Emergent topology in quenched quantum systems

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S. Sharma, S. Suzuki and A. Dutta, Phys. Rev. B **92**, 104306 (2015);
S. Sharma, U. Divakaran, A. Polkovnikov, A. Dutta, Phys. Rev. B **93**, 144306 (2016);
U. Divakaran, S. Sharma, A. Dutta, Phys. Rev. E **93** , 052133 (2016).

What happens after the quench in a closed quantum system?

- Linear growth of the entanglement entropy; collapse and revival of the Loschmidt echo; light cone like propagation of quantum correlations
- Thermalization
- Statistics of work done
- Periodically driven closed quantum systems
- Periodic steady state: heating effect
- Periodic driving: Entanglement entropy
- Follow the subsequent real time evolution: non-analyticities

Loschmidt overlap and the rate function of return probability

Sudden Quench:

Initial state $|\psi_0$: Ground state of the initial Hamiltonian H_i

Suddenly quench $H_i(\lambda_i)$ to $H(\lambda_f)$

Loschmidt Overlap: $L(t) = \langle\psi_0|\exp(-iH_f(\lambda_f)t|\psi_0\rangle$

Rate function: $I(t) = -\ln(|\langle\psi_0|e^{-iH(h_f)\mathbf{t}}|\psi_0\rangle|^2)/N \rightarrow$ Double quenches

Quantum dynamical phase transition: $\langle\psi_0|\exp(-iH_f(\lambda_f)t|\psi_0\rangle = 0$

Slow Ramping: Kibble-Zurek Scaling

Initial state $|\psi_0\rangle$: Ground state of the initial Hamiltonian H_i

Change λ_i to λ_f slowing using a protocol, e.g., $\lambda(t) = \lambda_i + (\lambda_f - \lambda_i)t/\tau$

The final state is ψ_f; not an eigenstate of the final Hamiltonian H_f

Set time $t = 0$

$L(t) = \langle\psi_0|\exp(-iH_f(\lambda_f)t|\psi_0\rangle$; $I(t) = -\ln(|\langle\psi_f|e^{-iH(h_f)\mathbf{t}}|\psi_f\rangle|^2)/N$

Role of slow quench is to prepare a desired initial state which evolves with the final Hamiltonian

What we call a dynamical phase transition?

- NO local order parameter; no universal exponents
- Quenched quantum systems
- Non-analyticities in dynamical free energy
- manifested in the subsequent real time evolution of the quenched system: rate function of the return probability
- Is there any connection with equilibrium QCP?

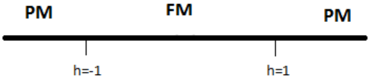
The simplest paramagntic model

$$H = -\sum_{<ij>} \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z$$

For $h > 1$, $\langle \sigma_i^x \rangle = 0$; Paramagnetic

For $h < 1$; $\langle \sigma_i^x \rangle \neq 0$; Ferromagnetic

• Quantum phase transitions at $\lambda = |h - 1| = 0$: Quantum critical point (QCP)



Studies have been generalized to various integrable and non-integrable models

Dutta, Aeppli, Chakrabarti, Divakaran, Rosenbaum and Sen, Cambridge University Press, 2015

Dynamical Phase Transition: Sudden quenches

- **Phase Transitions** $\rightarrow$ marked by non-analyticities in the free energy of a system.

CANNONICAL PARTITION FUNCTION

$$Z(\beta) = \mathrm{Tr} e^{-\beta H}$$

Equilibrium phase transition (EPT)

OVERLAP AMPLITUDE (sudden quenching)

$$G(t) = \langle \psi_0(\lambda_i) | e^{-iH(\lambda_f)t} | \psi_0(\lambda_i) \rangle$$

Dynamical Phase Transition (DPT)

Temperature driven EPT $\overset{\text{analouge}}{\longleftrightarrow}$ Real time evolution

Generalising to complex plane: $Z(z) = \langle \psi_0(\lambda_i) | e^{-zH(\lambda_f)} | \psi_0(\lambda_i) \rangle, \quad z \in \mathbb{C}$

$z = R$
$z = it$: overlap amplitude

Heyl et. al., PRL **110**, 135704 (2014); A. Silva, PRL **101**, 120603 (2008)

Dynamical Phase Transition (DPT)

Thermodynamic limit: **"Free Energy"**

$$f(z) = -\lim_{N\to\infty} \frac{\ln Z(z)}{N} = -\lim_{N\to\infty} \frac{\ln \langle \psi_0(\lambda_i)| e^{-zH(\lambda_f)} |\psi_0(\lambda_i)\rangle}{N}$$

- Non-analytic $f(z) \Rightarrow Z(z) \to 0$: find $z \Rightarrow$ marks DPTs

  $\Rightarrow$ Fisher zeros: z for which $Z(z) \to 0$

  $\Rightarrow$ Fisher zeros in complex z plane

  $\Rightarrow$ Real(z)=0: non-analyticities in time

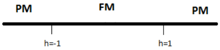
- Reflected in the non-analyticities in **Rate function**:

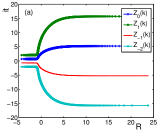
$$I(t) = -\frac{\ln(|\langle \psi_0(\lambda_i)| e^{-itH(\lambda_f)} |\psi_0(\lambda_i)\rangle|^2)}{N}$$

- Double Quench: Rate function of the return probability

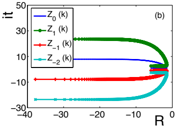
DPT ⇔ EPT in sudden quenching

Heyl et. al. PRL **110**, 135704

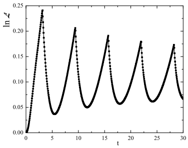




(a) Across the QCP: $h_i = 0.4$, $h_f = 1.3$.



(b) Within one phase: $h_i = 0.4$, $h_f = 0.8$.



Non-analytitcities appearing periodically; first derivative of the "free energy" is discontinuous

DPT: Sudden quenching

- Across the QCP or Equilibrium Phase Transition (EPT)

  $\Rightarrow$ Fisher zeros cross the imaginary time axis.

  $\Rightarrow$ At corresponding times non-analyticities appears.

  $\Rightarrow$ DPTs are present

- **Within one phase**

  $\Rightarrow$ Fisher zeros DO NOT cross the imaginary time axis.

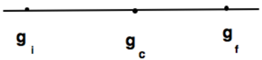
  $\Rightarrow$ NO non-analyticities.

  $\Rightarrow$ DPTs are ABSENT

There are indeed exceptions

First signature of DPTs

Are DPTs characteristic of integrability?

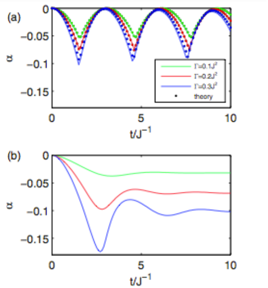


$H_{pollmann} = \sum_i \left[\sigma_i^x - \sigma_i^z \sigma_{i+1}^z + g(\cos\phi\sigma_i^x + \sin\phi\sigma_i^z)\right]$

- $g_c = g = 0 \Rightarrow H \rightarrow$ critical
- $\phi = 0, \pi \Rightarrow$ Transverse Ising chain $\rightarrow$ integrable $\rightarrow \nu = 1$
- $\phi \neq 0, \pi \Rightarrow$ non-integrable $\rightarrow$ except $g = 0 \rightarrow \nu = 8/15$
- $g \rightarrow g(t) = -t/\tau$ or $g = g_i + \frac{(g_f - g_i)t}{\tau}$
- $|\langle\psi_0(g_f, \tau)|e^{-iH_f t}|\psi_0(g_f, \tau)\rangle|^2 = e^{-\alpha(t)L}$
- $\alpha(t) \Rightarrow -I(t)$: Rate function

Pollmann, Mukerjee, Green and Moore, PRE **81**, 020101(R) (2010)

Integrable vs non-integrable:FM case



- $g_i = 0.5$ to $g_f = -0.5$. Fig. (a) $\phi = 0$ and Fig. (b) $\phi = \pi/32$.

- sharp **non-analyticities present** in $I(t)$ for $\phi = 0$, ramped across the QCP: **integrable case**.

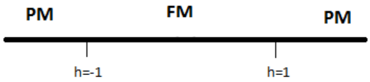
- No non-analyticities present for the non-integrable case

Anti-ferromagnetic version of the model, there exist DPTs: Sharma, Suzuki and Dutta, PRB (2015)

Integrable system: What happens to Fisher zeros in slow quenching?

$$H_k = \begin{pmatrix} h - \cos k & -i \sin k \\ i \sin k & -h + \cos k \end{pmatrix}$$

- $h \to \mathbf{t}/\tau \qquad \Rightarrow \qquad |\psi_i^g\rangle \to |\psi_f\rangle$



Rate function: $I(t) = -\ln(|\langle\psi_f|e^{-iH(h_f)\mathbf{t}}|\psi_f\rangle|^2)/N$

- The evolution in rate starts after the quenching process is finished.
- $|\psi_{k_f}\rangle = v_k|\psi_k^g(h_f)\rangle + u_k|\psi_k^e(h_f)\rangle$

S. Sharma, U. Divakaran, A. Polkovnikov, A. Dutta, PRB, 2016; U. Divakaran, S. Sharma, A. Dutta, PRE, 2016

Slow quenching: Rate function and Free energy

- Rate function: $I(t) = -\ln(|\langle\psi_f|e^{-H(h_f)it}|\psi_f\rangle|^2)/N$
  $= -\int_0^\pi \frac{dk}{2\pi} \log\left(1 - 4|u_k|^2|v_k|^2 \sin^2 \epsilon_k^f t\right)/N$
- Free energy: $f_k(z) = -\lim_{N\to\infty} \ln \langle\psi_f|e^{-zH_f}|\psi_f\rangle/N$
  $= -\int_0^\pi \frac{dk}{2\pi} \ln\left(|v_k|^2 + |u_k|^2 \exp(-2\epsilon_k^f z)\right)$
- Transition probability: $p_k = |u_k|^2 = |\langle\psi_{k_f}|\psi_k^e(h_f)\rangle|^2$

$$f_k(z) = -\int_0^\pi \frac{dk}{2\pi} \ln\left((1 - p_k) + p_k \exp(-2\epsilon_k^f z)\right)$$

Slow quenching: $z_n(k)$

- Free energy: $f_k(z) = -\int_0^\pi \frac{dk}{2\pi} \ln\left((1-p_k) + p_k \exp(-2\epsilon_k^f z)\right)$
- Non-analytic $f(z) \to$ Argument of log $= 0$

$$z_n(k) = \frac{1}{2\epsilon_k^f}\left(\ln\left(\frac{p_k}{1-p_k}\right) + i\pi(2n+1)\right)$$

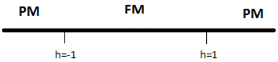
$$I(t) = -\int_0^\pi \frac{dk}{2\pi} \ln\left(1 + 4p_k(p_k-1)\sin^2 \epsilon_k^f t\right). \qquad (1)$$

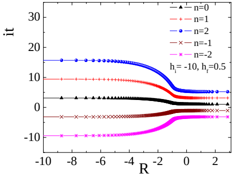
- $k_* \quad \Rightarrow \quad p_{k_*} = 1/2$ "infinite temperature" state

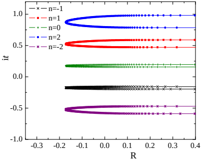
$$t_n^* = \frac{\pi}{\epsilon_{k_*}^f}\left(n + \frac{1}{2}\right)$$

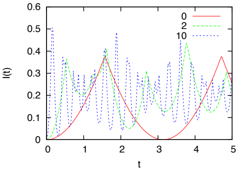
$$n = 0, 1, 2, \quad \cdots$$

DPTs in slow quenching



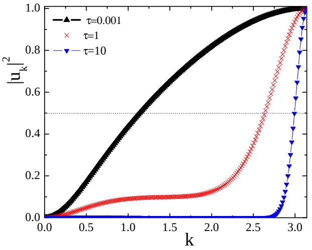




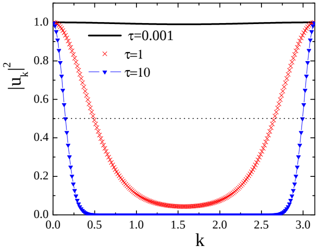


- two t_n^*s corresponding to two critical points $t_n^* = t_n^{\pm}$
- $h = -1$ occurs at $t_n^+ = 1/2\epsilon^f_{(\pi - k_*)}$
- $t_n^- = 1/2\epsilon^f_{k_*}$ at $h = 1$

Role of τ



τ=0.001
τ=1
τ=10
|u_k|^2
k
1.0
0.8
0.6
0.4
0.2
0.0
0.5
1.5
2.0
2.5
3.0



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|u_k|^2
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0.8
0.6
0.4
0.2
0.0
0.5
1.5
2.0
2.5
3.0

Topological aspect of DPT: Slow Quenching

• Quantum non-cyclic evolution of the system: Pancharatnam geometrical phase (PGP) phase

Dynamical Topological Order Parameter (DTOP)

• Characterizes topological properties of the real-time dynamics rather than of the instantaneous wave function or the instantaneous Hamiltonian

• Changes its value at DPT $\Rightarrow$ non-analyticities in Loschmidt overlap

• Recall: $|\psi_{k_f}\rangle = v_k|\psi_k^g(h_f)\rangle + u_k|\psi_k^e(h_f)\rangle$

$$L_k = \langle\psi_{f_k}|\exp(-iH_f t)|\psi_{f_k}\rangle = |r_k|\exp(i\phi_k)$$

• $\phi_k = \tan^{-1}\left(\frac{-|u_k|^2\sin(2\epsilon_k^f t)}{|v_k|^2+|u_k|^2\cos(2\epsilon_k^f t)}\right)$

• Dynamical phase: $\phi_k^{\rm dyn} = -\int_0^t ds\langle\psi_{f_k}(s)|H_f|\psi_{f_k}(s)\rangle = -2|u_k|^2\epsilon_k^f t$

J. C. Budich and M. Heyl, PRB **93**, 085416 (2016)

Properties of Dynamical Geometric Phase

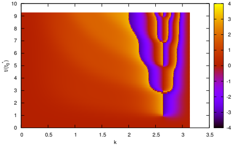
- Geometric phase: $\phi_k^G = \phi_k - \phi_k^{\mathrm{dyn}}$

$$\phi_k^G = \tan^{-1}\left(\frac{-|u_k|^2 \sin(2\epsilon_k^f t)}{|v_k|^2 + |u_k|^2 \cos(2\epsilon_k^f t)}\right) + 2|u_k|^2 \epsilon_k^f t$$

- As k goes from 0 to π, ϕ_k^G goes from $-\pi$ to π thus completing a full circle $\implies \phi_\pi^G - \phi_0^G = 0 \mod 2\pi$
- $\phi_k^G|_{k_*}$ is fixed to 0 or π for all values of t

$$\phi_k^G|_{k_*} = \tan^{-1}\left(-\tan(\epsilon_{k_*}^f t)\right) + 2|u_{k_*}|^2 \epsilon_{k_*}^f t,$$

- At DPTs $\phi_k(t)$ is ill-defined: Variation of ϕ_k^G as a function of k and t/t_0.



Dynamical Topological Quantum Number

- Quantized winding number

$$\nu_D = \frac{1}{2\pi} \oint_0^{\pi} \frac{\partial \phi_k^G}{\partial k}$$

Generalized Gauss's Law

- Serves as an order parameter

- ν_D may show an integer jump or a drop $\Rightarrow \Delta\nu_D(t^*)$

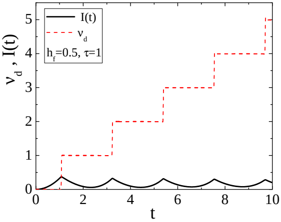
• integrating the full derivative of a periodic function (modulo 2π), the integral remains constant.

• unless there is some discontinuity in the phase, which should be manifested in the δ-function type contribution to the derivative of the geometric phase.

• such discontinuities develop only at DPT: Analyze the behavior of at k_*

$$\frac{\partial \phi_k^G}{\partial k}\Big|_{k_*} = 2\tan(\epsilon_{k_*}^f t)\frac{\partial |v_k|^2}{\partial k}\Big|_{k_*} + 2\frac{\partial |u_k|^2}{\partial k}\Big|_{k_*}(\epsilon_{k_*}^f t). \qquad (2)$$

ν_D vs Rate function: PM (non-topological) $\longrightarrow$ FM (topological)

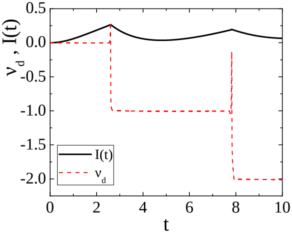


(d) • $h = t/\tau$ from $h_i = -10$ to $h_f = 0.5$

• system crosses the QCP at $h = -1$

• ν_D jumps by a factor of unity whenever there is a DPT

• $\mathrm{sgn}(\partial_k |u_k|^2|_{k_*})$ and hence $\Delta\nu_D$ is positive

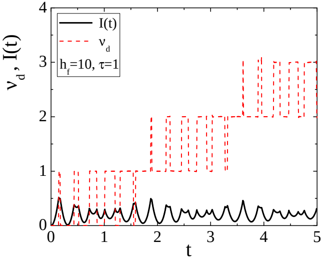


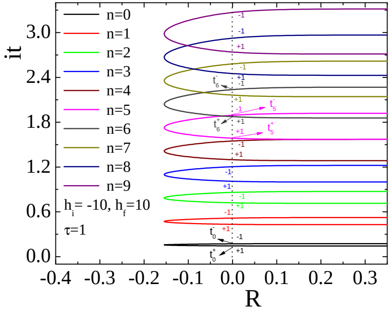
(e) • $h_i = 10$ to $h_f = 0.5$ crossing $h = 1$

• $\mathrm{sgn}(\partial_k |u_k|^2|_{k_*})$ and hence $\Delta\nu_D$ is negative

S. Sharma, U. Divakaran, A. Polkovnikov, A. Dutta, PRB (2016).

ν_D vs Rate function: PM $\longrightarrow$ FM$\longrightarrow$ PM





(f) • $h_i = -10$ to $h_f = 10$

• system crosses both the QCPs at $h = -1$ and $h = +1$

• ν_D oscillates between 0 and 1 $\Rightarrow$ $\nu_D = \pm 1$ at successive DPTs

• jump in ν_D by unity for some values of t

(g) • DPT at t_n^+ associated with a positive topological charge ($+1$)

• t_n^- denotes a DPT with a negative topological charge (-1)

• $t_6^+ < t_5^-$ $\Rightarrow$ there are two successive DPTs both with positive topological charge occurring at t_5^+ and t_6^+

• leads to jump in $\nu_D(t)$ by a factor of unity

Conclusion

• Aparrently, there exists a new class of phase transitions, known as *dynamical phase transitions* reflected in the non-analyticities in the rate-function

Integrable models

• Fisher-zeros and DPTs for a slow quenching depends on $p_k = 1/2$.

• Topological order parameter can be associated with the existence of DPTs in slow and sudden quenching.

Numerous open questions: higher dimension and topological structure, universality, diverging length scale, Role of integrability, Experimental signature

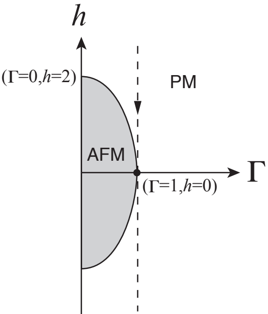
Model considered: AFM transverse Ising chain in a longitudinal field

Original Hamiltonian

$$H = \sum_i \sigma_i^z \sigma_{i+1}^z - \Gamma \sum_i \sigma_i^x - h \sum_i \sigma_i^z$$

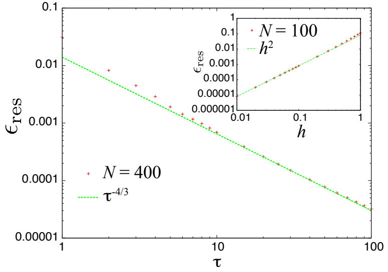
Longitudinal field: $h \to -t/\tau$ Transverse field: $\Gamma = 1$

- $h = 0$, $\Gamma = \Gamma_c = \pm 1$ $\to$integrable
- $h \neq 0 \to$ renders the model non-integrable



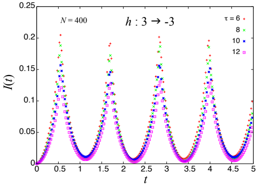
Scaling: Model considered

- Small h, $\Gamma = \Gamma_c = 1$: Gap $(\Delta E) \to \Delta E \sim h^{\nu_h z} = h^2$
- $z = 1 \Rightarrow \nu_h = 2$.



- **Main Fig.**: h changed linearly $-t/\tau$ to the QCP $(h = 0)$, $\epsilon_{\rm res} \sim \tau^{-\nu(d+z)/\nu z+1} = \tau^{-4/3}$
- **Inset**: $\epsilon_{\rm res} \sim h^{\nu_h(d+z)} \sim h^2$ for a sudden quench starting from the QCP

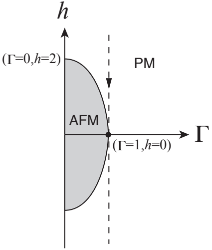
Our result: Slow quenching, non-integrable model

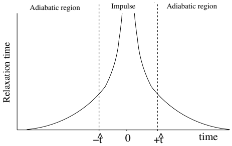


- Non-analyticities in the rate function.
- DPTs present in this non-integrable case.

Reason?

Slow quenching





- In slow quenching system stays in the instantaneous ground state: **Adiabatic region**
- Excitations occur close to critical point: **impulse region**
- System doesn't get time to respond and hence freezes: **impulse region**
- It's like a small sudden quenching close to the critical point.
- **Region of interest: Close to QCP.**

Effective integrable Hamiltonian

$$\tilde{H}_{\mathrm{eff}} = -\sum_i \tilde{\tau}_i^z \tilde{\tau}_{i+1}^z - \sum_i \tilde{\tau}_i^x + h^2 \sum_i \tilde{\tau}_i^x$$

$$H_k(h) = 2 \begin{pmatrix} (1-h^2) - \cos k & -i \sin k \\ i \sin k & -(1-h^2) + \cos k \end{pmatrix}$$

$\epsilon_k = 2\sqrt{\{(1-h^2) - \cos k\}^2 + \sin^2 k} \rightarrow$ 3 QCPs

$h = 0,\ k_c = 0$ and $h = \pm\sqrt{2}, k_c = \pi$

A.A.Ovchinnikov, D.V.Dmitriev, V.Ya.Krivnov and V.O.Cheranovskii, Phys. Rev. B **68**, 214406 (2003)

- Interested in $h = 0,\ k_c = 0 \to$ expand around $k \to 0$

$$H_k(h) = 2\begin{pmatrix} -h^2 + \frac{k^2}{2} & -ik \\ ik & h^2 - \frac{k^2}{2} \end{pmatrix}$$

- linear quenching $\longrightarrow$ reverse quenching problem in the original model

- $\epsilon_k = \sqrt{(h^2 - k^2/2)^2 + k^2}$

$\Rightarrow \Delta\epsilon_k = 2\epsilon_k|_{h=0} \sim k \ \Rightarrow \ z = 1$

$\Rightarrow \Delta\epsilon_k|_{k\to 0} \sim h^2 \Rightarrow \nu z = 2 \Rightarrow \nu = 2$

- $p_k = \mathcal{F}(k^2\tau^{4/3})$ $\quad\quad$ $\mathcal{F}$ scaling function

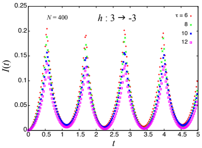
- $\epsilon_{\rm res} \sim \int dk k \mathcal{F}(k^2\tau^{4/3}) \sim \tau^{-4/3}$

KZ prediction $d = z = 1$, $\nu = 2$

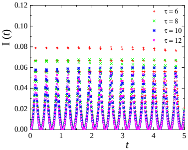
Our results: slow quenching

• Unique model example $\rightarrow$ work with **equivalent integrable model for $\tau \gg 1$**

• Enables us to explain **periodic occurrence of DPTs** for a slow quenching across the QCP in **non-integrable model**

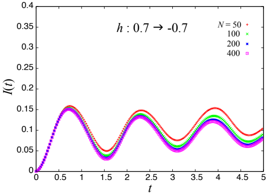


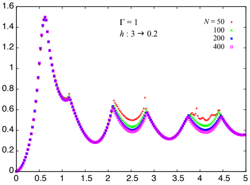
(a) Non-analyticities in the original Hamiltonian



(b) Non-analyticities in the effective Hamiltonian

Our results: sudden quenching





(c) • Following a sudden quench of small amplitude across the QCP of the original model

• Absence of DPTs

• Interaction term in the equivalent Hamiltonian does not change sign

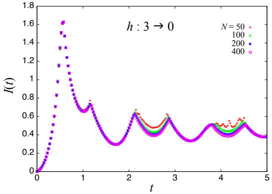
• Implies quenching does *not* take the system across the QCP of Hamiltonian

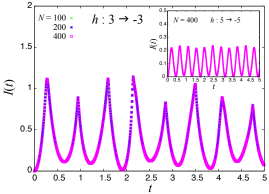
(d) • Presence of DPTs

• Quenching does not take the original Hamiltonian across a QCP

• But the nature of ground state changes: PM in direction of h for $h > 1$, PM in direction of Γ for $h < 1$.

Our results: sudden quenching





(e) • DPTs present

• Nature of the ground state changes

(f) • **Main fig**: Presence of DPTs

• generic feature of a sudden quench across the QCP

• **Inset**: initial and final Hamiltonians essentially reduce to an assembly of non-interacting spins

• Leading to Rabi oscillations between two fully polarized states

Conclusion

• There exists a new class of phase transitions, known as *dynamical phase transitions* reflected in the non-analyticities in the rate-function

Non-integrable model

• Contrary to the observation of Pollmann *et al.* DPTs are observed in a slow quenching of a non-integrable model

• The model can be mapped to an integrable problem for a slow quenching.

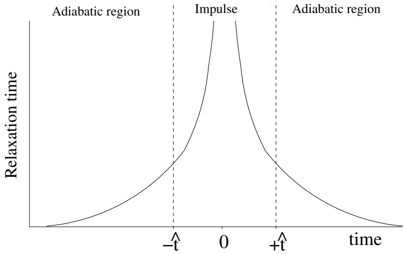
Two-level problem

• Fisher-zeros and DPTs for a slow quenching depends on $p_k = 1/2$.

• Topological order parameter can be associated with the existence of DPTs in slow and sudden quenching.

The occurrence and absence of DPTs are still to be explored in many situations

A quick derivation of KZ scaling



Consider a linear driving $\lambda = t/\tau$

Non-adiabatic effect dominates when

$$\xi_\tau \sim \left(\frac{\hat{t}}{\tau}\right)^{-\nu z} = \frac{\lambda}{\dot{\lambda}} = \tau$$

- • Characteristic time scale: $\hat{t} \sim \tau^{z\nu/(z\nu+1)}$; $\xi \sim \tau^{\nu/(z\nu+1)}$
- • Defect density $n \sim \frac{1}{\xi^d} \sim \tau^{-\nu d/\nu z+1}$