial\tilde{Y}$ is Moebius.

Question: For X, Y CAT(-1) spaces, does a Moebius map $F : \partial X \to \partial Y$ extend to an isometry $f : X \to Y$?

Theorem

(Bourdon) For X a rank one symmetric space with maximum of sectional curvatures equal to -1, Y a CAT(-1) space, any Moebius embedding $F : \partial X \to \partial Y$ extends to an isometric embedding $f : X \to Y$.

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Marked length spectrum and Moebius maps

Theorem

(Otal) X, Y closed negatively curved n-manifolds have same marked length spectrum $\Leftrightarrow F : \partial\tilde{X} \to \partial\tilde{Y}$ is Moebius.

Question: For X, Y CAT(-1) spaces, does a Moebius map $F : \partial X \to \partial Y$ extend to an isometry $f : X \to Y$?

Theorem

(Bourdon) For X a rank one symmetric space with maximum of sectional curvatures equal to -1, Y a CAT(-1) space, any Moebius embedding $F : \partial X \to \partial Y$ extends to an isometric embedding $f : X \to Y$.

Conformal maps and geodesic conjugacies

X, Y complete, simply connected manifolds with $K \leq -1$, then any conformal map $F : \partial X \to \partial Y$ induces a topological conjugacy of geodesic flows $\phi : T^1 X \to T^1 Y$:

Given $v \in T^1 X$ tangent to (ξ, η), define $\phi(v)$ to be the unique $w \in T^1 Y$ tangent to $(F(\xi), F(\eta))$ satisfying

$$dF_{\rho_x, \rho_y}(\xi) = 1$$

where $x = \pi(v), y = \pi(w)$.

Flowing v, w for time t scales visual metrics at $\xi, F(\xi)$ by same factor e^t, so ϕ preserves time along geodesics.

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If F is Moebius, then ϕ is flip-equivariant:

$$dF_{\rho_x,\rho_y}(\xi)dF_{\rho_x,\rho_y}(\eta) = \frac{\rho_y(F(\xi),F(\eta))^2}{\rho_x(\xi,\eta)^2} = 1$$

so

$$dF_{\rho_x,\rho_y}(\xi) = 1 \text{ iff } dF_{\rho_x,\rho_y}(\eta) = 1$$

hence

$$\phi(-v) = -\phi(v)$$

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The integrated Schwarzian of a conformal map

For a conformal map $F : \partial X \to \partial Y$, measure failure of $\phi : T^1 X \to T^1 Y$ to be flip-equivariant:

The **integrated Schwarzian** of F is the function $S(F) : \partial^2 X \to \mathbb{R}$ defined by

$$\begin{aligned} S(F)(\xi,\eta) &:= \text{signed distance between foot of } \phi(v) \text{ and foot of } \phi(-v) \\ &= -\log(dF_{\rho_x,\rho_y}(\xi) dF_{\rho_x,\rho_y}(\eta)) \end{aligned}$$

where v tangent to (ξ,η), $x \in (\xi,\eta)$, $y \in (F(\xi),F(\eta))$ (independent of choices of v, x, y).

Cocycle: $S(F \circ G) = S(F) \circ (G,G) + S(G)$

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Integrated Schwarzian and cross-ratio distortion

Conformal map $F : \partial X \to \partial Y$ said to be C^1 conformal if dF_{ρ_x, ρ_y} continuous.

Theorem

(B.) Let X be a simply connected complete Riemannian manifold with sectional curvatures satisfying $-b^2 \leq K \leq -1$ for some $b \geq 1$, and let Y be a proper geodesically complete CAT(-1) space. Let $F : U \subset \partial X \to V \subset \partial Y$ be a C^1 conformal map between open subsets U, V. Then

$$\log \frac{[F(\xi), F(\xi'), F(\eta), F(\eta')]}{[\xi, \xi', \eta, \eta']}$$

$$= \frac{1}{2}\left(S(F)(\xi, \eta) + S(F)(\xi', \eta') - S(F)(\xi, \eta') - S(F)(\xi', \eta)\right)$$

for all $(\xi, \xi', \eta, \eta') \in \partial^4 U$. In particular F is Moebius if and only if $S(F) \equiv 0$.

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Almost-isometric extension of Moebius maps

A map $f : X \to Y$ between metric spaces is a (K, ϵ)-quasi-isometry if

$$\frac{1}{K} d(x,y) - \epsilon \leq d(f(x), f(y)) \leq K d(x,y) + \epsilon$$

for all $x, y \in X$, and if all points of Y are within bounded distance from the image of f.

Theorem

(B.) X, Y proper, geodesically complete CAT(-1) spaces. Then any Moebius map $F : \partial X \to \partial Y$ extends to a $(1, \log 2)$-quasi-isometry $f : X \to Y$ with image $\frac{1}{2}\log 2$-dense in Y.

Theorem

(B.) If X, Y are in addition metric trees, then f can be taken to be a surjective isometry.

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Complete, simply connected, negatively curved manifolds: infinitesimal rigidity

Theorem

(B.) (X, g_0) *complete, simply connected manifold with* $K(g_0) \leq -1$*. Suppose* (g_t) *smooth 1-parameter family of metrics on* X *such that* $K(g_t) \leq -1$ *and* $g_t \equiv g_0$ *outside a compact* $C \subset X$*, so* $id : (X, g_0) \to (X, g_t)$ *extends to a homeomorphism* $\hat{id}_t : \partial X_{g_0} \to \partial X_{g_t}$*.*

Suppose $\hat{id}_t$ *is Moebius for all* t*. Then* $\hat{id}_t$ *extends to an isometry* $f_t : (X, g_0) \to (X, g_t)$ *for all* t*.*

Complete, simply connected, negatively curved manifolds: local rigidity

Theorem

(B.) (X, g_0) *complete, simply connected manifold with* $K(g_0) \leq -1$*. Given compact* $C \subset X$*, there exists* $\epsilon > 0$ *such that if* g *is a metric on* X *satisfying*

(1) $K(g) \leq -1, g \equiv g_0$ *outside* C*,*

(2) $Vol_g(C) = Vol_{g_0}(C)$*,*

(3) $||g - g_0||_{C^3} < \epsilon$*,*

then if $\hat{id} : \partial X_{g_0} \to \partial X_g$ *is Moebius, then* $\hat{id}$ *extends to an isometry* $f : (X, g_0) \to (X, g)$*.*

Proof of almost-isometric extension

(Z, ρ_0) compact, diameter one, antipodal metric space

Space of Moebius metrics:

$\mathcal{M}(Z, \rho_0) := \{\rho \text{ metric on } Z \,|\, [\,]_\rho = [\,]_{\rho_0}, \rho \text{ diameter one, antipodal}\}$

$$d_{\mathcal{M}}(\rho_1, \rho_2) := \max_{\xi \in Z} \log \frac{d\rho_2}{d\rho_1}(\xi)$$

Step 1. X CAT(-1) space, then the map

$$i_X : X \to \mathcal{M}(\partial X)$$
$$x \mapsto \rho_x$$

is an **isometric embedding**.

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Step 3. Moebius map $f : \partial X \to \partial Y$ induces an isometry $f_* : \mathcal{M}(\partial X) \to \mathcal{M}(\partial Y)$. Compose maps:

$$X \to \mathcal{M}(\partial X) \to \mathcal{M}(\partial Y) \to Y$$

(where last map is a nearest point projection).

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Proof of infinitesimal rigidity

Step 1. $g_t \equiv g_0$ outside compact C implies $\hat{id}_t : \partial X_{g_0} \to \partial X_{g_t}$ **locally Moebius**, hence conformal.

Step 2. The integrated Schwarzian measures difference between lengths of bi-infinite geodesics:

$$S(\hat{id}_t)(\xi, \eta) = \lim_{p \to \xi, q \to \eta} \left(d_{g_t}(p, q) - d_{g_0}(p, q) \right)$$

Step 3. First variation of integrated Schwarzian given by ray-transform:

$$\frac{d}{dt}_{|t=0} S(\hat{id}_t)(\xi, \eta) = \frac{1}{2} I(\dot{g}_0)(\xi, \eta) = \frac{1}{2} \int_{(\xi,\eta)} \dot{g}_0$$

Step 4. (Sharafutdinov) Kernel of ray transform equals infinitesimally trivial deformations:

$$I(\dot{g}_t) \equiv 0 \Leftrightarrow \dot{g}_t = \mathcal{L}_{X_t} g_t \text{ for some vector field } X_t$$

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Step 5. Integrate time-dependent vector field (X_t) to get 1-parameter family of isometries $f_t : (X, g_0) \to (X, g_t)$.