Nori uniformization of algebraic stacks

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Plan

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Algebraic stacks

Most stacks $\mathfrak{X}$ (in this talk) come equipped with an epimorphism $X_0 \to \mathfrak{X}$, where X_0 is an algebraic space.

The groupoid $X_1 := X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$ enables to reconstruct $\mathfrak{X}$.

For instance if G is a group scheme acting on a scheme X, the quotient stack $[X/G]$ is associated to the groupoid $(\mathrm{pr}_1, a) : X \times G \rightrightarrows X$.

A stack morphism $\mathfrak{Y} \to \mathfrak{X}$ is *representable* if for any cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathfrak{Y} & \longrightarrow & \mathfrak{X} \end{array}$$

where X is an algebraic space, Y is an algebraic space as well.

A stack $\mathfrak{X}$ is *algebraic* is $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable and there exists a smooth atlas $U \to \mathfrak{X}$, where U is an algebraic space.

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Étale coverings

Let $\mathfrak{X}$ a connected algebraic stack. A *finite étale cover* of $\mathfrak{X}$ is by definition a *representable* finite étale morphism $\mathfrak{Y} \to \mathfrak{X}$. The corresponding category is denoted by $\mathrm{Cov}(\mathfrak{X})$.

Theorem (B.Noohi 2004)

The category $\mathrm{Cov}(\mathfrak{X})$ *is a Galois category.*

So any geometric point $\overline{x} : \mathrm{spec}\,\Omega \to \mathfrak{X}$ defines as usual an étale fundamental group

$$\pi_1(\mathfrak{X}, \overline{x}) := \mathrm{Aut}(\overline{x}^*)$$

where $\overline{x}^* : \mathrm{Cov}(\mathfrak{X}) \to \mathrm{Sets}$ is the corresponding fiber functor.

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Uniformization criterion

A connected algebraic stack $\mathfrak{X}$ is called *uniformizable* if there exists $Y \to \mathfrak{X}$ in Cov X, where Y is a (non empty) algebraic space.

By functoriality, for any geometric point $\overline{x}$, any $\alpha \in \mathrm{Aut}(\overline{x})$ defines a *hidden loop* α^* in $\pi_1(\mathfrak{X}, \overline{x})$. So one gets morphisms :

$$\omega_{\overline{x}} : \mathrm{Aut}(\overline{x}) \to \pi_1(\mathfrak{X}, \overline{x}) .$$

Theorem (Noohi 2004)

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The origins

Let X/k be a scheme, endowed with a rational point $x \in X(k)$. Nori considers triples $(G, Y \to X, y)$, where G/k is a finite group scheme; $Y \to X$ is a G-torsor, and $y \in Y(k)$ lifts x.

Definition

X admits a fundamental group scheme if there exists a triple $(\pi(X,x), \widetilde{X}_x, \tilde{x})$ which is pro-universal among these triples.

Theorem (Nori 1976)

If X is reduced and connected, then X admits a fundamental group scheme.

We want first to improve on two points :

- work with an algebraic stack rather than a scheme,
- get rid of the rational base point.

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Gerbes

Theorem (B.-Vistoli 2015)

Let $\mathfrak{X}/k$ be an algebraic stack. If $\mathfrak{X}$ is geometrically {connected and reduced} then there exists a morphism $\mathfrak{X} \to \pi_{\mathfrak{X}/k}$ which is (pro)-universal among morphisms to finite gerbes.

To any group scheme G, one can associate its *classifying stack* $\mathrm{B}\,G$ defined by $\mathrm{B}\,G(S) = \{G\text{-torsors over } S\}$.

Definition

A stack Γ is an *affine gerbe* if $\Gamma \neq \emptyset$, two objects are locally isomorphic, and the automorphism group of any object is affine.

$$\text{neutralized affine gerbes } k \longleftrightarrow \text{ affine group schemes}/k$$

$$(\Gamma, x \in \Gamma(k)) \to \mathrm{Aut}_k(x)$$

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Residual gerbes and Nori-uniformization

For any geometric point $\overline{x} : \operatorname{spec}\Omega \to \mathfrak{X}$ of an algebraic stack, there is an associated *residual gerbe* obtained by an epi-mono factorization : $\operatorname{spec}\Omega \to \mathcal{G}_{\overline{x}} \to \mathfrak{X}$.

Definition

Let $\mathfrak{X}/k$ be an algebraic stack. $\mathfrak{X}$ is *Nori-uniformizable* if there exists a G-torsor $Y \to \mathfrak{X}$, where Y is an algebraic space, and G/k is a finite group scheme.

Theorem (I.Biswas-B. 2017)

Let $\mathfrak{X}/k$ be an algebraic stack of finite type, with finite inertia, and geometrically {connected and reduced}. Then $\mathfrak{X}/k$ is Nori-uniformizable if and only if for any $x \in |\mathfrak{X}|_0$, the morphism

$$\mathcal{G}_x \to \mathfrak{X} \to \pi_{\mathfrak{X}/k} \text{ is faithful.}$$

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Sketch of a proof of ‘only if’ statement

Proof.

A stack morphism $f : \mathfrak{Y} \to \mathfrak{X}$ is *faithful* is for any point $y : S \to \mathfrak{Y}$, the induced morphism of S-groups $\mathrm{Aut}_{\mathfrak{Y}}(y) \to \mathrm{Aut}_{\mathfrak{X}}(f(y))$ is a monomorphism. When $\mathfrak{X}$ et $\mathfrak{Y}$ are algebraic, this is equivalent to : f is representable.

Now the fact that $\mathfrak{X}$ is Nori-uniformizable exactly means that there is a representable morphism $\mathfrak{X} \to \mathrm{B}\, G$, where G is a finite group scheme (incidentally, one could replace $\mathrm{B}\, G$ by any finite gerbe).

Since the morphism $\mathcal{G}_x \to \mathfrak{X}$ is also representable, so is the composite morphism $\mathcal{G}_x \to \mathfrak{X} \to \mathrm{B}\, G$, hence it is faithful. By the universal property of the fundamental gerbe, $\mathcal{G}_x \to \pi_{\mathfrak{X}/k}$ factors this composite morphism, hence must be faithful as well.

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Essentially finite bundles

The Tannaka correspondence over a field k :

$$\text{Affine gerbes} \longleftrightarrow \text{Tannaka categories}$$

$$\Gamma \rightarrow \mathrm{Vect}_k\,\Gamma$$

$$(S \rightarrow \{\text{fibre functors of } \mathcal{C}_S\}) \leftarrow \mathcal{C}$$

Definition

1. *(Weil) A locally free sheaf $\mathcal{E}$ is finite if there exists $f \neq g$ in $\mathbb{N}[t]$ so that $f(\mathcal{E}) \simeq g(\mathcal{E})$.*
2. *(Nori, B.-Vistoli) $\mathcal{E}$ is essentially finite if it is the kernel of a morphism between two finite locally free sheaves.*

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Nori uniformization : tannakian criterion

Theorem (B.-Vistoli 2015)

If $\mathfrak{X}/k$ is a proper algebraic stack that is geometrically {connected and reduced}, then pull-back along $\mathfrak{X} \to \pi_{\mathfrak{X}/k}$ identifies $\mathrm{Vect}\,\pi_{\mathfrak{X}/k}$ *with* $\mathrm{EF\,Vect}(\mathfrak{X})$.

Corollary (I.Biswas-B. 2017)

If $\mathfrak{X}/k$ is moreover of finite type, with finite inertia, then $\mathfrak{X}/k$ is Nori-uniformizable if and only if for any $x \in |\mathfrak{X}|_0$, any $V \in \mathrm{Rep}(\mathcal{G}_x)$ is a subquotient of $\mathcal{E}_{|\mathcal{G}_x}$, for some $\mathcal{E} \in \mathrm{EF\,Vect}(\mathfrak{X})$.

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- Uniformization of algebraic stacks
  - Representable morphisms
  - Fundamental groups
  - Hidden loops
- Nori uniformization
  - Nori fundamental gerbe
  - Residual gerbes
  - Tannakian interpretation
- Application to tame torsors
  - Tamely ramified torsors
  - Parabolic bundles
  - Stack of roots

Kummer torsors

We fix a scheme X/k and D a simple normal crossings divisor on X, meaning $D = \cup_{i \in I} D_i$ is the union of a finite family of irreducible, smooth divisors, crossing normally. We denote the corresponding family by $\mathbf{D} = (D_i)_{i \in I}$ and add to our data a family $\mathbf{r} = (r_i)_{i \in I}$ of positive integers.

Definition

Let $x \in |X|_0$ and $U = \operatorname{spec} R$ be an open affine neighbourhood. For each $i \in I_x = \{i \in I / x \in |D_i|\}$, let $s_i \in R$ be a local equation for the Cartier divisor D_i. The corresponding *Kummer morphism* on U with ramification data $(\mathbf{D}, \mathbf{r})$ is the morphism

$$Z = \operatorname{spec} R[\mathbf{t}]/[\mathbf{t}^{\mathbf{r}} - \mathbf{s}] \to \operatorname{spec} R\,,$$

where the algebra is the tensor product over R of $R[t_i]/t_i^{r_i} - s_i$ for all i in I_x.

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Torsors locally induced by Kummer torsors

Given $x \in |X|_0$, we set $(\mathbf{r}_x)_i$ as r_i if $i \in I_x$ and 1 otherwise; this defines a multi-index $\mathbf{r}_x$.

Definition

Let G/k be a finite abelian group scheme. A *tamely ramified G-torsor* with ramification data $(\mathbf{D}, \mathbf{r})$ is the data of a scheme Y endowed with an action of G and a finite and flat G-invariant morphism $Y \to X$ such that for each closed point x of X, there exists a monomorphism $\mu_{\mathbf{r}_x} \to G$ such that in a *fppf* neighbourhood of x in X, the morphism $Y \to X$ is isomorphic to $Z \times^{\mu_{\mathbf{r}_x}} G$, where $Z \to \operatorname{spec} R$ is a Kummer torsor.

Example

If $\#I = 1$, and $G = \mu_r$, one recovers the notion of *uniform cyclic cover* (this amounts to the data of $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on X and $\alpha : \mathcal{L}^{\otimes r} \simeq \mathcal{O}_X(-D)$ is an isomorphism).

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If $\#I = 1$, and $G = \mu_r$, one recovers the notion of *uniform cyclic cover* (this amounts to the data of $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on X and $\alpha : \mathcal{L}^{\otimes r} \simeq \mathcal{O}_X(-D)$ is an isomorphism).

Plan

Simpson's definition

We endow the set $\frac{1}{\mathbf{r}}\mathbb{Z}^I = \prod_{i \in I} \frac{\mathbb{Z}}{r_i}$ with the component-wise partial order.

Definition

A *parabolic vector bundle* on $(X, \mathbf{D})$ with weights in $\frac{1}{\mathbf{r}}\mathbb{Z}^I$ consists of two compatible pieces of data :

- a functor $\mathcal{E}_\cdot : \left(\frac{1}{\mathbf{r}}\mathbb{Z}^I\right)^0 \to \mathrm{Vect}\, X$ and,
- for each integral multi-index $\mathbf{l}$ in $\mathbb{Z}^I$, a natural transformation $\mathcal{E}_{\cdot + \mathbf{l}} \simeq \mathcal{E}_\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\mathbf{l} \cdot \mathbf{D})$.

Given a closed point x of D, and $\mathbf{l} \in \mathbb{Z}^I$, we will write that a parabolic vector bundle $\mathcal{E}_\cdot$ on $(X, \mathbf{D})$ *admits* $\mathbf{l}/\mathbf{r}$ *as a weight at* x if the monomorphism $\left(\mathcal{E}_{(\mathbf{l}+\mathbf{1})/\mathbf{r}}\right)_x \to \left(\mathcal{E}_{\mathbf{l}/\mathbf{r}}\right)_x$ is not an isomorphism.

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Existence of torsors with prescribed ramification

Parabolic vector bundles (with arbitrary weights) form a tensor category denoted by $\mathrm{Par}(X, \mathbf{D})$. One can define (essentially) finite objects as before, leading to a category $\mathrm{EF}\,\mathrm{Par}(X, \mathbf{D})$.

Theorem (I.Biswas-B., work in progress)

Assume that X is proper, of finite type, geometrically {connected and reduced}. The two following statements are equivalent :

1. *There exists a finite abelian group scheme G/k and a tamely ramified G-torsor $Y \to X$ with ramification data $(\mathbf{D}, \mathbf{r})$,*
2. *for each closed point x of D, and for all $\mathbf{l} \in \mathbb{Z}^I$, such that $0 \leq \mathbf{l} < \mathbf{r}_x$, there exists an object $\mathcal{E}.$ in $\mathrm{EF}\,\mathrm{Par}(X, \mathbf{D})$ with abelian monodromy and weights in $\frac{1}{\mathbf{r}}\mathbb{Z}^I$, such that $\mathcal{E}.$ admits $\mathbf{l}/\mathbf{r}$ as a weight at x.*

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Plan

Vistoli's definition

Definition (Vistoli 2008)

The *stack of roots* $\sqrt[\mathbf{r}]{\mathbf{D}/X}$ is the stack classifying **r**-th roots of $(\mathcal{O}_X(\mathbf{D}), s_{\mathbf{D}})$.

Theorem (I.Biswas (1997), B.(2007))

$\mathrm{Vect}(\sqrt[\mathbf{r}]{\mathbf{D}/X}) \simeq \mathrm{Par}_{\frac{1}{\mathbf{r}}\mathbb{Z}^I}(X, \mathbf{D})$.

Proposition

Let Y be a k-scheme endowed with an action of a finite abelian k-group scheme G and let $Y \to X$ be a finite, flat, and G-invariant morphism to a scheme. Equivalence :

1. *$Y \to X$ factors through a G-torsor $Y \to \sqrt[\mathbf{r}]{\mathbf{D}/X}$,*
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Sketch of a proof : tame to stacky

Start with a tamely ramified G-torsor $Y \to X$. For each point x of X, there exists a *fppf* neighbourhood $X' = \operatorname{spec} R \to X$ of x such that $Y' \to X'$ is induced by a $\mu_{\mathbf{r}_x}$-cover $Z' \to X'$ along $\mu_{\mathbf{r}_x} \hookrightarrow G$. It follows that $[Z'/\mu_{\mathbf{r}_x}] \simeq [Y'/G]$ above X'.
Moreover, $[Z'/\mu_{\mathbf{r}_x}] \simeq \mathfrak{X}'$ above X' (where $\mathfrak{X} = \sqrt[\mathbf{r}]{\mathbf{D}/X}$) . So we get local morphisms $Y' \to \mathfrak{X}'$ and we have to show that they glue into a global one. The gluing data comes from the following :

Lemma

Let ϕ be an automorphism of $\mathfrak{X} = \sqrt[\mathbf{r}]{\mathbf{D}/X}$ above X. There exists a unique isomorphism $\phi \simeq \mathrm{id}$.

The fact that this is a descent data results from the fact that the action of G on Y is faithful.

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Tame stacks

An algebraic stack [...] $\mathfrak{X}$ with finite inertia admits a moduli space $p : \mathfrak{X} \to X$. It is a *tame* stack if $p_* : \mathrm{Qcoh}(\mathfrak{X}) \to \mathrm{Qcoh}(X)$ is exact.

Lemma

Let G/S be an affine group scheme, $\mathfrak{X}$ be a tame stack with moduli space X and let $Y \to \mathfrak{X}$ be a G-torsor . Then Y is trivial locally on X if and only if $\forall x \in |\mathfrak{X}|_0$, $Y_{|\mathcal{G}_x} \to \mathcal{G}_x$ is trivial.

Proposition

Let G/S be an abelian affine group scheme, $\mathfrak{X}$ be a tame stack with moduli space X and consider two G-torsors $Y_i \to \mathfrak{X}$ for $i \in \{1, 2\}$. Then $Y_1 \to \mathfrak{X}$ is isomorphic to $Y_2 \to \mathfrak{X}$ fppf locally on X if and only if for each closed point $x \in |\mathfrak{X}|_0$, the restrictions $Y_{1|\mathcal{G}_x} \to \mathcal{G}_x$ and $Y_{2|\mathcal{G}_x} \to \mathcal{G}_x$ are isomorphic.

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Assume that $Y \to \sqrt[\mathbf{r}]{\mathbf{D}/X} = \mathfrak{X}$ is a G-torsor. We will show that $Y \to X$ is a tamely ramified G-torsor.

In a Zariski neighbourhood of $x \in X$, $\mathfrak{X} \simeq [Z/H]$ where $H = \mu_{\mathbf{r}_x}$, and $Z \to X$ is a H-Kummer cover, with ramification data $(\mathbf{D}, \mathbf{r})$. It follows that the morphism

$$\mathcal{G}_x \xrightarrow{i} \mathfrak{X} \xrightarrow{Z} \mathrm{B}\, H$$

is an isomorphism. One thus gets a morphism

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which is representable since Y is a scheme. This defines a group monomorphism $H \to G$, and one can then conclude by applying the proposition to the two G-torsors on $\mathfrak{X}$ given by $Y_1 = Y$ and $Y_2 = Y \circ i \circ Z$.

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