Relative Canonical Bundles for families of Calabi-Yau manifolds, twisted Hodge Bundles, and Positivity

ICTS Bangalore
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Hodge metric

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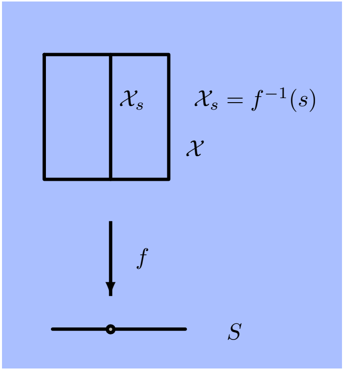

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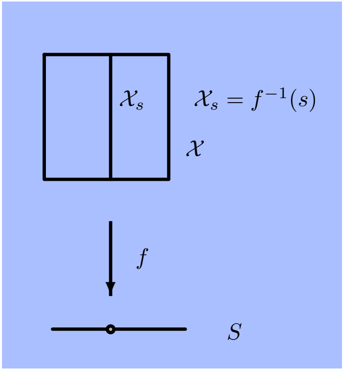

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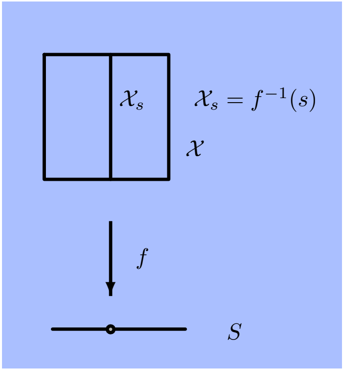

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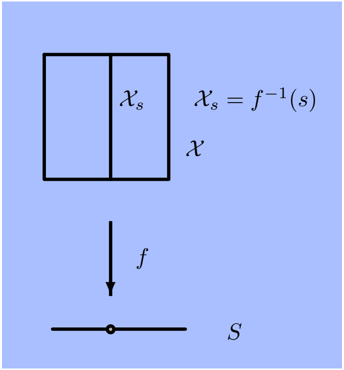

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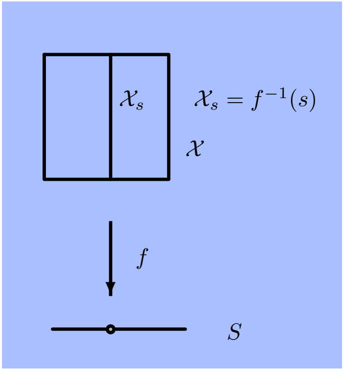

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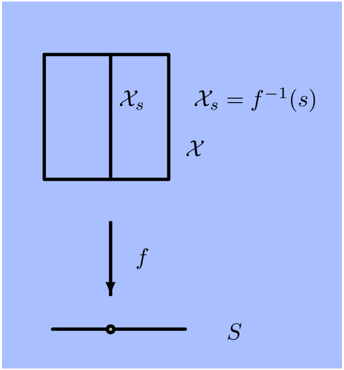

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Hodge metric on $f_* \Omega^n_{\mathcal{X}/S}$

L^2-inner product/norm of canonical forms η :

$$(-1)^n \int_{\mathcal{X}_s} \eta \wedge \overline{\eta}$$

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Griffiths: Curvature of Hodge bundle

Kodaira-Spencer classes: obstructions against splitting of

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$$\begin{array}{rcl} \rho : T_{S,s_0} & \longrightarrow & H^1(X, \mathcal{T}_X) \\ \dfrac{\partial}{\partial s} & \mapsto & [A_s] = \left[A^{\alpha}_{\overline{\beta}}(z) \dfrac{\partial}{\partial z^{\alpha}} \overline{dz^{\beta}} \right] \end{array}$$

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The cup product

$$\begin{aligned} A \otimes \phi & \mapsto A \cup \phi \\ H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s}) \otimes H^0(\mathcal{X}_s, \Omega^n_{\mathcal{X}_s}) & \to H^1(\mathcal{X}_s, \Omega^{n-1}_{\mathcal{X}_s}) \end{aligned}$$

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induces

$$\sigma_0 : f_*\Omega^n_{\mathcal{X}/S} \to R^1 f_* \Omega^{n-1}_{\mathcal{X}/S} \otimes \mathcal{T}_S^{\vee}$$

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Curvature Θ of $f_*\Omega^n_{\mathcal{X}/S}$

$$(\Theta e, e') = (\sigma_0 e, \sigma_0 e')$$

with hermitian metric induced by flat metric on $R^n f_*\mathbb{C}$, i.e. integration over $\mathcal{X}_s$,

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or for $\partial/\partial s, \partial/\partial s' \in T_{S,s}$

$$R(\partial/\partial s, \partial/\partial s', e, e') = (A \cup e, A' \cup e')$$

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Notation

$$\begin{aligned} \omega_X^n &= g\,dV \text{ Ricci-flat volume form} \\ 0 &= Ric(\omega_X) = -\sqrt{-1}\partial\overline{\partial}\log(\omega_X^n) \end{aligned}$$

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Let X Kähler.
A **polarization**

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Yau's theorem states the unique existence of a unique Ricci-flat Kähler form ω_X in any Kähler class λ_X.

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Known:

ω_{WP} is **Kähler.**

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G. Sch. '85 for hol. symplectic manifolds
Tian '86 for general CY manifolds

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Consequence

$$\omega_{WP}(s) = -\sqrt{-1}\partial\overline{\partial}\log(-1)^n \int_{\mathcal{X}_s} (\phi \wedge \overline{\phi})$$

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Now turn to $\mathcal{K}_{\mathcal{X}/S}$ rather than $f_*(\mathcal{K}_{\mathcal{X}/S})$.

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Curvature of $\mathcal{K}_{\mathcal{X}/S}$

Let $f : \mathcal{X} \to S$ be a (polarized) family of Calabi-Yau manifolds with family of Ricci-flat relative volume forms

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$$\Theta_{\mathcal{X}} = \frac{1}{\mathrm{vol}(\mathcal{X}_s)} f^*\omega_{WP}$$

Question Can the relative Kähler-Einstein/Ricci-flat form $\omega_{\mathcal{X}/S}$ be represented by a closed, real $(1,1)$-form on the total space?

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Proposition (A. Fujiki - G.Sch. '90)

$(f : \mathcal{X} \to S, \lambda_{\mathcal{X}/S})$ a polarized family of Calabi-Yau manifolds, and $\omega_{\mathcal{X}_s} \in \lambda_{\mathcal{X}_s}$ the Kähler-Einstein forms. Then locally with respect to S there exists a d-closed $(1,1)$-form $\omega_{\mathcal{X}}$ on the total space $\mathcal{X}$ such that

$$\omega_{\mathcal{X}}|\mathcal{X}_s = \omega_{\mathcal{X}_s}.$$

If the polarization can be represented by a closed, real $(1,1)$-form, then $\omega_{\mathcal{X}}$ can be chosen globally as a $(1,1)$-form.
Let $n = \dim \mathcal{X}_s$. Then such a form is uniquely determined by the equation

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back1 back2

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Theorem (M. Braun-G. Sch. '16)

Let $\mathcal{X}$ be Kähler $(f : \mathcal{X} \to S, \lambda_{\mathcal{X}/S})$ holomorphic, polarized family of Calabi-Yau manifolds.
Assume that the Green's functions of $\Box$ for functions on fibers $\mathcal{X}_s$, $s \in S$ are uniformly bounded from below (by a negative constant).

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back

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Cheeger '70, Cheeger - Yau '80

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Twisted Hodge bundles, canonically polarized case

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A compact complex manifold X is called *canonically polarized*, if $\mathcal{K}_X$ is positive (ample).

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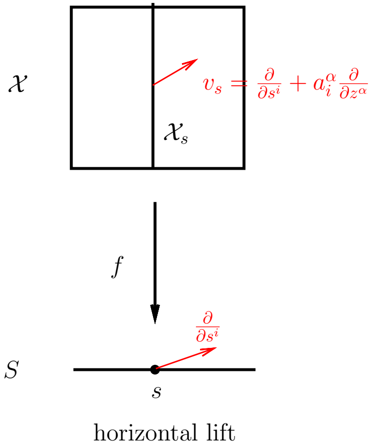
Given $\omega_{\mathcal{X}}$ positive definite on fibers



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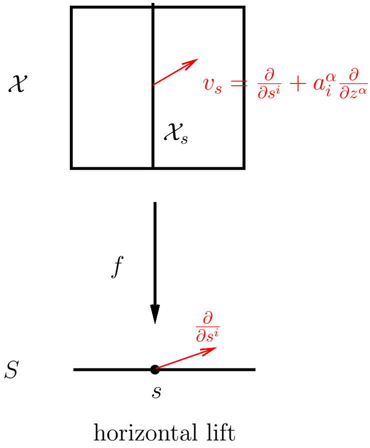

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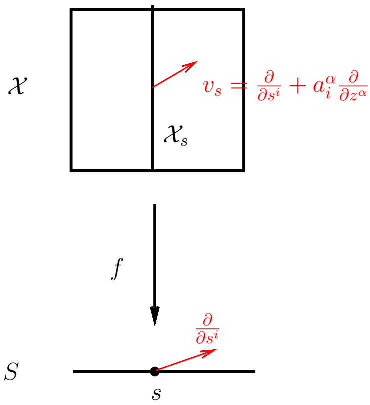

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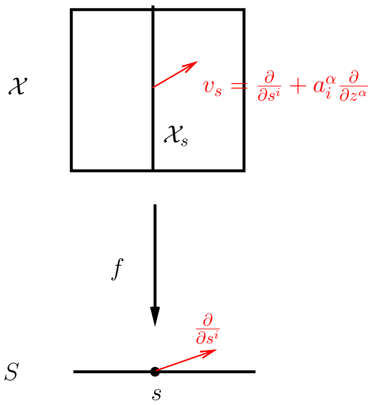

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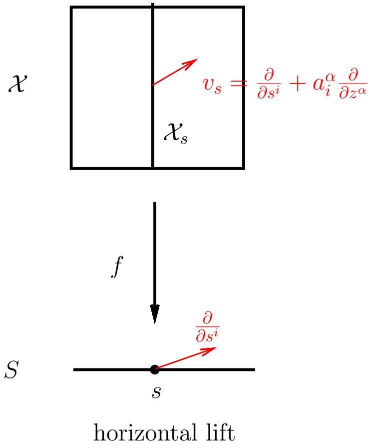

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The only contribution in (1), which may be negative, originates from the harmonic parts in the third term. It equals

$$
-\int_{\mathcal{X}_s} H(A\cup\overline{\psi})\overline{H(A_j\cup\overline{\psi})}gdV.
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Using the above positivity of the relative canonical bundle, we see that the Corollary follows from

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Further results by Sh. Takayama - Chr. Mourougane and K. Liu - X. Yang.

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Let $f : \mathcal{X} \to S$ be a holomorphic family of compact complex manifolds, and (L, h) be a *relatively positive line bundle* on $\mathcal{X}$.

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$p = n$ yields curvature of $f_*(\mathcal{K}_{\mathcal{X}/S}\otimes L)$

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Corollary

Theorems of Bo Berndtsson and G. Sch.

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Use Lie-derivatives for differential forms with values in hermitian line bundles.

Methods (Calabi-Yau manifolds)

Calabi's Theorem '57

Holmorphic 1-forms and holomorphic vector fields on Calabi-Yau manifolds are parallel.

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$$(z, s) \mapsto s$$

$$z = (z^1, \ldots, z^n)$$

$$s = (s^1, \ldots, s^r)$$

components z^α, s^i

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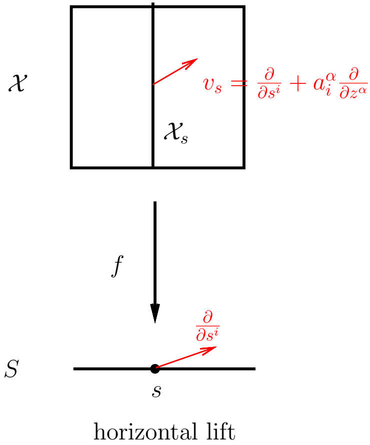
Geodesic curvature



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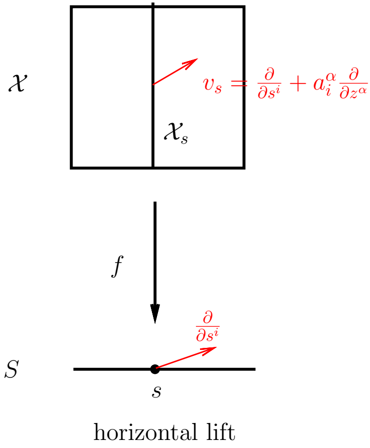

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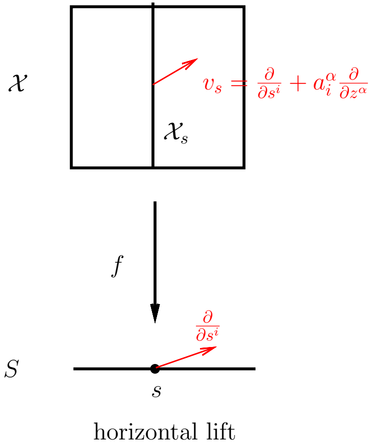

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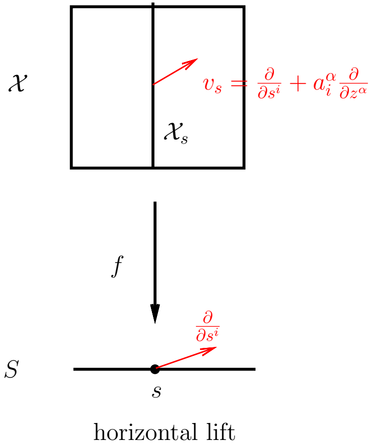

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Fact

Fact Curvature form of $\mathcal{K}_{\mathcal{X}/S}$:

$$\Theta = -\sqrt{-1}\partial\overline{\partial}\log g$$

$$= -\sqrt{-1}\left(\Theta_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}} + \Theta_{i\bar{\jmath}}\, ds^i \wedge ds^{\bar{\jmath}} + \Theta_{i\overline{\beta}}\, ds^i \wedge dz^{\overline{\beta}} + \Theta_{\alpha\bar{\jmath}}\, dz^\alpha \wedge ds^{\bar{\jmath}}\right)$$

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$$\Theta_{i\overline{\beta}}\,g^{\overline{\beta}\alpha}\,\partial_{\alpha} = \overline{\partial}^{*}A^{\alpha}_{i\overline{\beta}}\,\partial_{\alpha}dz^{\overline{\beta}}$$

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$\Rightarrow$Fact.

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On the other hand (cf. Proposition)

$$0 = \int_{\mathcal{X}/S} \omega_{\mathcal{X}}^{n+1} = \sqrt{-1} \left(\int_{\mathcal{X}/S} \varphi_{i\bar{\jmath}} g \, dV \right) ds^i \wedge ds^{\bar{\jmath}}$$

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$$G_s(A_i \cdot A_{\bar{\jmath}}) \geq -c \cdot vol(\mathcal{X}_s)\, \Theta_{i\bar{\jmath}}$$

(in the sense of matrices/hermitian forms)

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Theorem

Claim

For a suitable constant $c' > 0$ the form

$$\widetilde{\omega}_{\mathcal{X}} = \omega_{\mathcal{X}} + c' f^* \omega_{WP}$$

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Namely, with $c' = c \cdot vol(\mathcal{X}_s) + 1$

$$\begin{aligned} \widetilde{\omega}_{\mathcal{X}}^{n+1} &= (\omega_{\mathcal{X}} + c' f^* \omega_{WP})^{n+1} = \sqrt{-1}(\varphi_{i\bar{\jmath}} + c'\Theta_{i\bar{\jmath}}) ds^i \wedge ds^{\bar{\jmath}} \\ &\geq \sqrt{-1}\Theta_{i\bar{\jmath}} ds^i \wedge ds^{\bar{\jmath}}, \end{aligned}$$

whereas $\widetilde{\omega}_{\mathcal{X}}|\mathcal{X}_s = \omega_{\mathcal{X}_s} > 0$. □

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