Aspects of non-equilibrium dynamics of quantum systems

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Polkovnikov, Sengupta, Silva and Vengalattore, Rev. Mod. Phys. 83, 863 (2011)

Dutta, Aeppli, Chakrabarti, Divakaran, Rosenbaum and Sen, *Quantum phase transitions in transverse field spin models: from statistical physics to quantum information* (Cambridge University Press, Cambridge, 2015)

Outline

- Quenching across a quantum critical point and Kibble-Zurek scaling of defect density
- Quenching across a quantum critical line and non-linear quenching
- Transverse field Ising model in one dimension and Kitaev model in two dimensions
- Effects of topology and interactions on quenching
- Periodic driving, Floquet edge modes and topological invariants
- Manipulating the Dirac dispersion in graphene and dynamical localization

Defect formation due to quench

It was pointed out long ago that taking a system across a critical temperature can produce a variety of defects

In the early universe, cooling may have led to formation of topological defects (like domains, strings or magnetic monopoles) due to spontaneous symmetry breaking of some scalar field

Zel'dovich, Kobzarev and Okun, JETP 40, 1 (1974); Kibble, J. Phys. A 9, 1387 (1976)

A quench across the normal-superfluid transition in ^4He can lead to domains with different phases of the Bose condensate, or to the production of vortices

Zurek, Nature 317, 505 (1985)

Review: Zurek, Phys. Rep. 276, 177 (1996)

The density of defects depends on the rate of quenching

In the case of ^4He, Zurek predicted that the density of vortices will scale with the quenching time as $1/\sqrt{\tau}$

Kibble-Zurek scaling

Hand waving derivation:

Suppose that we vary the temperature T linearly in time to take it across the critical value T_c. So the time dependence is $T - T_c = t/\tau$

A defect is formed when T is close to T_c, say, when the correlation time $\xi_t \sim (T - T_c)\,\tau$

The correlation length ξ is given by $\xi_t \sim \xi^z$ and also by $\xi \sim (T - T_c)^{-\nu}$, where ν and z are called the correlation length exponent and dynamical critical exponent respectively

Hence $\xi^z \sim \xi^{-1/\nu}\,\tau$, and $\xi \sim \tau^{\nu/(z\nu+1)}$

In d dimensions, the volume associated with a point defect is $v \sim \xi^d$. Hence the defect density is given by

$$n = 1/v \sim 1/\tau^{d\nu/(z\nu+1)}$$

It turns out that the same scaling law holds for quenching across a quantum critical point at zero temperature

Quantum critical point (QCP)

A quantum system at zero temperature, i.e., in its ground state, may undergo a phase transition as some parameter γ in the Hamiltonian is varied

Suppose that $\gamma = \gamma_c$ is the QCP, and the system is ordered (disordered) for $\gamma < \gamma_c$ ($> \gamma_c$)

If $M_{\vec{r}}$ denotes the order parameter field, then

$$\langle 0|M_{\vec{r}}|0\rangle = m$$

where $m \neq 0$ ($= 0$) for $\gamma < \gamma_c$ ($> \gamma_c$)

Quantum critical point

The ground state of a quantum system may undergo a phase transition as a parameter in the Hamiltonian is varied

Example: the Ising model in a transverse field in one dimension

$$H = -\sum_n [\, \sigma_n^z \sigma_{n+1}^z + \gamma\, \sigma_n^x \,], \quad \text{where} \quad \sigma_n^a \quad \text{are Pauli matrices}$$

There is a QCP at $\gamma_c = 1$ related to the finite temperature critical point of the classical Ising model in two dimensions (solved by Onsager)

For $\gamma = 0,$ the ground state of the system has ferromagnetic order with all spins having $\sigma^z = 1$ or -1

The order parameter $\mathcal{O} = |\langle \sigma_n^z \rangle|$ goes from 1 to 0 as γ goes from 0 to $\gamma_c,$ and is 0 for $\gamma \geq \gamma_c$

For $\gamma = \infty,$ all spins have $\sigma^x = 1$ in the ground state of the system

Transverse Ising model $\cdots$

The two-spin correlation function $\langle \sigma_0^z \sigma_n^z \rangle - \langle \sigma_0^z \rangle^2$ goes to zero exponentially as $\exp(-|n|/\xi)$ as $|n| \to \infty$, where the correlation length ξ diverges near the QCP as $| \gamma - \gamma_c |^{-\nu}$

Consider the spectrum of the low-lying excitations $\omega(k)$

At the QCP $\gamma = \gamma_c$, $\omega(k)$ vanishes at some momentum k_c as $| k - k_c |^z$

Near the QCP, the gap $\Delta E = \omega(k_c)$ between the ground state and the first excited state vanishes as $\Delta E \sim | \gamma - \gamma_c |^{z\nu}$

These relations define two critical exponents ν and z
(there are other critical exponents, such as $\langle \sigma_0^z \rangle \sim |\gamma_c - \gamma|^{1/8}$, but we don't need those exponents at the moment)

For the transverse Ising model, $\omega(k) = 2\sqrt{(\gamma - 1)^2 + 4\gamma \cos^2(k/2)}$ (shown below), so that $\gamma_c = 1$, $k_c = \pi$. ω goes to zero linearly as $|\gamma - \gamma_c|$ for $k = k_c$ and as $|k - k_c|$ for $\gamma = \gamma_c$. So the critical exponents are $z = \nu = 1$

Transverse Ising model $\cdots$

$$H \;=\; -\sum_n \left[\, \sigma_n^z \sigma_{n+1}^z \;+\; \gamma\, \sigma_n^x \,\right]$$

For $\gamma \to \infty,$ the ground state is

and the lowest excited state has a spin pointing in the wrong direction

For $\gamma \to 0,$ the ground states are

and

and the lowest excited state is a domain wall (a topological defect)

Quenching in transverse Ising model

What happens if we change γ from ∞ to 0 in a time τ ?

For $\gamma \to 0,$ the ground states are

and

But due to quenching at a finite rate, the state actually observed as $\gamma \to 0$ will have some defects

Some of the defects are topological (domain walls), while others are not

Defect scaling law

How does the defect density depend on the quenching time τ ?

Consider a linear quench with $\gamma = -t/\tau$, where $-\infty < t < 0$

Main result: For the transverse Ising model, if τ is much larger than the inverse of the band width of the low-energy excitations, then the density of defects n scales as $1/\sqrt{\tau}$

Reason: When one quenches across a QCP, there are necessarily a number of low-energy modes for which the quenching is not adiabatic. These modes give rise to defects

Zurek, Dorner and Zoller, Phys. Rev. Lett. 95, 105701 (2005)
Polkovnikov and Gritsev, Nature Physics 4, 477 (2008)

Jordan-Wigner transformation

The model can be solved exactly by mapping spin-1/2's to fermions using the Jordan-Wigner transformation

Lieb, Schultz and Mattis, Ann. Phys. 16, 407 (1961)
Kogut, Rev. Mod. Phys. 51, 659 (1979)

We define

$$a_n = \left[\prod_{m=-\infty}^{n-1} \sigma_m^z\right] \sigma_n^+,$$

$$a_n^\dagger = \left[\prod_{m=-\infty}^{n-1} \sigma_m^z\right] \sigma_n^-,$$

where $\sigma_n^\pm = (1/2)\,(\sigma_n^x \pm i\sigma_n^y)$

Then $\{a_m, a_n\} = 0$ and $\{a_m, a_n^\dagger\} = \delta_{mn}$

These operators create and annihilate spinless fermions

Hamiltonian

In terms of the fermion operators, the Hamiltonian

$$H \;=\; -\sum_n \left[\, \sigma_n^x \sigma_{n+1}^x \;+\; \gamma\, \sigma_n^z \,\right]$$

becomes

$$H \;=\; -\sum_n \left[\, 2\gamma a_n^\dagger a_n + a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1} + a_n^\dagger a_{n+1}^\dagger + a_{n+1} a_n \,\right]$$

Define the Fourier transforms

$$a_k \;=\; \frac{1}{\sqrt{N}} \sum_n a_n\, e^{-ikn} \qquad \text{and} \qquad a_n \;=\; \frac{1}{\sqrt{N}} \sum_{-\pi<k<\pi} a_k\, e^{ikn}$$

where N is the number of sites. Then the Hamiltonian becomes

$$H \;=\; 2 \sum_{0<k<\pi} \left[-(\gamma + \cos k)\, (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \;-\; i \sin k\, (a_k a_{-k} - a_{-k}^\dagger a_k^\dagger) \right]$$

Energy spectrum

The Hamiltonian is

$$H \;=\; 2 \sum_{0<k<\pi} \; [-(\gamma + \cos k)\; (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \;-\; i \sin k \; (a_k a_{-k} - a_{-k}^\dagger a_k^\dagger)]$$

Thus the system decouples into non-interacting fermions with momenta $\pm\, k$

For each pair of momenta $\pm\, k,$ there are four states: $\mid \phi >$ (empty state), $\mid k >, \;\mid -k >$ (one-fermion states), and $\mid k, \; -k >$ (two-fermion state)

The one-fermions states have zero energy, while the states $\mid \phi\rangle$ and $i \mid k, \; -k\rangle$ are governed by the 2×2 Hamiltonian

$$h_k \;=\; \begin{pmatrix} -4(\gamma + \cos k) & 2 \sin k \\ 2 \sin k & 0 \end{pmatrix},$$

The ground state lies in this two-dimensional subspace, and the energy spectrum is $\omega(k) = 2\sqrt{(\gamma - 1)^2 + 4\gamma \cos^2(k/2)}$

As γ varies with time, only these two states can mix with each other

Quenching of γ

Thus, we only have to deal with a two-level system for each value of $\pm k$

Damski, Phys. Rev. Lett. 95, 035701 (2005)

For $\gamma = -t/\tau$, the Hamiltonian is given by

$$h_k = \begin{pmatrix} 2(t/\tau - \cos k) & 2\sin k \\ 2\sin k & -2(t/\tau - \cos k) \end{pmatrix},$$

plus a constant

If we start in the ground state of this system at $t = -\infty$,
which state do we reach at $t = 0$?

This is the famous Landau-Zener problem

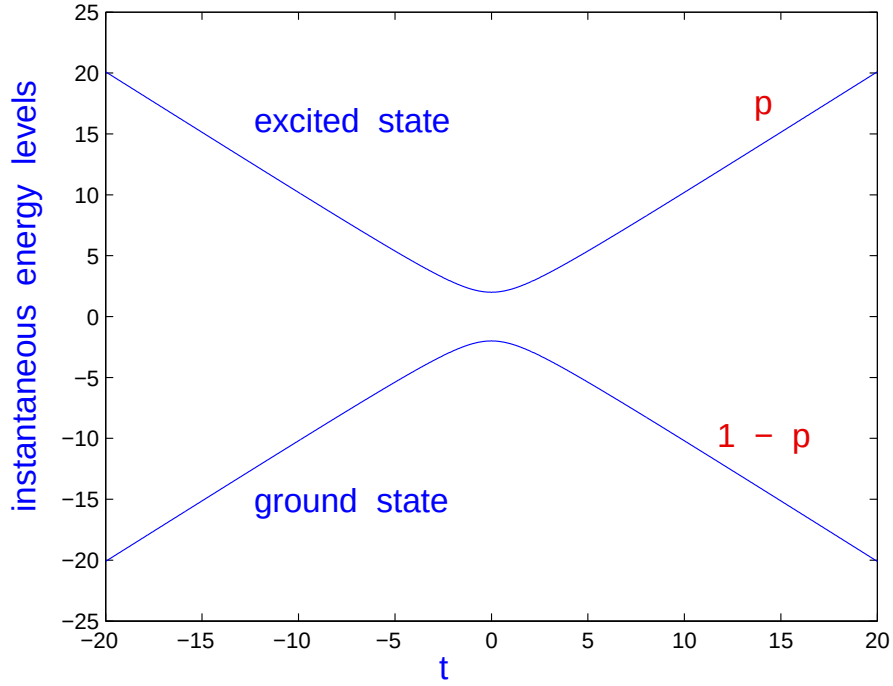
Zener, Proc. R. Soc. London Ser A 137, 696 (1932)
Landau and Lifshitz, Quantum Mechanics: Non-relativistic Theory
(Pergamon, Oxford, 1965)

Landau-Zener problem

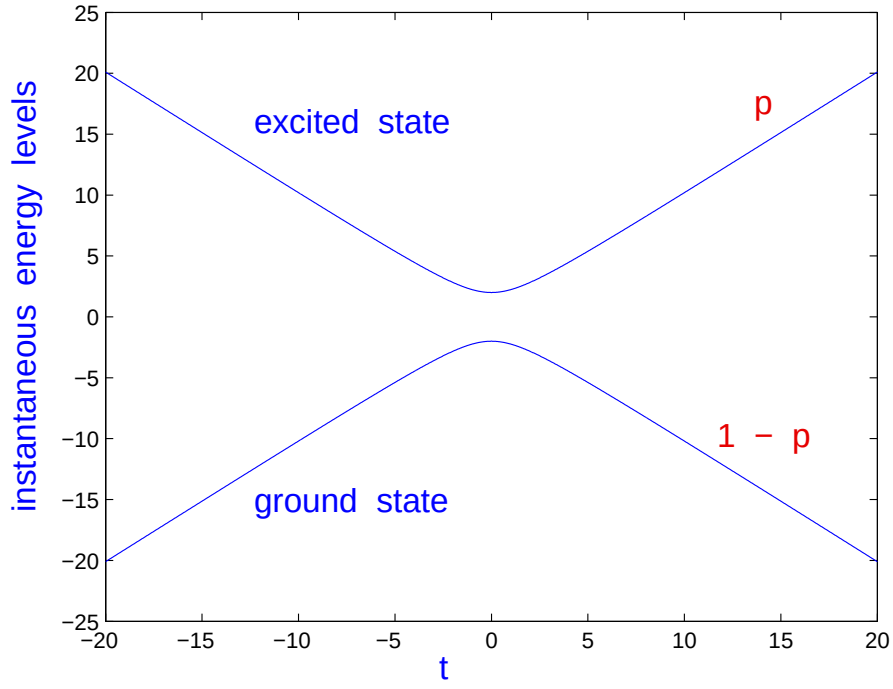
Given a two-level system with a time-dependent Hamiltonian

$$H = \begin{pmatrix} t/\tau & b \\ b & -t/\tau \end{pmatrix},$$

the instantaneous eigenvalues are given by $\pm \sqrt{b^2 + t^2/\tau^2}$



Landau-Zener transition



If we start in the ground state at $t \to -\infty,$ the probability of ending in the excited state at $t \to \infty$ is given by $p = \exp\,[\,-\pi b^2 \tau\,]$

Landau and Zener, 1932

If we do not begin or end at $t = \pm\infty,$ we get $p \sim 1/(b^4\tau^2)$

The cross-over between the two results occurs if the initial or final times are of the order of $b\tau$

Scaling argument for p

The probability of ending in the excited state is $p = \exp\,[\,-\pi b^2 \tau\,]$

Note that $p \to 0$ or 1 as $\tau \to \infty$ (adiabatic) or 0 (sudden quench)

A simple scaling argument shows that p must be a function of $b\sqrt{\tau}$.
The Schrödinger equation is

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} t/\tau & b \\ b & -t/\tau \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Multiplying throughout by $\sqrt{\tau}$ and re-defining $t' = t/\sqrt{\tau},$ we obtain

$$i\frac{\partial}{\partial t'}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} t' & b\sqrt{\tau} \\ b\sqrt{\tau} & -t' \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Hence, if we start with $\psi_1(t' = -\infty) = 1,$ then $p = |\psi_1(t' = \infty)|^2$ must be a function of the single parameter $b\sqrt{\tau}$ which must $\to 0$ or 1 as $b\sqrt{\tau} \to \infty$ or 0

Defect scaling law

Returning to the Hamiltonian for the transverse Ising model

$$H \;=\; 2 \sum_{0<k<\pi} \; [-(\gamma + \cos k)\; (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \;+\; \sin k\; (a_k a_{-k} + a_{-k}^\dagger a_k^\dagger)],$$

the total defect density is

$$n \;=\; \int_0^\pi \; \frac{dk}{2\pi}\, p_k \;=\; \int_0^\pi \; \frac{dk}{2\pi}\; \exp\; [\; -2\pi\tau \sin^2 k\;]$$

For large τ, the integral is dominated by the contributions from $k = 0$ and π

We then get $n \;\sim\; \int_0^\infty \; dk\; \exp\; [\; -2\pi\tau k^2\;] \;\sim\; \frac{1}{\sqrt{\tau}}$

The power law for n versus τ arises because the quench takes the system across a QCP where the energy vanishes at some values of k.
So no matter how slowly we quench across this point, there are low-energy modes (with energies $\lesssim 1/\sqrt{\tau}$) for which the quenching is not adiabatic

General defect scaling law

A 'hand waving' derivation for the defect scaling for quenching across a QCP for a system in d dimensions, with critical exponents ν and z :

By analogy with the two-level analysis for the transverse Ising model, we assume a Hamiltonian of the form

$$H = \begin{pmatrix} \Delta E \; sign(t) & |k|^z \\ |k|^z & -\Delta E \; sign(t) \end{pmatrix},$$

where $\Delta E \sim |\gamma - \gamma_c|^{z\nu}$, and we assume a quench of the form $\gamma - \gamma_c \sim t/\tau$

We now try a scaling argument as before. Multiplying the Schrödinger equation $i\partial\psi/\partial t = H\psi$ by $\tau^{z\nu/(z\nu+1)}$ and re-defining $t' = t/\tau^{z\nu/(z\nu+1)}$, we see that the probability of ending in the excited state is given by a function $p_k(k\tau^{\nu/(z\nu+1)})$

Hence the defect density is given by $n \sim \int_0^\infty d^d k \, p_k \sim \frac{1}{\tau^{d\nu/(z\nu+1)}}$

For $d = \nu = z = 1$, we recover $n = 1/\sqrt{\tau}$

General defect scaling law $\cdots$

For translation invariant systems, the relation

$$n \sim \frac{1}{\tau^{d\nu/(z\nu+1)}}$$

has been derived by Polkovnikov, Phys. Rev. B 72, 161201(R) (2005)
using first-order perturbation theory and scaling arguments for the dispersion of the low-lying excitations $\omega(k)$ and the derivative $\langle\, k \mid \frac{\partial}{\partial\gamma} \mid 0 \,\rangle$.
The existence of two-level systems is not necessary

The 'hand waving' argument for this is similar in spirit to the Kibble-Zurek argument given earlier: the defects are produced by a region in momentum space with volume k^d, where $(\gamma - \gamma_c)\tau \sim t \sim 1/\omega$, with $k \sim (\gamma - \gamma_c)^\nu$ and $\omega \sim k^z$.
Then we find that the density of defects $n \sim k^d \sim 1/\tau^{d\nu/(z\nu+1)}$

For $d \geq 2(z + 1/\nu)$, we get $n \sim 1/\tau^2$ due to contributions from all momenta, not just small momenta. This is because in practice $\gamma - \gamma_c$ only goes between two finite positive and negative values, not $\pm\infty$, and the excitation probability is then of the order of $1/\tau^2$

General defect scaling law $\cdots$

The Kibble-Zurek scaling relation is

$$n \sim \frac{1}{\tau^{d\nu/(z\nu+1)}}$$

We now discuss some generalizations of this scaling relation:

(i) quenching across a gapless surface in momentum space

(ii) quenching along a critical line in parameter space

(iii) non-linear quenching

Each of these is found to modify the power law

Gapless surface in momentum space

Suppose that at the point $\gamma = \gamma_c,$ the energy vanishes on a surface of $d - m$ dimensions in momentum space, rather than at an isolated point

Then the momentum integration which appears in the expression for the defect density will be over m dimensions instead of d dimensions

Hence, we will get

$$n \sim \int_0^\infty d^m k \, p_k(k\tau^{\nu/(z\nu+1)}) \sim \frac{1}{\tau^{m\nu/(z\nu+1)}}$$

There is actually a model where this happens: the Kitaev model which has $d = 2,\ m = 1,\ \nu = z = 1.$ Thus $n \sim 1/\sqrt{\tau}$ instead of $1/\tau$ as it should have been for a two-dimensional model with $\nu = z = 1$

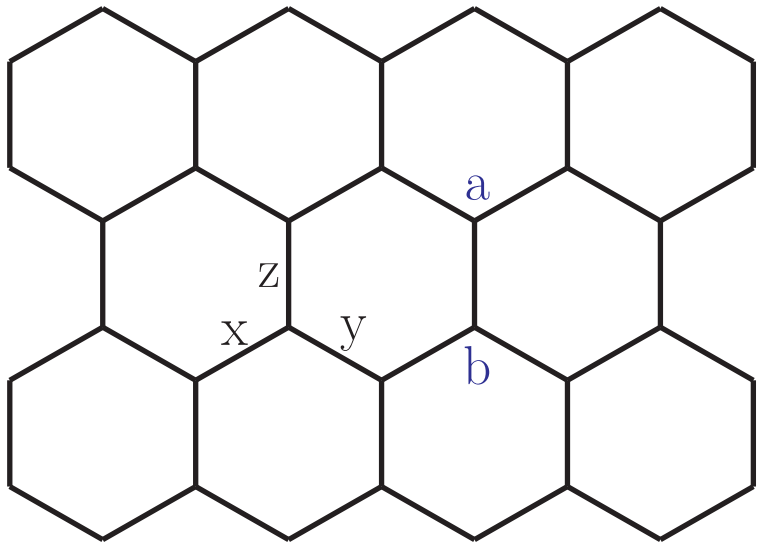
Kitaev model

This is the only known spin model in two dimensions which is exactly solvable
Kitaev, Ann. Phys. 321, 2 (2006)

The model has spin-1/2's on a honeycomb lattice, with highly anisotropic couplings between nearest neighbors. The Hamiltonian is

$$H \;=\; \sum_{j+l=\text{even}} \left(J_1\, \sigma^x_{j,l}\sigma^x_{j+1,l} \;+\; J_2\, \sigma^y_{j-1,l}\sigma^y_{j,l} \;+\; J_3\, \sigma^z_{j,l}\sigma^z_{j,l+1}\right)$$

Can assume that all couplings $J_i \;\geq\; 0$



Jordan-Wigner transformation

The Kitaev model can be solved exactly by mapping it to Majorana fermions by a Jordan-Wigner transformation, even though it is a model in two dimensions

$$a_{\vec{n}} = \left[\prod_{\vec{m}=-\infty}^{\vec{n}-1} \sigma^z_{\vec{m}} \right] \sigma^y_{\vec{n}} \, (\sigma^x_{\vec{n}}) \text{ for even (odd) numbered chains}$$

$$b_{\vec{n}} = \left[\prod_{\vec{m}=-\infty}^{\vec{n}-1} \sigma^z_{\vec{m}} \right] \sigma^x_{\vec{n}} \, (\sigma^y_{\vec{n}}) \text{ for even (odd) numbered chains}$$

depending on whether $\vec{n}$ lies on the A or B sub-lattice

These operators satisfy the anticommutation relations

$$\{ a_{\vec{m}} \, , \, a_{\vec{n}} \} = \{ b_{\vec{m}} \, , \, b_{\vec{n}} \} = 2 \, \delta_{\vec{m}\vec{n}}, \quad \text{and} \quad \{ a_{\vec{m}} \, , \, b_{\vec{n}} \} = 0$$

These are Majorana operators since they are Hermitian

The string of $\sigma^z_{\vec{m}}$'s is chosen to go along the x and y bonds, towards the right (left) on even (odd) numbered chains

Jordan-Wigner transformation

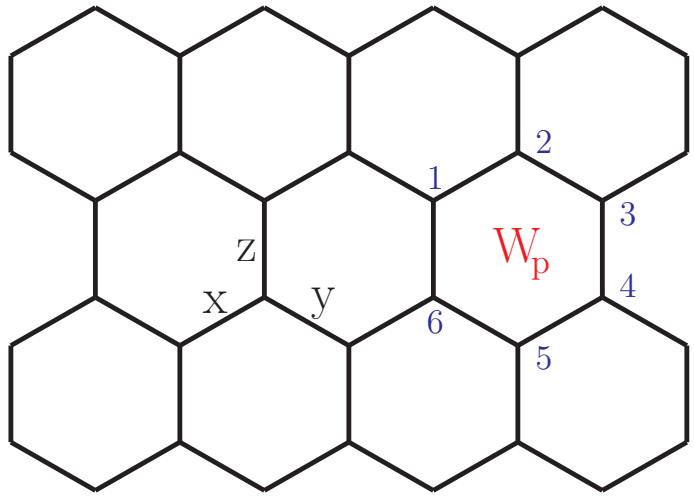
$$H = \sum_{j+l=\text{even}} (J_1 \, \sigma^x_{j,l}\sigma^x_{j+1,l} + J_2 \, \sigma^y_{j-1,l}\sigma^y_{j,l} + J_3 \, \sigma^z_{j,l}\sigma^z_{j,l+1})$$

The xx and yy interactions become local and quadratic in the Majorana fermion operators under the Jordan-Wigner transformation

The zz interaction would normally become non-local and quartic in the fermion operators

But in this model, this remains local and only couples fermions on nearest neighbor sites due to a large number of conserved quantities

Conserved quantities



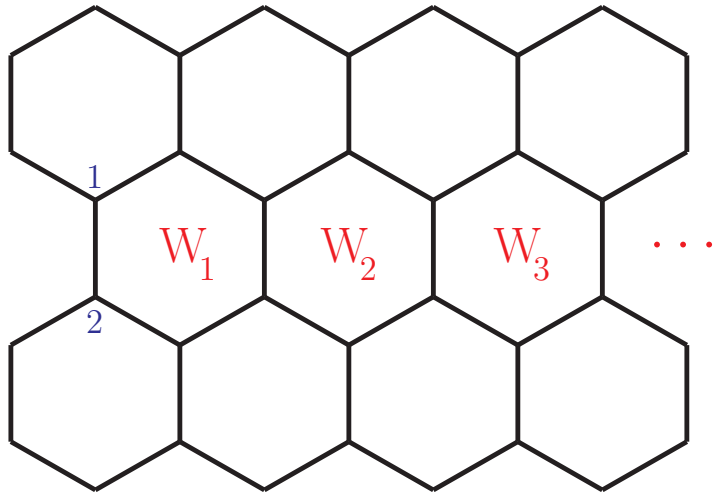
The model has a conserved quantity W associated with each hexagon:

$$W = \sigma_1^y \, \sigma_2^z \, \sigma_3^x \, \sigma_4^y \, \sigma_5^z \, \sigma_6^x$$

Hence there are $2^{N/2}$ decoupled sectors corresponding to the values of $W = \pm 1$ in the $N/2$ different hexagons (the number of sites is N)

Because of these conserved quantities, the zz interactions become local in terms of the Majorana fermions

Kitaev model $\cdots$



If $1/2$ lies on an even/odd numbered chain, then $\sigma_1^z \sigma_2^z \sim a_1\, b_1\, W_1\, W_2\, W_3\, \cdots$

In any particular sector with some given values of W_i, the zz interaction reduces to a product of two fermion operators. The ground state turns out to lie in a sector in which all the $W_i = 1$. In that sector, we find that

$$H = i \sum_{\vec{n}} [\, J_1\, b_{\vec{n}}\, a_{\vec{n}-\vec{M}_1} + J_2\, b_{\vec{n}}\, a_{\vec{n}+\vec{M}_2} + J_3\, b_{\vec{n}}\, a_{\vec{n}}\,] \,,$$

where $\vec{M}_1 = \frac{\sqrt{3}}{2}\hat{i} + \frac{3}{2}\hat{j}$ and $\vec{M}_2 = \frac{\sqrt{3}}{2}\hat{i} - \frac{3}{2}\hat{j}$

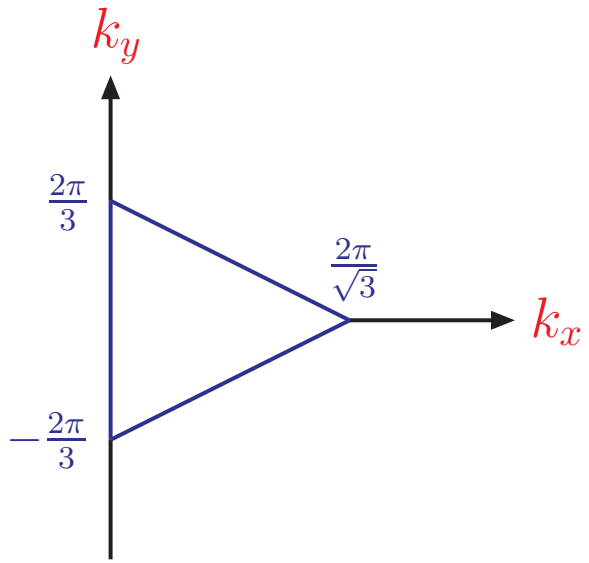
Brillouin zone

Define the Fourier transforms

$$a_{\vec{n}} = \sqrt{\frac{4}{N}} \sum_{\vec{k}} [\, a_{\vec{k}}\, e^{i\vec{k}\cdot\vec{n}} + a^{\dagger}_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{n}}\,],$$

$$b_{\vec{n}} = \sqrt{\frac{4}{N}} \sum_{\vec{k}} [\, b_{\vec{k}}\, e^{i\vec{k}\cdot\vec{n}} + b^{\dagger}_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{n}}\,],$$

where $\vec{k}$ runs over only half the Brillouin zone (due to their Majorana mature) which looks as follows:



Hamiltonian

The Hamiltonian of the Kitaev model is

$$
\begin{aligned}
H &= \sum_{\vec{k}} \left(a^{\dagger}_{\vec{k}} \;\; b^{\dagger}_{\vec{k}} \right) H_{\vec{k}} \begin{pmatrix} a_{\vec{k}} \\ b_{\vec{k}} \end{pmatrix}, \\
H_{\vec{k}} &= 2\left[J_3 + J_1 \cos(\vec{k}\cdot\vec{M}_1) + J_2 \cos(\vec{k}\cdot\vec{M}_2)\right]\sigma^2 \\
&\quad + 2\left[J_1 \sin(\vec{k}\cdot\vec{M}_1) - J_2 \sin(\vec{k}\cdot\vec{M}_2)\right]\sigma^1,
\end{aligned}
$$

where $\vec{M}_1 = \frac{\sqrt{3}}{2}\hat{i} + \frac{3}{2}\hat{j}$ and $\vec{M}_2 = \frac{\sqrt{3}}{2}\hat{i} - \frac{3}{2}\hat{j}$

This is a system of non-interacting Majorana fermions

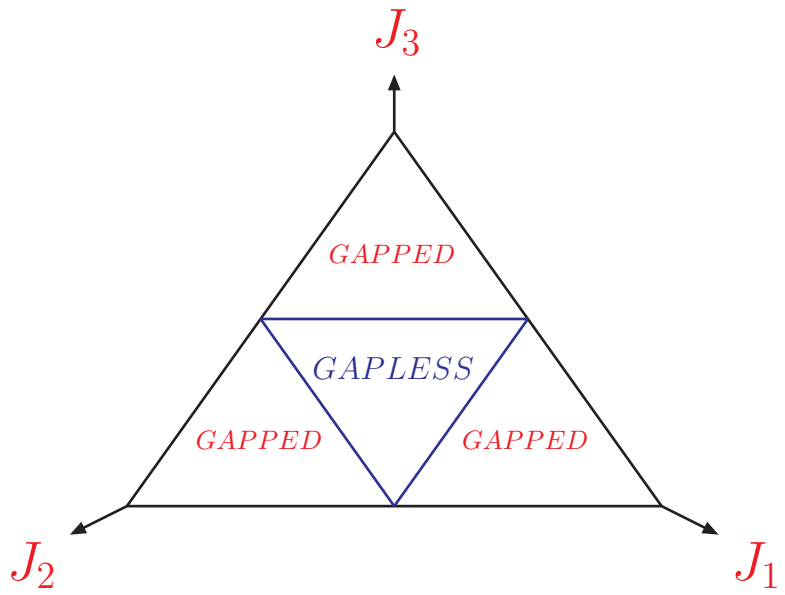
Depending on the values of $J_1,\ J_2,\ J_3,$ there may or may not be a gap between the ground state and the first excited state

Phase diagram of Kitaev model

If $J_1 < J_2 + J_3, \quad J_2 < J_3 + J_1$ and $J_3 < J_1 + J_2,$ the system is gapless along some lines in half the Brillouin zone

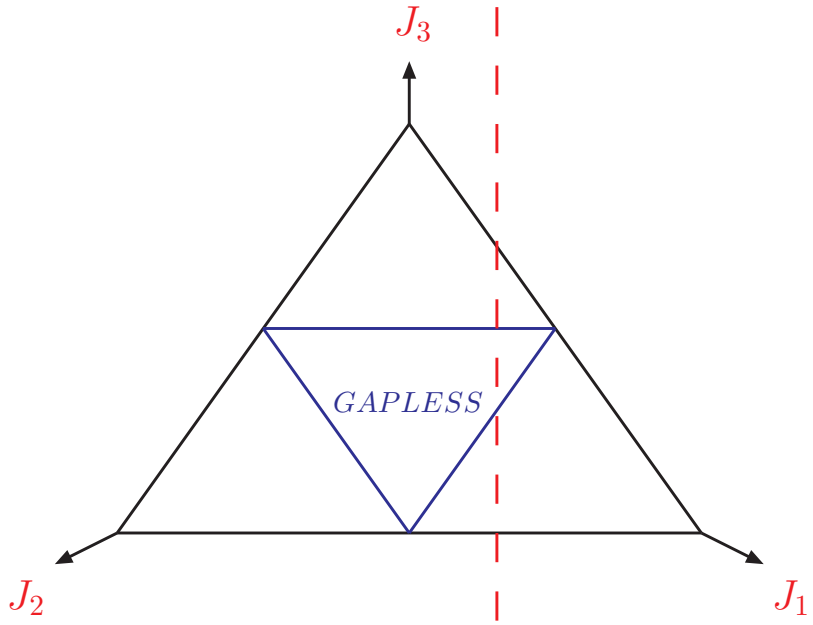
For all other values of $(J_1,\ J_2,\ J_3),$ the system is gapped

The phase diagram can be shown in terms of points in an equilateral triangle satisfying $J_1 + J_2 + J_3 = 1$ (the value of J_i is the distance from the opposite side)



Quenching in the Kitaev model

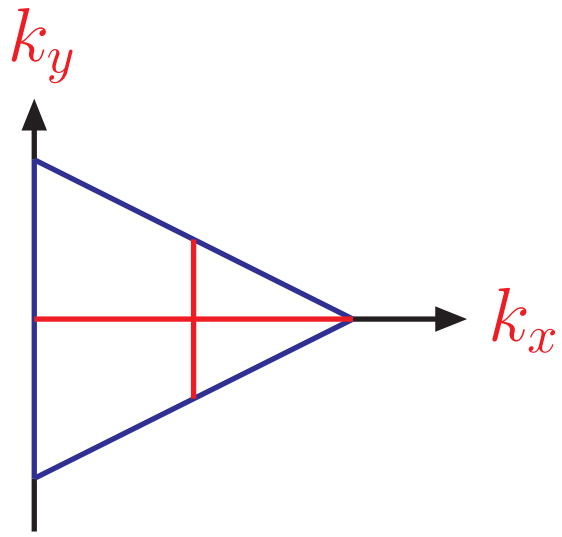
Let us hold $J_1,\ J_2$ fixed, and vary J_3 in time as $Jt/\tau,$ from $t = -\infty$ to $t = \infty$ (as shown by the red dotted line). Then the system will pass through the gapless region for some time



Sengupta, Sen and Mondal, Phys. Rev. Lett. 100, 077204 (2008)
Mondal, Sen and Sengupta, Phys. Rev. B 78, 045101 (2008)

Scaling of defect density

In the gapless region, the energy of the low-lying excitations typically vanishes on some lines in half the Brillouin zone as indicated in red below



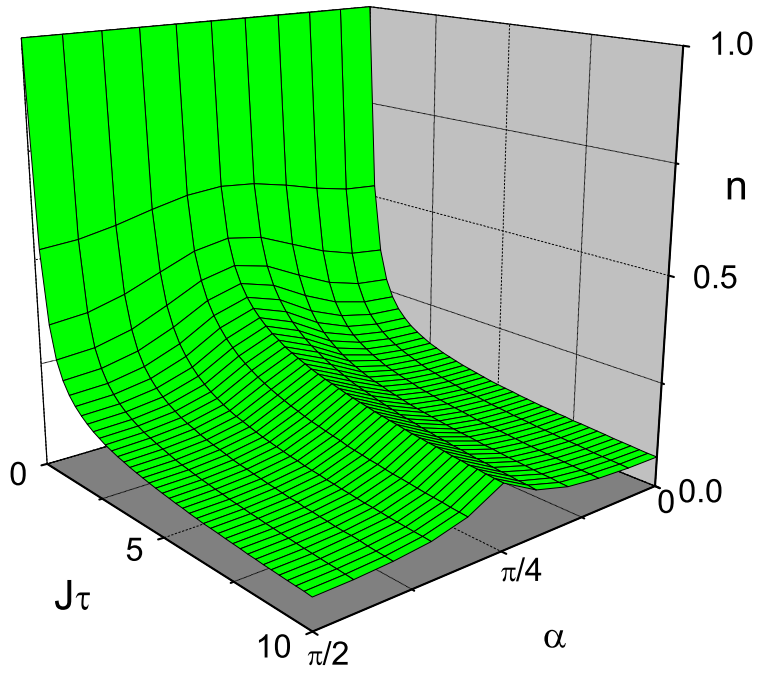
Thus the Kitaev model has $d = 2$ but $m = 1$. Also, $\nu = z = 1$

Hence the defect density scales as $n \sim 1/\sqrt{\tau}$ instead of $1/\tau$

$$n = \frac{3\sqrt{3}}{4\pi^2} \int\int d^2\vec{k}\ p_{\vec{k}}$$

$$p_{\vec{k}} = e^{-2\pi\tau\ [J_1 \sin(\vec{k}\cdot\vec{M}_1) - J_2 \sin(\vec{k}\cdot\vec{M}_2)]^2/J}$$

Plot of defect density



n versus $J\tau$ and $\alpha = \tan^{-1}(J_2/J_1)$

The defect density is maximum when $\alpha = \pi/4,$ i.e., when $J_1 = J_2$ because this is when the system stays in the gapless phase for the longest time

Quenching along a critical line

A different situation arises if one quenches along a critical line in parameter space. In terms of a two-level system, suppose that the Hamiltonian for the modes with momenta $\pm k$ is

$$H = \begin{pmatrix} |k|^a \, t/\tau & |k|^z \\ |k|^z & -|k|^a \, t/\tau \end{pmatrix}$$

A scaling argument then shows that the defect density scales as $n \sim \frac{1}{\tau^{d/(2z-a)}}$ for a system in d dimensions

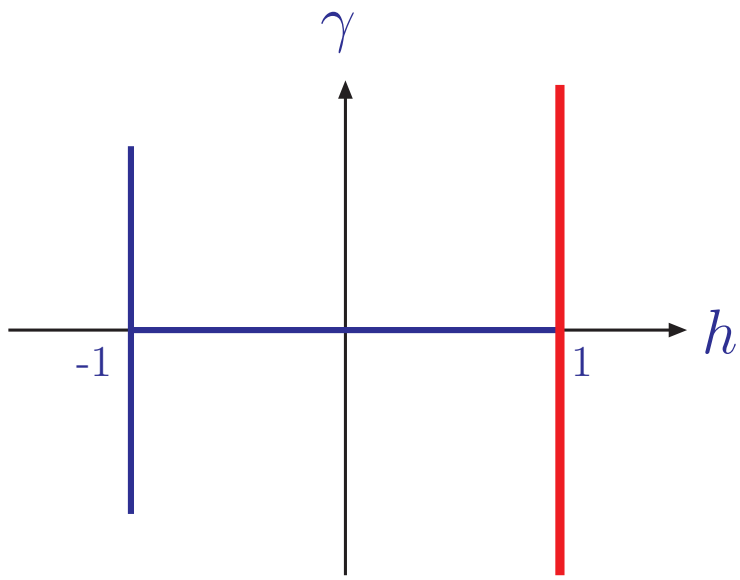
Example: the spin-1/2 XY chain with a transverse magnetic field. The Hamiltonian is

$$H = -\sum_n [\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \gamma(\sigma_n^x \sigma_{n+1}^x - \sigma_n^y \sigma_{n+1}^y) + h\sigma_n^z]$$

Mukherjee, Divakaran, Dutta and Sen, Phys. Rev. B 76, 174303 (2007)
Divakaran, Dutta and Sen, Phys. Rev. B 78, 144301 (2008)

Quenching along a critical line $\cdots$

$$H = -\sum_n [\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \gamma(\sigma_n^x \sigma_{n+1}^x - \sigma_n^y \sigma_{n+1}^y) + h\sigma_n^z]$$

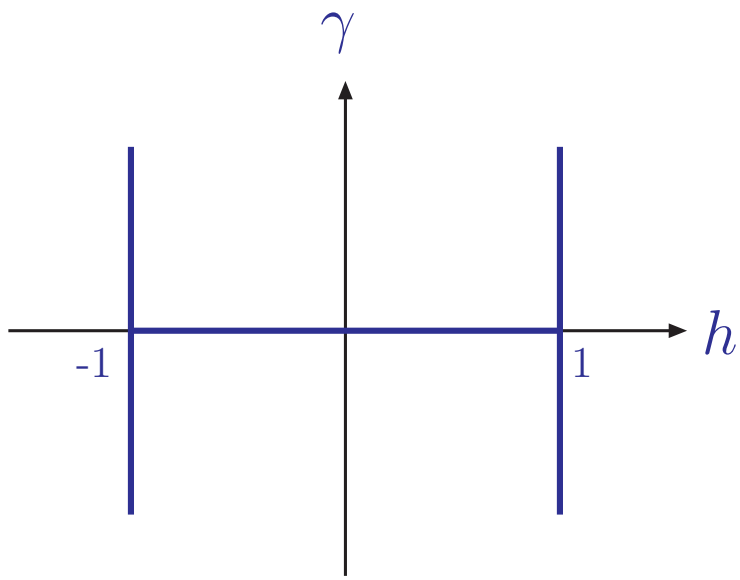


The critical lines are given by $h = -1, \quad h = 1$ and $-1 \leq h \leq 1, \ \gamma = 0$

If we quench along the red line $h = 1,$ we get $d = 1, \ z = 2, \ a = 1$

Hence the defect density scales as $n \sim 1/\tau^{d/(2z-a)} \sim 1/\tau^{1/3}$

Different quenching possibilities



Quenching along one of the blue vertical lines $h = \pm 1$ gives $n \sim 1/\tau^{1/3}$

The quenching procedure discussed earlier was to keep γ fixed at a non-zero value and cross one of the lines $h = \pm 1$. This gives $n \sim 1/\tau^{1/2}$

Finally, quenching through one of the multicritical points at $h = \pm 1,\ \gamma = 0$ gives $d = 1,\ z = 3,\ a = 0$. Hence $n \sim 1/\tau^{d/(2z-a)} \sim 1/\tau^{1/6}$

Divakaran, Mukherjee, Dutta and Sen, J. Stat. Mech. (2009) P02007

Anisotropic critical point

It may be possible to engineer systems in optical lattices where the dispersion at a QCP is anisotropic in momentum space. For instance, a semi-Dirac system in two dimensions would have the Hamiltonian

$$H_k = \begin{pmatrix} \mu & iv_F|k_y| + k_x^2/(2m) \\ -iv_F|k_y| + k_x^2/(2m) & -\mu \end{pmatrix}$$

If the chemical potential μ is slowly quenched through zero as t/τ, the defect density will be given by

$$n \sim \int\int dk_x dk_y \exp\left[-\pi\tau\left(v_F^2 k_y^2 + \frac{k_x^4}{(2m)^2}\right)\right] \sim \frac{1}{\tau^{3/4}}$$

Dutta, Singh and Divakaran, EPL 89, 67001 (2010)
Hikichi, Suzuki and Sengupta, Phys. Rev. B 82, 174305 (2010)

Non-linear quenching

We can change the quenching parameter γ through a QCP in a non-linear way. A 'hand waving' way of doing this is to take

$$H = \begin{pmatrix} \Delta E \; sign(t) & |k|^z \\ |k|^z & -\Delta E \; sign(t) \end{pmatrix},$$

where $\Delta E \sim |\gamma - \gamma_c|^{z\nu}$, and we set $|\gamma - \gamma_c| = |t/\tau|^{\alpha}$

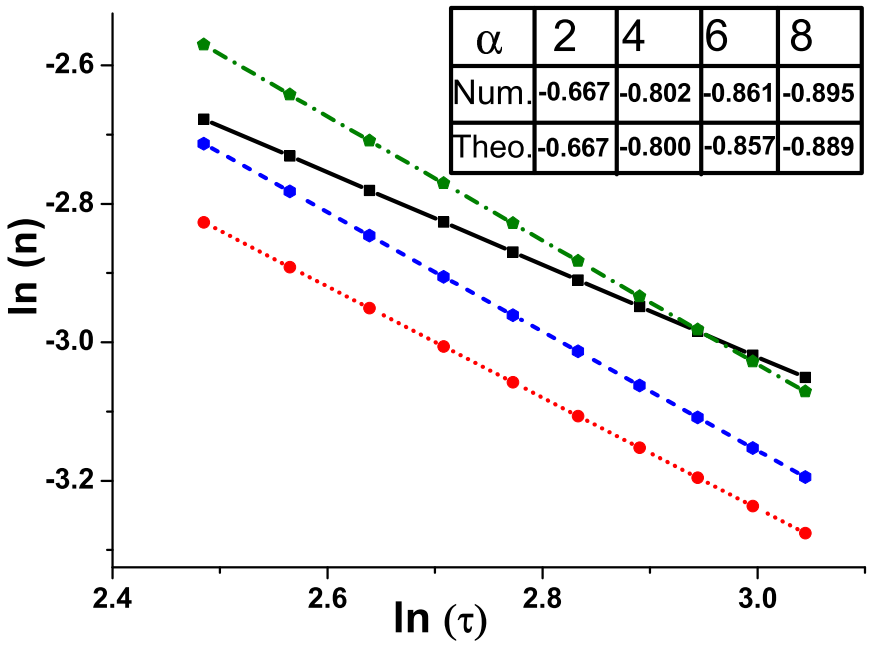
Then a scaling argument will show, for a system in d dimensions, that the defect density goes as $n \sim \frac{1}{\tau^{d\nu\alpha/(z\nu\alpha+1)}}$

The power law is modified as if we have linear quenching but $\nu \to \nu\alpha$

For $d = \nu = z = 1$, we obtain $n \sim \frac{1}{\tau^{\alpha/(\alpha+1)}}$

Sen, Sengupta and Mondal, Phys. Rev. Lett. 101, 016806 (2008)
Mondal, Sengupta and Sen, Phys. Rev. B 79, 045128 (2009)

Non-linear quenching $\cdots$



α	2	4	6	8
Num.	-0.667	-0.802	-0.861	-0.895
Theo.	-0.667	-0.800	-0.857	-0.889

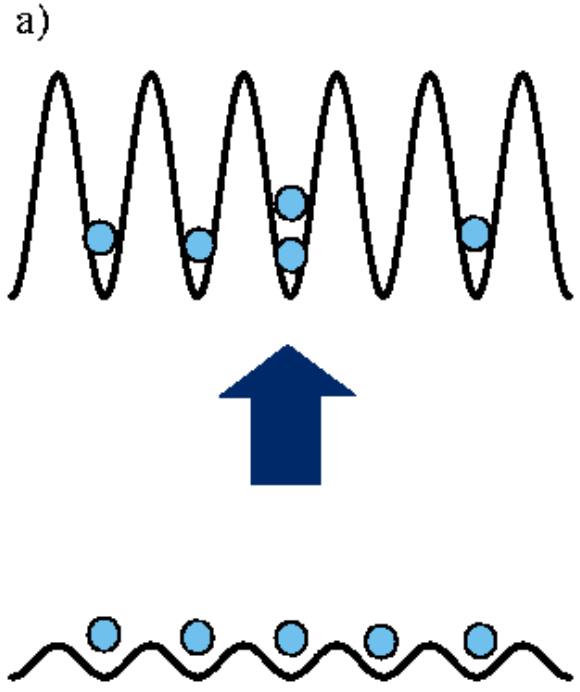
Plots of $\ln(n)$ versus $\ln(\tau)$ for the one-dimensional Kitaev model (with $J_1 + J_2$ held fixed and $J_1 - J_2$ varied as $|t/\tau|^{\alpha} sign(t)$) for $\alpha = 2$ (black line), $\alpha = 4$ (red), $\alpha = 6$ (blue) and $\alpha = 8$ (green)

The slopes of these lines agree reasonably with the theoretical values of $-\alpha/(\alpha + 1)$ as shown in the table

Quenching in a Luttinger liquid

The loading of interacting bosons onto a one-dimensional optical lattice provides an example of quenching in a Luttinger liquid

Suppose that the periodic potential of the optical lattice is changed in time as $V(x,t) = V(t)\ \cos(2\pi x/a),$ where the lattice spacing a is commensurate with the bosonic density. The potential couples right and left moving particles and gives rise to a 'mass'



Luttinger liquid $\cdots$

In terms of the bosonic field variable $\phi,$ the action is of the sine-Gordon form

$$S = \frac{1}{2} \int \int dx dt \left[\frac{1}{v} \left(\frac{\partial \phi}{\partial t} \right)^2 - v \left(\frac{\partial \phi}{\partial x} \right)^2 + V(t) \cos(2\sqrt{\pi K}\phi) \right]$$

where the Luttinger parameter K is governed by the interaction strength. $K \to 1$ for strongly repulsive interactions, while $K \to \infty$ for weakly repulsive interactions

The 'cosine' term has scaling dimension K since the equal-time correlation function

$$\langle \cos(2\sqrt{\pi K}\phi(x,0)) \cos(2\sqrt{\pi K}\phi(0,0)) \rangle \sim \frac{1}{|x|^{2K}}$$

The coefficient of the cosine term, $V(t),$ has scaling dimension $2 - K$ since the action must be dimensionless and x and t have scaling dimension -1 each. So if the cosine term gives rise to a mass gap $m,$ we must have $V(t) \sim m^{2-K}.$ The corresponding correlation length must scale as $\xi \sim 1/m \sim V(t)^{-1/(2-K)}.$ Hence the correlation length exponent is $\nu = 1/(2-K)$

Quenching in a Luttinger liquid

If the periodic potential is changed slowly as $V(t) = \alpha t,$ the Kibble-Zurek expression implies, with $d = z = 1,$ that the number of excitations must scale as

$$n \sim \alpha^{d\nu/(z\nu+1)} \sim \alpha^{1/(3-K)}$$

for $0 < K < 2$

$K = 2$ $(\nu = \infty)$ is the Kosterlitz-Thouless point

For $K > 2,$ the cosine term is irrelevant, and the excitations receive contributions from all modes, not just the low-momentum modes.
Then one finds that $n \sim \alpha$

De Grandi, Barankov and Polkovnikov, Phys. Rev. Lett. 101, 230402 (2008)

Ising chain in a general magnetic field

The defect density or residual energy scales as $1/\tau^{d\nu/(z\nu+1)}$. For the Ising chain at the QCP, i.e,

$$H = -\sum_n [\, \sigma_n^z \sigma_{n+1}^z + \sigma_n^x \,]$$

the dynamical critical exponent $z = 1$ because it is described by a 'Lorentz invariant' theory in which the frequency and wave number scaling in the same way

But the correlation length exponent ν depends on how the theory is perturbed from the QCP

So far we have only considered perturbation by a transverse field $\delta H = \gamma_x \sum_n \sigma_n^x$ for which $\nu = 1$. But if we perturb it by a longitudinal field $\delta H = \gamma_z \sum_n \sigma_n^z$, then $\nu = 8/15$

This can be seen as follows. Since the longitudinal magnetization vanishes as $\langle \sigma_n^z \rangle \sim \gamma_x^{1/8}$, the scaling dimension of σ_n^z is $1/8$. Hence the scaling dimension of γ_z must be $15/8$, so that the change in the action $\delta S = \int\int dx dt\, \gamma_z\, \sigma^z(x,t)$ is dimensionless

If ξ_z is the correlation length corresponding to the longitudinal perturbation, we must have $\gamma_z \sim 1/\xi_z^{15/8}$, or $\xi_z \sim \gamma_z^{-8/15}$. Hence $\nu = 8/15$

Ising chain in a magnetic field $\cdots$

The defect density or residual energy scales as $1/\tau^{d\nu/(z\nu+1)}$

If we quench through the QCP in a general direction with both γ_x and γ_z going through zero, the defect density will have terms scaling as $1/\tau^{1/2}$ and $1/\tau^{8/23}$ respectively. The latter one dominates for large τ

So if we consider the Hamiltonian

$$H = -\sum_n [\, \sigma_n^z \sigma_{n+1}^z + \sigma_n^x + g\,(\cos\phi\, \sigma_n^x + \sin\phi\, \sigma_n^z)\,]$$

and quench g through zero, the defect density will scale as $1/\tau^{1/2}$ if $\phi = 0,\ \pi$ and as $1/\tau^{8/23}$ for any other value of ϕ

Pollmann, Mukerjee, Green and Moore, Phys. Rev. E 81, 020101(R) (2010)

Entanglement entropy

If a system consists of two parts A and B and is in a state $|\Psi\rangle$,
the entanglement entropy between the two parts is defined to be

$$E(\Psi) \;=\; -\; tr(\rho_A\; \log_2 \rho_A)$$

where ρ_A is the density matrix of part A when part B is integrated out.
(Interchanging A and B gives the same value for $E(\Psi)$)

Namely, if

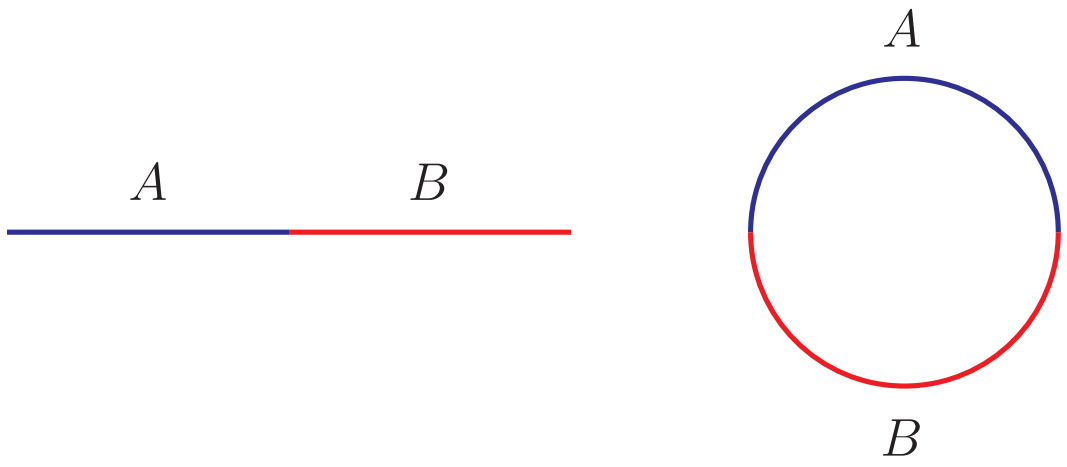
$$|\Psi\rangle \;=\; \sum_{ij}\; c_{ij}\; |A_i\rangle \;\otimes\; |B_j\rangle$$

then $(\rho_A)_{ii'} \;=\; \sum_j\; c_{ij}\; c^*_{i'j}$, and $E(\Psi) \;=\; -\; \sum_k\; \lambda_k\; \log_2 \lambda_k$ where
λ_k are the eigenvalues of ρ_A

Entanglement entropy $\cdots$

For a one-dimensional system close to a QCP which is described by a conformal field theory, the entanglement entropy between the two halves of an infinitely long system is given by $(c/6)\ \log_2 \xi$ and $(c/3)\ \log_2 \xi$ for open and periodic boundary conditions respectively, where c is the central charge of the conformal field theory and ξ is the correlation length

The factor of 2 between the two expressions can be understood from the picture

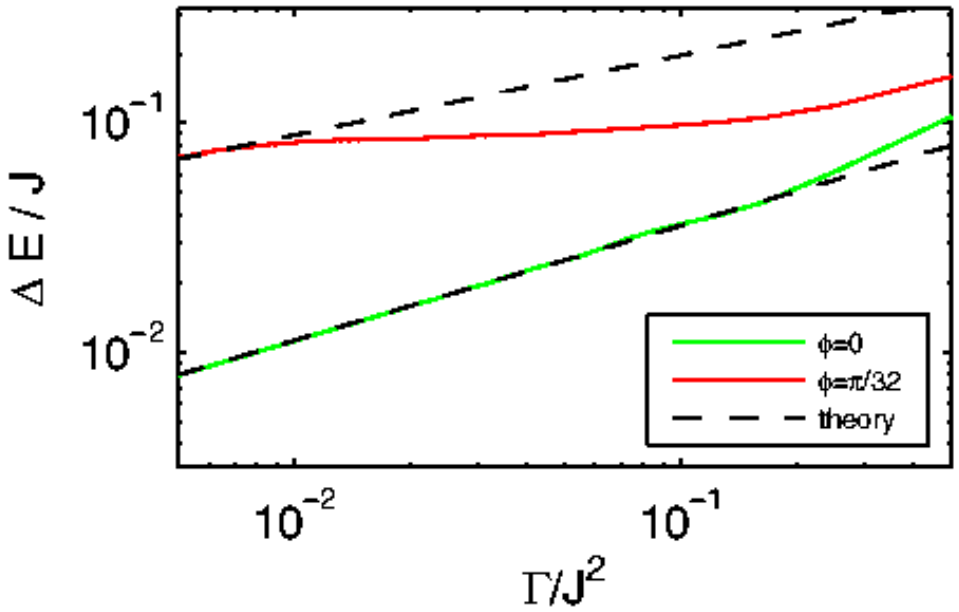


Vidal, Latorre, Rico and Kitaev, Phys. Rev. Lett. 90, 227902 (2003)

Entanglement entropy due to quench

If the system is quenched through a QCP, an entanglement entropy is generated which is given by the above expression, where $\xi \sim 1/n \sim \tau^{\nu/(\nu+1)}$ (we have set $d = z = 1$). Hence, for an open chain, $\Delta E_{quench} = \frac{c\,\nu}{6\,(\nu+1)} \log_2 \tau$

For the transverse Ising chain in a magnetic field, we get different behaviors depending on whether $\nu = 1$ or $8/15$, i.e., whether $\phi = 0$ or $\neq 0$ ($c = 1/2$ for this system)



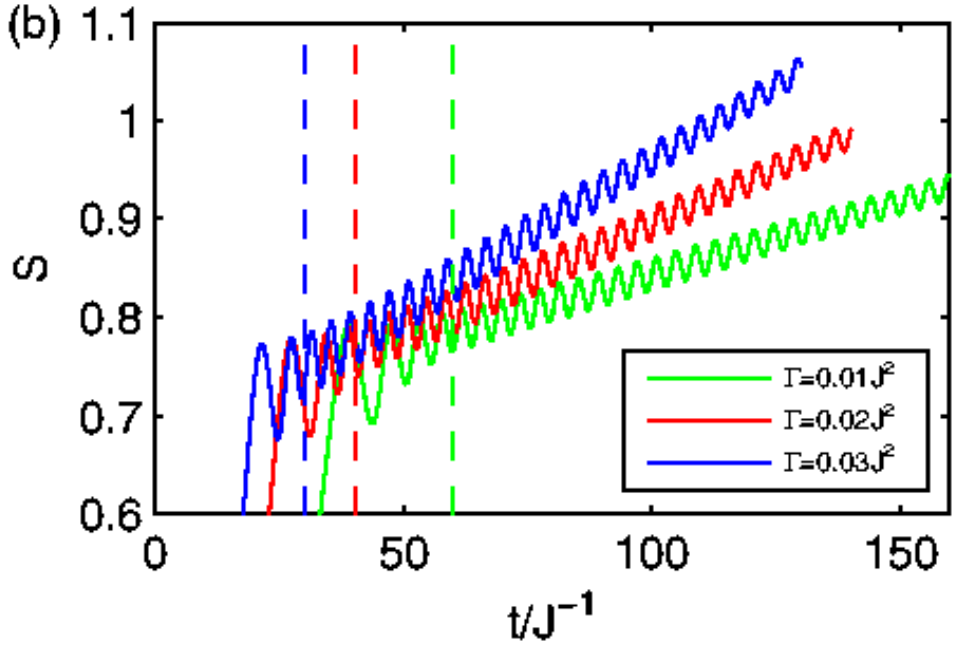
Entropy versus $\Gamma = 1/\tau$ for quenching from $g_i = 0.5$ to $g_f = -0.5$

Pollmann et al., Phys. Rev. E 81, 020101(R) (2010)

Evolution of entropy with time

If the Hamiltonian is held fixed after the quench, the entanglement entropy grows linearly in time for the following reason. Just after the quench, a number of defects are produced; pairs of quasiparticle excitations (defects) which are produced close to each other are entangled. As these excitations move away from each other with constant velocity in opposite directions, the entropy increases linearly with the size of the entangled region

Calabrese and Cardy, Phys. Rev. Lett. 96, 136801 (2006)



Quenching from $g_i = 0.4$ to $g_f = -0.4$ with $\phi = 0$

Pollmann et al., Phys. Rev. E 81, 020101(R) (2010)

Loschmidt echo

The oscillations seen in the entanglement entropy as a function of time can be understood thus. Consider quenching an integrable system which can be written as a product of two-level systems parameterized by the wave number k. In one of these sub-systems, let the probabilities of being in the ground and excited states be $1 - p_k$ and p_k. Now these states evolve with the final Hamiltonian; if their energy difference is ΔE_k, the overlap between the state just after the quench and after an additional time t is

$$\langle \psi_k(0) | \psi_k(t) \rangle = \begin{pmatrix} \sqrt{p_k} & \sqrt{1-p_k} \end{pmatrix} \begin{pmatrix} \sqrt{p_k} \\ \sqrt{1-p_k}\, e^{-i\Delta E_k t} \end{pmatrix}$$

The square of the overlap between the state of the entire system just after the quench and after a time t is then given by

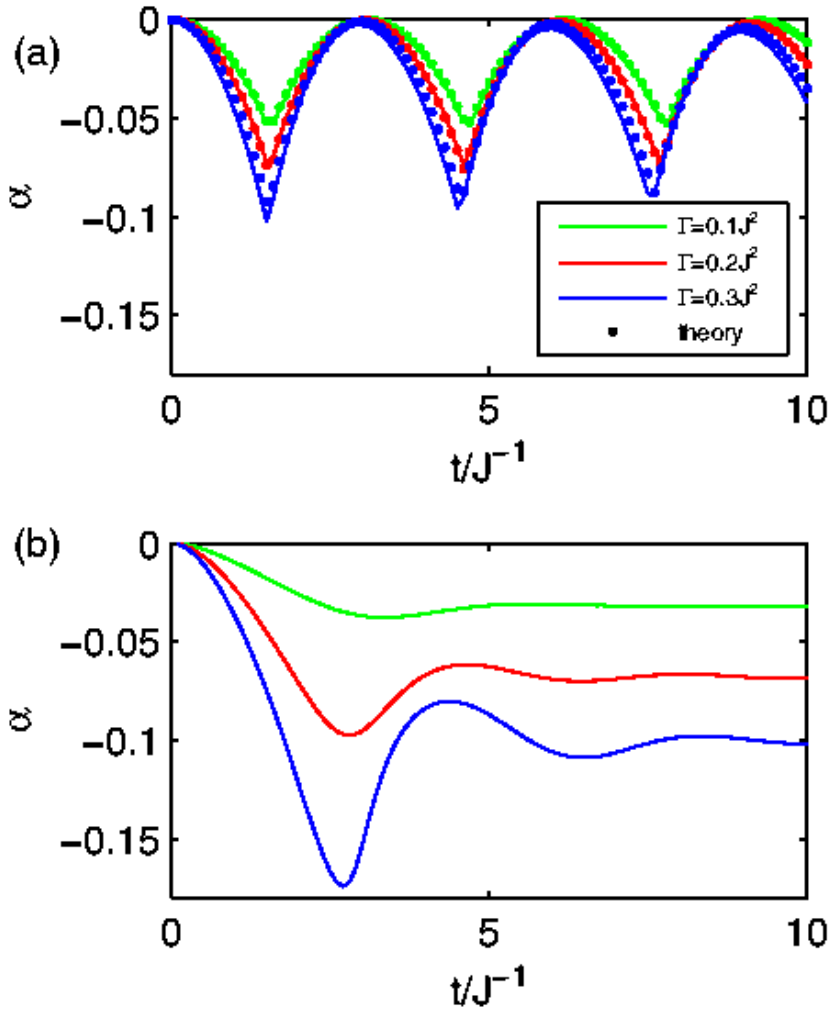
$$\begin{aligned} |\langle \Psi(0) | \Psi(t) \rangle|^2 &= \otimes |\langle \psi_k(0) | \psi_k(t) \rangle|^2 \\ &= \exp\left[-L \int \frac{dk}{2\pi} \log\left[1 - 4 p_k (1-p_k) \sin^2(\Delta E_k t/2) \right] \right] \end{aligned}$$

where L is the system size. The mode for which $p_k = 1/2$ gives rise to cusps in the above expression whenever $\sin^2(\Delta E_k t/2 = 1$

This phenomenon only occurs if the system decomposes into a product of two-level systems. It does not occur if the system is non-integrable

Loschmidt echo $\cdots$

The overlap squared $= e^{\alpha L}$ versus of time for an integrable system $(\phi = 0)$ and a non-integrable system $(\phi = \pi/32)$ after quenching from $g_i = 0.5$ to $g_f = -0.5$



Pollmann et al., Phys. Rev. E 81, 020101(R) (2010)

Thermalization

In many cases it is found that a system evolves into an effectively thermal state following a rapid quench. For instance, expectation values of operators are be given by a sum over states n weighted by the Boltzmann factor $e^{-\beta E_n}$ with some effective inverse temperature β

Rigol, Dunjko and Olshanii, Nature 452, 854 (2008)

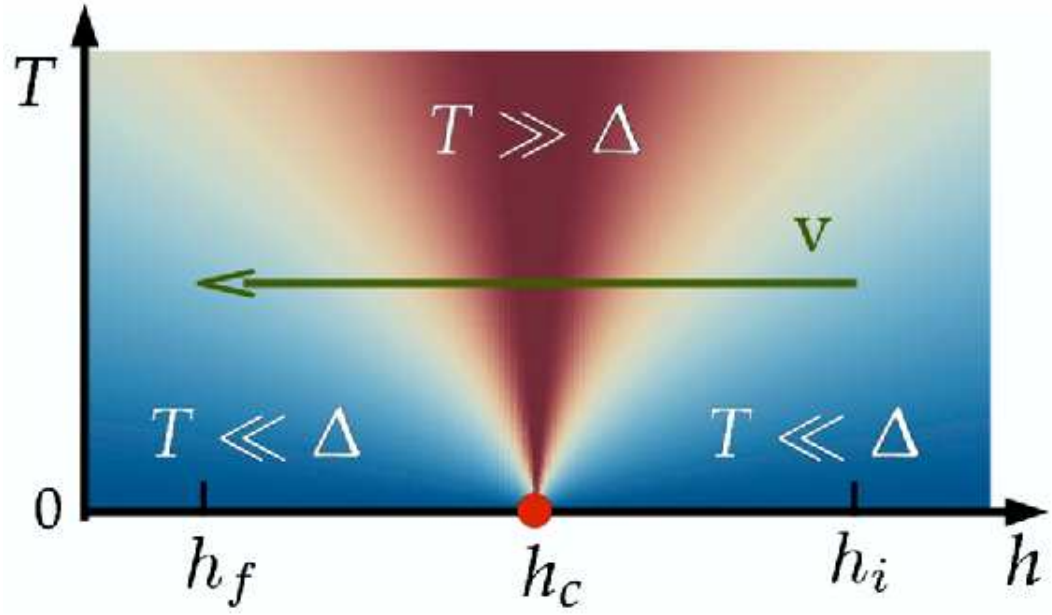
This gets modified if the system is integrable, i.e., has a number of conserved quantities C_1, C_2, $\cdots$. Then the weights of the different states must include a factor which takes into account the values of the conserved quantities, i.e., $e^{-\beta E_n - \lambda_1 C_{1n} - \lambda_2 C_{2n} - \cdots}$, where λ_i is a Lagrange multiplier which ensures that the expectation value of C_i is equal to the value that it had initially

Rigol, Dunjko, Yurovsky and Olshanii, Phys. Rev. Lett. 98, 050405 (2007)

In an integrable model of interacting fermions in one dimension (Luttinger model), the final distribution is given by a momentum-dependent temperature

Iucci and Cazalilla, Phys. Rev. A 80, 063619 (2009)

Effect of finite temperature



Patane, Silva, Amico, Fazio and Santoro, Phys. Rev. Lett. 101, 175701 (2008)

At finite temperatures, the effect of the QCP remains visible in a cone shaped region whose width grows as $T \sim \Delta E \sim |h - h_c|^{z\nu}$

In this region, defects are produced thermally at a rate T^θ, where θ is an exponent which depends on the form of the thermal bath. If we cross this region with a velocity v, the time spent there is $\sim |h - h_c|/v \sim T^{1/(z\nu)}/v$, and the density of defects formed from each initial state is $T^{\theta+1/(z\nu)}/v$

Effect of finite temperature

The total density of defects formed at finite temperature is therefore given by $(\int d^d k)\, T^{\theta+1/(z\nu)}/v$.
The region of k which is excited is of order $T^{1/z}$. Hence $\int d^d k \sim T^{d/z}$

Thus the density of thermally produced defects is of order $T^{\theta+1/(z\nu)+d/z}/v$

This is to be compared to the density of defects produced by quantum effects at zero temperature given by $1/\tau^{d\nu/(z\nu+1)} \sim v^{d\nu/(z\nu+1)}$, since $v \sim 1/\tau$

The relative magnitude of the two gives the cross-over velocity

$$v_c \sim T^{(\theta+1/(z\nu)+d/z)(z\nu+1)/(d\nu+z\nu+1)}$$

For $v \ll v_c$, the defect production is dominated by thermal effects, while for $v \gg v_c$, it is dominated by zero temperature quantum excitations

For the transverse Ising chain coupled to an Ohmic bath, $\theta = 2$ and $v_c \sim T^{8/3}$

Adiabatic theorem

If we start in the ground state of a quantum system and slowly vary its Hamiltonian at a rate $1/\tau$ to go across a quantum critical point (QCP), the state reached eventually differs from the final ground state in having some low-energy excitations

Kibble-Zurek scaling:

The density of point-like excitations scales as $1/\tau^{d\nu/(z\nu+1)}$, where d, ν, z are the spatial dimensionality, correlation length exponent and dynamical critical exponent at the QCP

In the adiabatic limit $\tau \to \infty$, the system reaches the final ground state and no excitations are produced

This is the adiabatic theorem of quantum mechanics

Failure of adiabatic theorem

The theorem fails if the system has topological order and the initial and final ground states lie in different topological sectors

Suppose that the different topological sectors are labeled by a topological quantity. The ground state may have a degeneracy, with the different ground states lying in different sectors

It may happen that there is a topological sector which contains a ground state G on one side of a QCP but does not contain a ground state on the other side

Topological blocking

Suppose that the Hamiltonian is varied in time in such a way that the time evolution does not change the topological sector

Then a system which begins in the ground state G on one side of the QCP will never reach a ground state on the other side, no matter how slow the quenching across the QCP is

We call this topological blocking

This is insensitive to local perturbations or terms which break local symmetries since topological sectors are not connected to each other by local operators

Ising chain in a transverse field

One-dimensional Ising model in a transverse magnetic field has the Hamiltonian

$$H = - J \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x - h \sum_{i=1}^{N} \sigma_i^z$$

The system is ordered (ferromagnetic) if $h < J$, and disordered (paramagnetic) if $h > J$. There is a QCP at $h = J$

This maps, via the Jordan-Wigner transformation, to a system of spinless fermions

$$\begin{aligned} H &= - J \sum_{i=1}^{N-1} (c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}) + J T_z (c_N^\dagger - c_N)(c_1^\dagger + c_1) \\ &\quad + h \sum_{i=1}^{N} (2 c_i^\dagger c_i - 1) \end{aligned}$$

Ising chain in a transverse field

The Hamiltonian commutes with the parity of the total fermion number $N_F = \sum_i c_i^\dagger c_i$

$$T_z \;=\; \prod_i \sigma_i^z \;=\; (-1)^{N_F}$$

For an even number of sites $N,$ the fermionic system has periodic (antiperiodic) boundary conditions if the number of fermions is odd (even), i.e., if $T_z \;=\; -1\,(1)$

After Fourier transforming, we find that the allowed momenta include $k = 0,\, \pi$ if $T_z \;=\; -1,$ and do not include $k = 0,\, \pi$ if $T_z \;=\; 1$

In the ferro phase, the ground state is doubly degenerate, with one ground state each in the two sectors $T_z \;=\; 1$ and $T_z \;=\; -1$

In the para phase, there is a unique ground state lying in the sector with $T_z \;=\; 1.$ The lowest energy state in the sector with $T_z \;=\; -1$ is separated from the ground state by a finite gap

Hamiltonian in momentum space

In momentum space, the fermionic Hamiltonian is

$$H = \sum_{0<k<\pi} \begin{bmatrix} c_k^\dagger & c_{-k} \end{bmatrix} h_k \begin{bmatrix} c_k \\ c_{-k}^\dagger \end{bmatrix}$$

$$h_k = (2h - 2J\cos k)\,\tau^z + 2J \sin k\,\tau^x$$

If the momenta $k = 0,\ \pi$ are present, the Hamiltonian contains two more terms, $(2h - 2J)\, c_0^\dagger c_0 + (2h + 2J)\, c_\pi^\dagger c_\pi$

Now we quench from the ferro to the para phase by varying h from 0 to $2J$ as

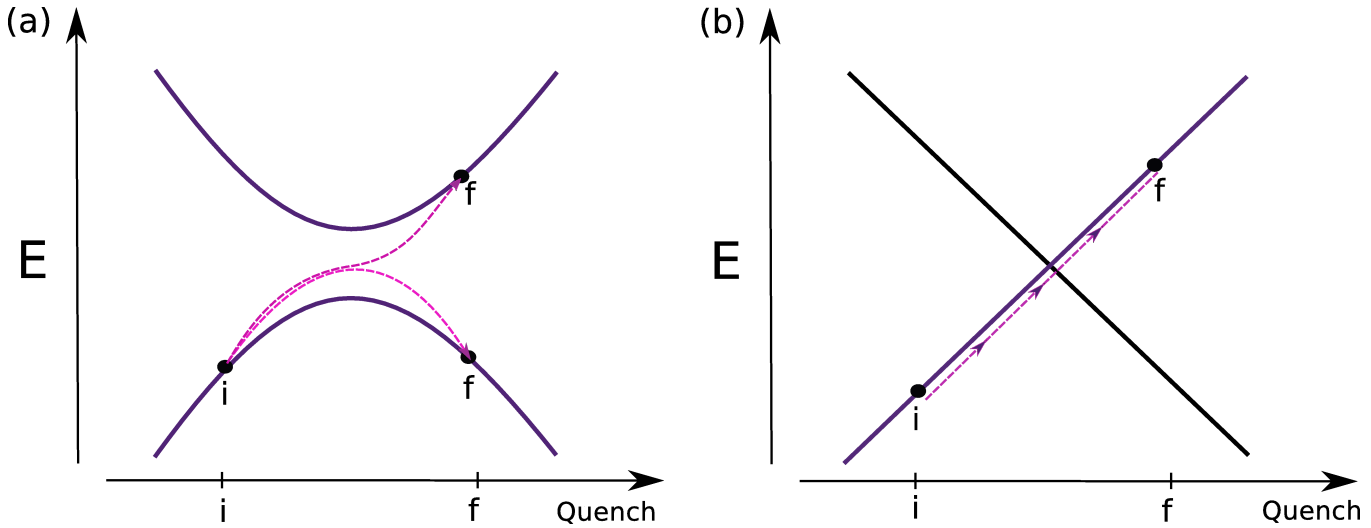
$$h(t) = 2J\,\frac{t}{T} \quad \text{for} \quad 0 < t < T$$

The modes with $k = 0,\ \pi$ do not evolve at all. All the other modes are described by two-level systems of the Landau-Zener form

Landau-Zener problem

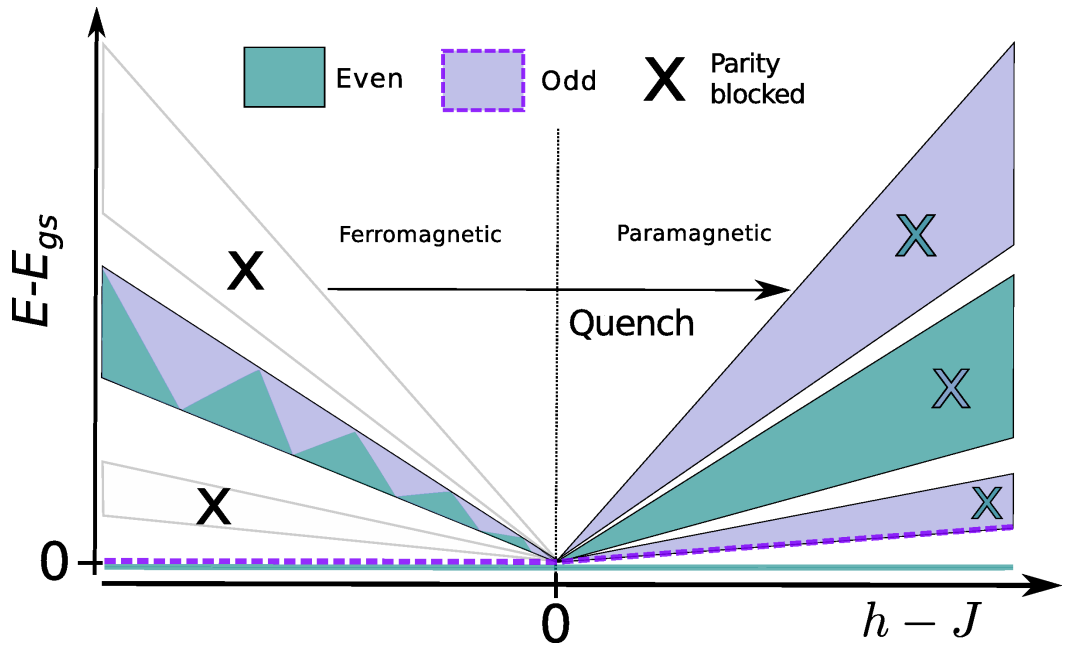
Upon passing through the QCP, the two-level systems labeled by $\pm k$ get excited with a probability p_k given by the Landau-Zener formula. Integrating p_k over $0 < k < \pi$ gives a density of excitations with the Kibble-Zurek scaling $1/(JT)^{1/2}$ if the quenching is slow, i.e., if $JT \gg 1$

However, the mode at $k = 0$ does not evolve and remains in the same state; hence this has an excitation probability equal to 1



Topological blocking

So, if the system contains the momentum $k = 0,$ i.e., if the number of fermions is odd, the system necessarily goes to an excited state even for very slow quenching through the QCP. The ground state $- - >$ ground state evolution is blocked in the odd fermion sector



Kells, Sen, Slingerland and Vishveshwara, Phys. Rev. B 89, 235130 (2014)

Periodically varying potentials

Potentials varying periodically in time can lead to many interesting effects in quantum systems

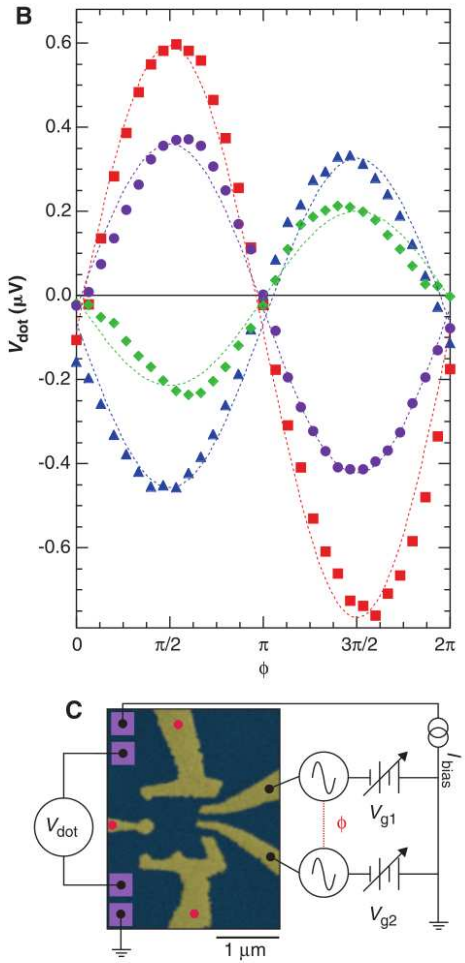
They can lead to pumping of charge between two reservoirs at the same chemical potential and temperature. Two sinusoidal potentials with a phase difference of ϕ pump charge proportional to $\sin\phi$ (can be related to geometric phase)

$f = 10MHz$, dot resistance $\sim 13k\Omega$

$T = 330mK$

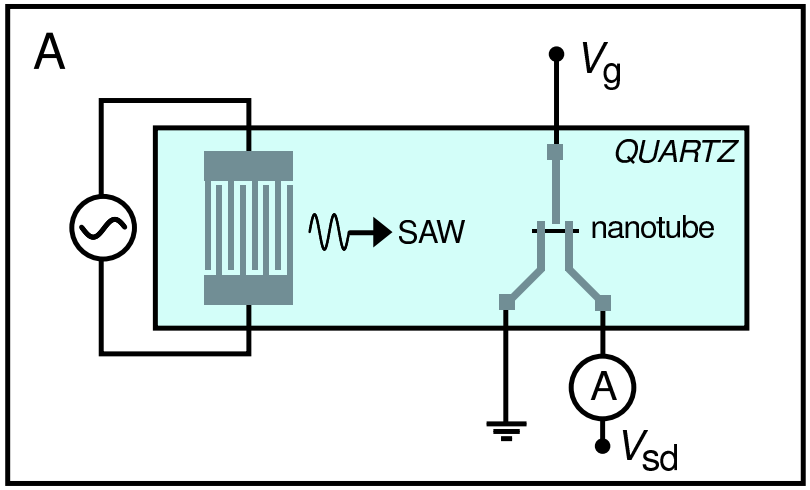
Different curves are for different magnetic fields up to $80mT$

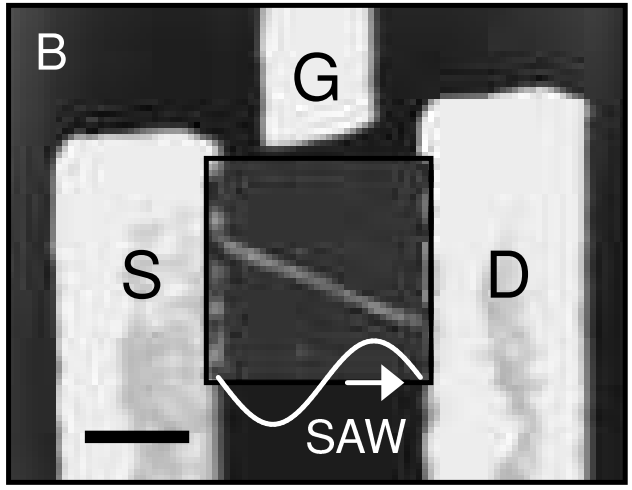
Switkes et al., Science 283, 1905 (1999)



SAW on a carbon nanotube

A surface acoustic wave travelling on a piezoelectric substance (quartz) sets up an 'electrical potential wave' which moves along the length of the carbon nanotube





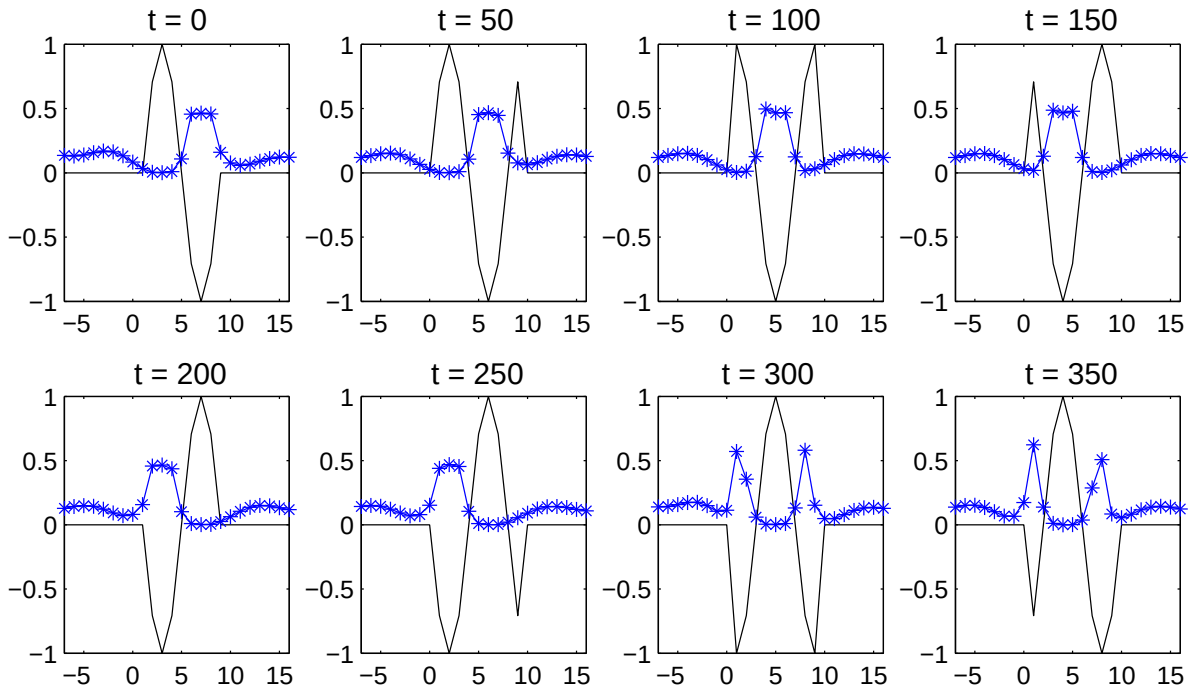
$f = 3.2GHz,\ \lambda = 1\mu m,\ \Delta Q \sim 40$ per cycle, $I \sim 20nA,\ T = 5K$

Possible use as a standard current if ΔQ is exactly quantized

Leek et al., Phys. Rev. Lett. 95, 256802 (2005)

A mechanism of charge pumping

Consider a ‘potential wave’ which periodically travels from one end of the wire to the other. Then the minima of the potential can trap electrons and carry them from one end to the other



$\omega = \pi/200$, wave length $= \pi/8$, $\phi_{n+1} - \phi_n = \pi/4$

Electron numbers and potentials for pumping at eight sites

Agarwal and Sen, J. Phys. Condens. Matter 19, 046205 (2007)

Adiabatic scattering theory

Consider a system connected to N leads. Suppose that the system is periodically driven with a very long time period T, and $S(t)$ is the instantaneous (frozen) scattering matrix given by a $N \times N$ unitary matrix. Namely,

$$\begin{pmatrix} \psi_{1,out} \\ \psi_{2,out} \\ \cdots \\ \psi_{N,out} \end{pmatrix} = S \begin{pmatrix} \psi_{1,in} \\ \psi_{2,in} \\ \cdots \\ \psi_{N,in} \end{pmatrix}$$

Then the number of electrons flowing into lead j over one time period is given by

$$\Delta Q_j = \frac{i}{2\pi} \int_0^T (dSS^\dagger)_{jj}$$

Büttiker, Thomas and Pretre, Z. Phys. B 94, 133 (1994)

Adiabatic scattering theory

If the instantaneous scattering matrix is completely reflecting, S will be a diagonal matrix with matrix elements of the form $S_{jj} = e^{i\theta_j}$. Hence the number of electrons flowing into lead j is

$$\Delta Q_j = \frac{i}{2\pi} \int_0^T dt\, (dSS^\dagger)_{jj} = -\frac{1}{2\pi} \int_0^T d\theta_j$$

This must be an integer since θ_j is periodic in time. Quantized charge pumping occurs if the instantaneous potential has zero transmission probability at all times !

If there are two periodically varying weak potentials $V_1(t)$ and $V_2(t)$, one can show that the number of electrons flowing into any of the leads is proportional to the area enclosed by the closed curve traced out by these potentials in one time period

$$\Delta Q_j \sim \oint dV_1\, V_2$$

If $V_1 = a_1 \sin(\omega t)$ and $V_2 = a_2 \sin(\omega t + \phi)$, then $\Delta Q_j \sim a_1 a_2 \sin(\phi)$

Brouwer, Phys. Rev. B 58, R10135 (1998)

Periodically varying potentials

Potentials varying periodically in time can produce states at the edges of a non-topological system

Lindner, Refael and Galitski, Nature Phys. 7, 490 (2011)

We saw that

$$H = -\sum_n \left[\gamma \left(f_n^\dagger f_{n+1} + f_{n+1}^\dagger f_n\right) - \Delta \left(f_n f_{n+1} + f_{n+1}^\dagger f_n^\dagger\right) + \mu\, f_n^\dagger f_n\right]$$

can have either topological phases with end modes or non-topological phases with no end modes

We will vary μ periodically in time and see if this can generate end modes

We will consider periodic kicks

$$\mu = c_0 + c_1 \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Floquet theory for periodic kicks

$$i\frac{\partial\psi}{\partial t} = [H_0 + V_0 \sum_{n=-\infty}^{\infty} \delta(t-nT)]\,\psi$$

Define the Floquet operator as the symmetrized product

$$U = \exp[-\frac{i}{2}V_0]\,\exp[-iH_0T]\,\exp[-\frac{i}{2}V_0]$$

The wave functions must be eigenvectors of the Floquet operator with eigenvalues $e^{i\theta_n}$

U and $e^{i\theta_n}$ can be found numerically

If we write $\theta_n = -\epsilon_n T$, ϵ_n is called the quasienergy

Finding Majorana modes

The Hamiltonian for a N-site system

$$H(t) = -\sum_n \left[\gamma(f_n^\dagger f_{n+1} + f_{n+1}^\dagger f_n) - \Delta(f_n f_{n+1} + f_{n+1}^\dagger f_n^\dagger) + \mu(t) f_n^\dagger f_n\right]$$

can be written in terms of Majorana fermions as

$$H(t) \;=\; i \sum_{r,s=1}^{2N} v_r \; M_{rs}(t) \; v_s$$

where $v_r \;=\; a_1,\; b_1,\; \cdots,\; a_N,\; b_N,$ and $M_{rs}\;(t)$ is a real antisymmetric matrix.
The Heisenberg equations take the form

$$\frac{dv_r}{dt} \;=\; i\,[H, v_r] \;=\; 4\sum_s M_{rs} v_s$$

The $2N$-dimensional column $v(t)$ satisfies $v(T) = U\; v(0)$

where $U \;=\; \mathcal{T}\; \exp[4\int_0^T dt\; M(t)]$

Finding Majorana modes $\cdots$

Since M is a real antisymmetric matrix, $U = \mathcal{T} \exp[4 \int_0^T dt\, M(t)]$ is a real orthogonal matrix

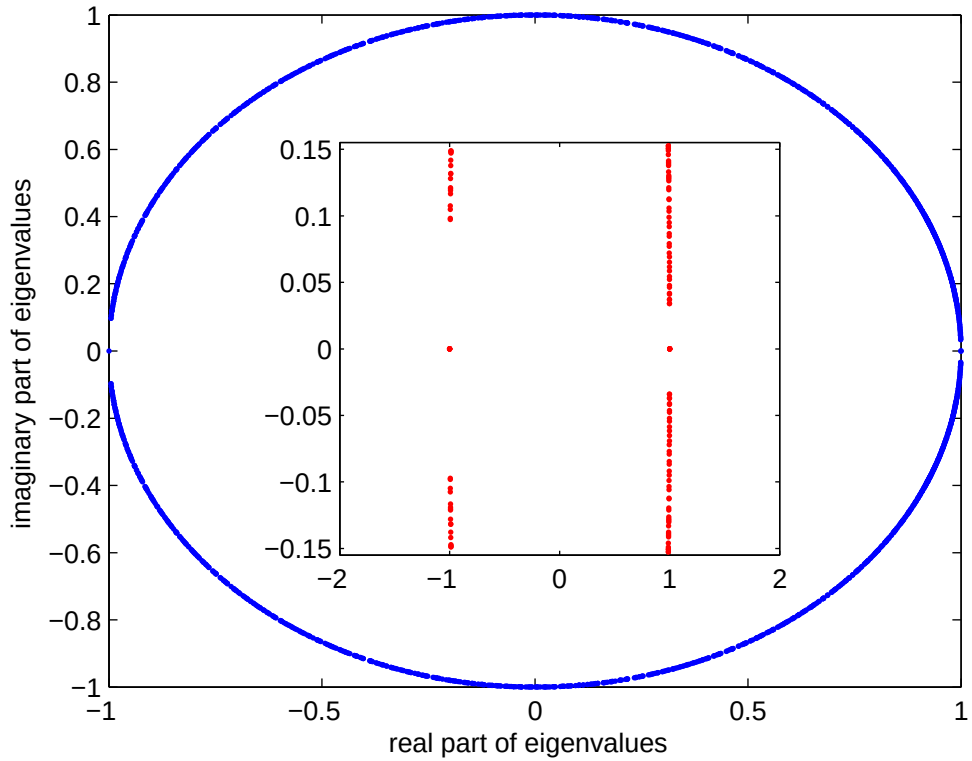
The eigenvalues of U generally come in pairs $e^{\pm i\theta}$, since $Ux = e^{i\theta}x$ implies $Ux^* = e^{-i\theta}x^*$

If there is a single mode ψ at one end of a long chain, we must have $e^{i\theta} = e^{-i\theta}$, namely, $e^{i\theta} = \pm 1$, and $\psi = \psi^*$

Thus, a Majorana end mode has Floquet eigenvalue equal to $\pm\ 1$

If Floquet eigenvalues equal to $\pm\ 1$ exist, they must be separated from all the other eigenvalues by a gap. Namely, if the eigenvalues are plotted on a unit circle, there must be a gap $\Delta\theta$ between the eigenvalues at $\theta = 0$ or π and all the other eigenvalues

Gap in Floquet eigenvalue spectrum



Floquet eigenvalue spectrum for a $1000-$site system with
$\gamma = 1,\ \Delta = -1,\ \omega = 1,\ \mu(t) = 2.5\ +\ 0.2\ \sum_n \delta(t - 2\pi n/\omega)$

There are four eigenvalues each at the centres of the gaps at $+1$ and -1

Thakurathi, Patel, Sen and Dutta, Phys. Rev. B 88, 155133 (2013)

Inverse participation ratio

A convenient way of distinguishing the end modes from the bulk modes is through the inverse participation ratio

Normalize the j^{th} eigenvector of U, called $\psi_j(m)$, so that

$$\sum_m |\psi_j(m)|^2 = 1$$

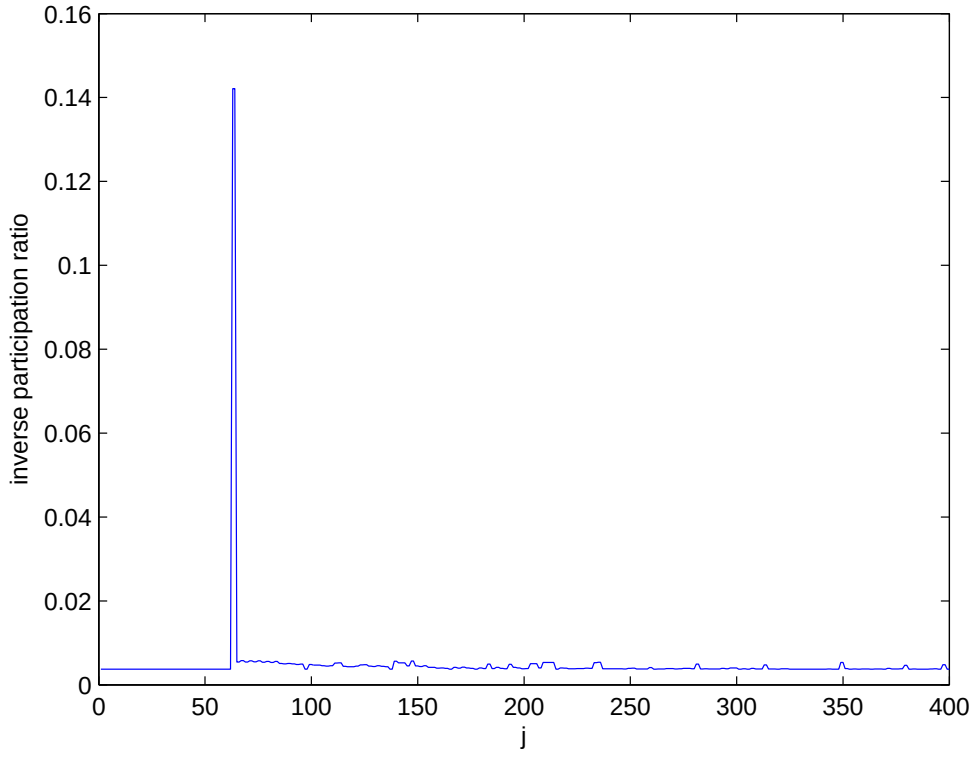
We define the inverse participation ratio of the state as

$$I_j = \sum_m |\psi_j(m)|^4$$

If $\psi_j(m)$ is spread over L sites, then $|\psi_j(m)|^2 \sim 1/L$ at those sites. Hence $I_j \sim 1/L$

So the more localized states will have the larger values of I_j

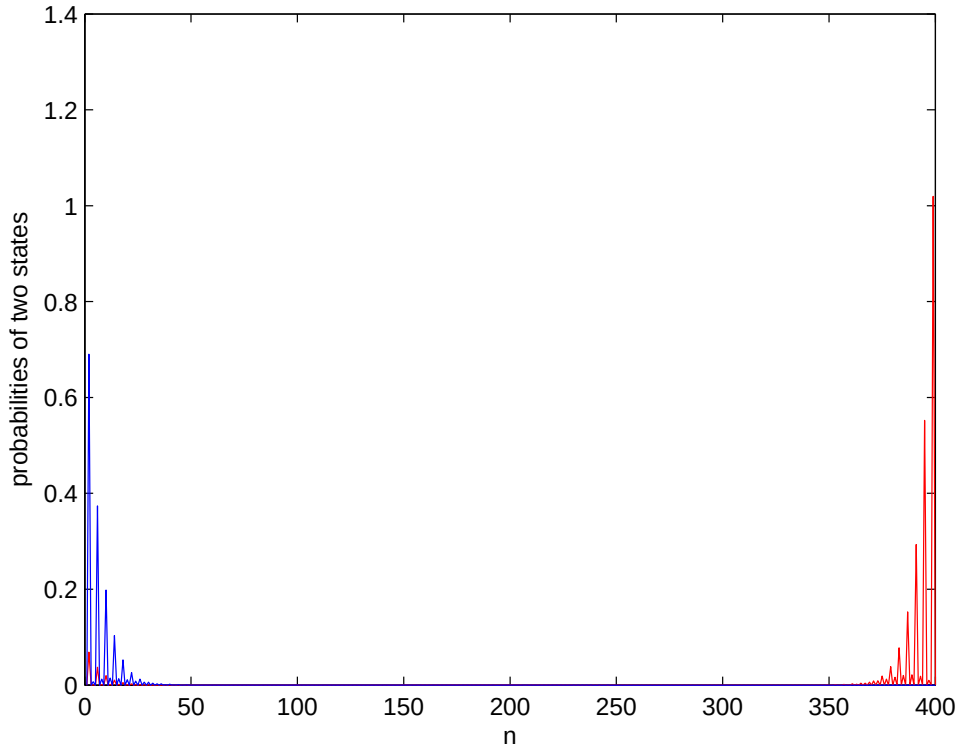
End modes for periodic kicks



Inverse participation ratios of eigenvectors for a $200-$site system with
$\gamma = 1,\ \Delta = -1,\ \omega = 12,\ \mu(t) = 2.5\ +\ 0.2\ \sum_n \delta(t - 2\pi n/\omega)$

Note: $\mu > 1$ is the non-topological phase for a time-independent Hamiltonian

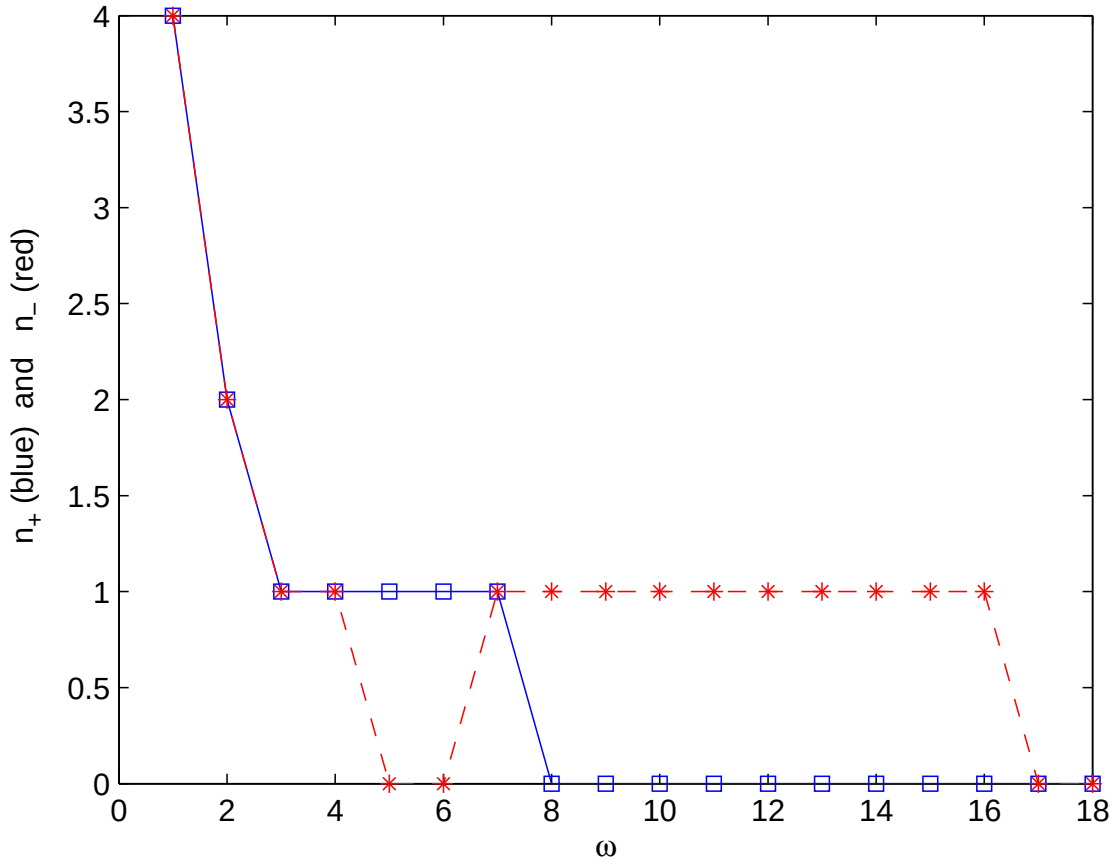
End modes for periodic kicks



Probabilities of the two states with the largest inverse participation ratios.
Their Floquet eigenvalues are equal to -1

They are Majorana modes – the wave functions $\psi_j(m)$ are real

Number of end modes versus ω



Plot of numbers n_+ and n_- of Floquet eigenvalues equal to $+1$ and -1 versus ω, for a $200-$site system with $\gamma = 1,\ \Delta = -1,\ \mu(t) = 2.5\ +\ 0.2\ \sum_n \delta(t - 2\pi n/\omega)$

Topological invariant

Is there a topological invariant which can predict the numbers of end modes with Floquet eigenvalues equal to $+1$ and -1 ?

We can find a Floquet operator for each momentum k. For

$$H(t) = -\sum_n \left[\gamma(f_n^\dagger f_{n+1} + f_{n+1}^\dagger f_n) - \Delta(f_n f_{n+1} + f_{n+1}^\dagger f_n^\dagger) + \mu(t) f_n^\dagger f_n\right]$$

$$\mu(t) = c_0 + c_1 \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

we have

$$U_k = e^{ic_1\sigma^z}\, e^{-i2T\,[(\gamma\cos k - c_0)\,\sigma^z + \Delta \sin k\,\sigma^y]}\, e^{ic_1\sigma^z}$$

This can be re-written as a single exponential

$$U_k = e^{-i(a_{2,k}\,\sigma^y + a_{3,k}\,\sigma^z)}$$

where the $a_{i,k}$ are some functions of k

Floquet operator U_k

Given

$$U_k \;=\; e^{ic_1\sigma^z}\; e^{-i2T\;[(\gamma\cos k - c_0)\;\sigma^z\;+\;\Delta\sin k\;\sigma^y]}\; e^{ic_1\sigma^z}$$

the coefficients $a_{2,k}$ and $a_{3,k}$ in

$$U_k \;=\; e^{-i(a_{2,k}\;\sigma^y\;+\;a_{3,k}\;\sigma^z)}$$

are not defined uniquely. To define them uniquely, we impose the constraint

$$0 < a_{2,k}^2 \;+\; a_{3,k}^2 < \pi$$

This can be imposed if U_k is not equal to $\pm\; I.$ We find that this is true for all values of $\omega \;=\; 2\pi/T$ except

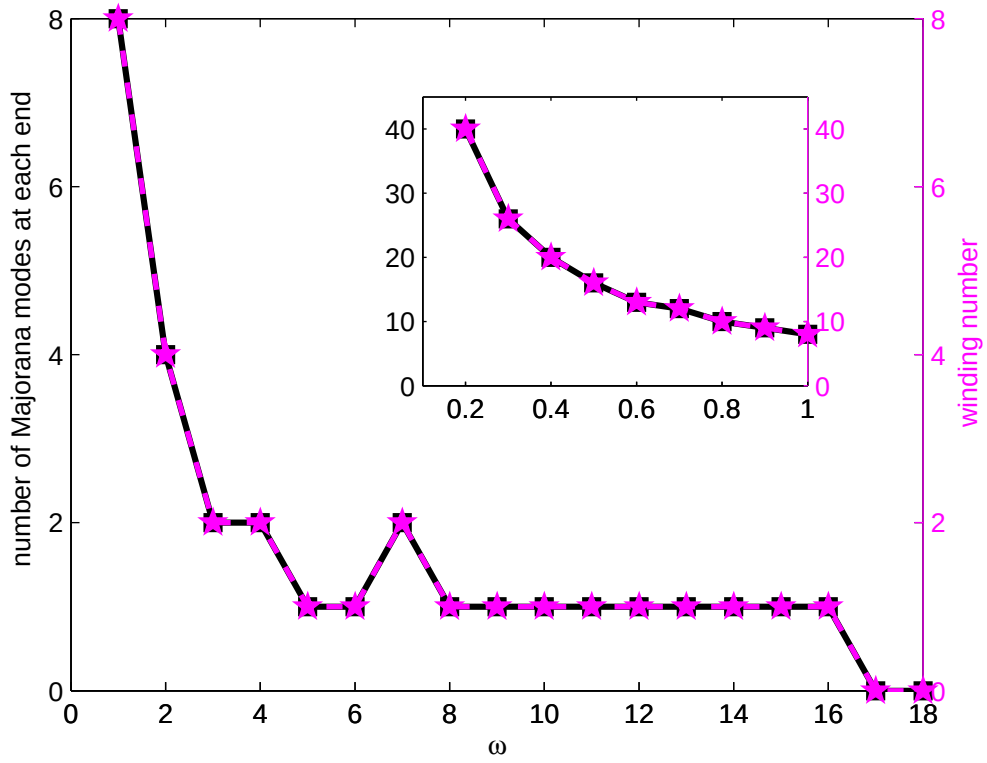
$$\omega \;=\; \frac{4\pi(c_0 \pm \gamma)}{n\pi - 2c_1}$$

where n is an integer

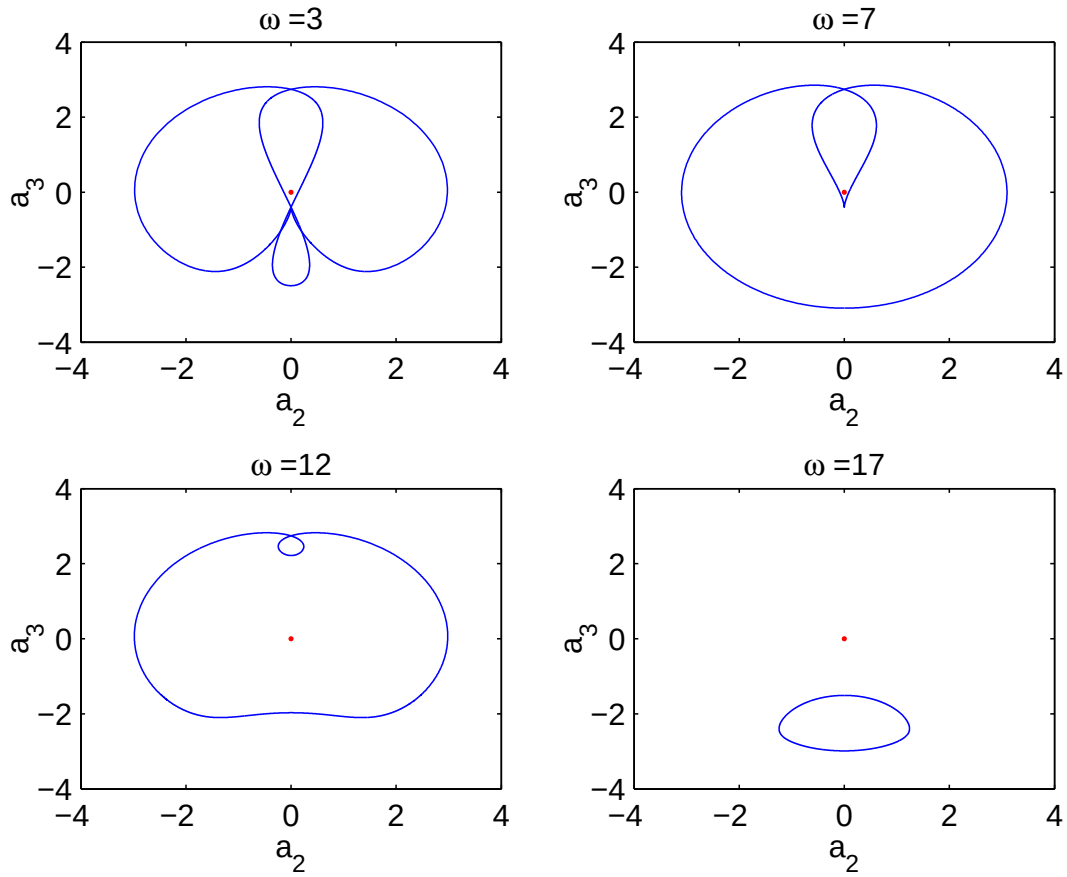
Topological invariant

Given $U_k = e^{-i(a_{2,k}\,\sigma^y + a_{3,k}\,\sigma^z)}$, we have a closed curve in two-dimensional space, $(a_{2,k}, a_{3,k})$, as k goes from $-\pi$ to π

The winding number of this curve exactly matches the total number of Majorana modes at each end of the system



Winding number



Closed curves for four different frequencies for a $200-$site system with $\gamma = 1,\ \Delta = -1,\ \mu(t) = 2.5\ +\ 0.2\ \sum_n \delta(t - 2\pi n/\omega).$

For $\omega\ =\ 3,\ 7,\ 12,\ 17,$ the winding numbers are $2,\ 2,\ 1,\ 0$ respectively

However, the winding numbers do not give the numbers of end modes with Floquet eigenvalues equal to $+1$ and -1 separately

A new topological invariant

Is there a topological invariant which will separately give the numbers of Floquet eigenvalues equal to $+1$ and -1 ?

Yes !

For $k = 0$ and $\pi,$ we have

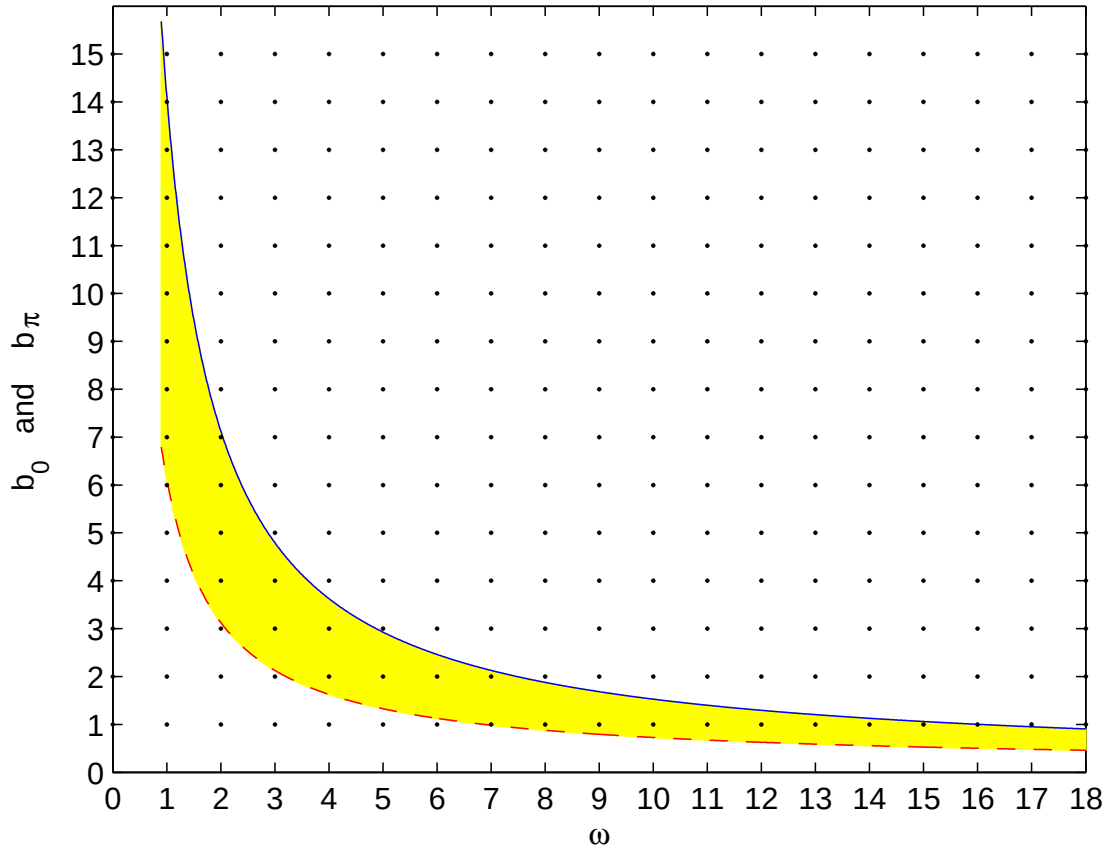
$$U_0 \;=\; e^{i[2T(c_0-\gamma)\,+\,2c_1]\;\sigma^z} \qquad \text{and} \qquad U_\pi \;=\; e^{i[2T(c_0+\gamma)\,+\,2c_1]\;\sigma^z}$$

Define

$$b_0 \;=\; \frac{4(c_0-\gamma)}{\omega} \;+\; \frac{2c_1}{\pi} \qquad \text{and} \qquad b_\pi \;=\; \frac{4(c_0+\gamma)}{\omega} \;+\; \frac{2c_1}{\pi}$$

We see that $U_0\;(U_\pi)$ becomes equal to $\pm\; I$ when $b_0\;(b_\pi)$ becomes equal to an integer $n.$ At these values of $\omega,$ the quasienergy bands becomes gapless at $k = 0\;(\pi),$ and a Majorana end mode with Floquet eigenvalue equal to $(-1)^n$ appears or disappears

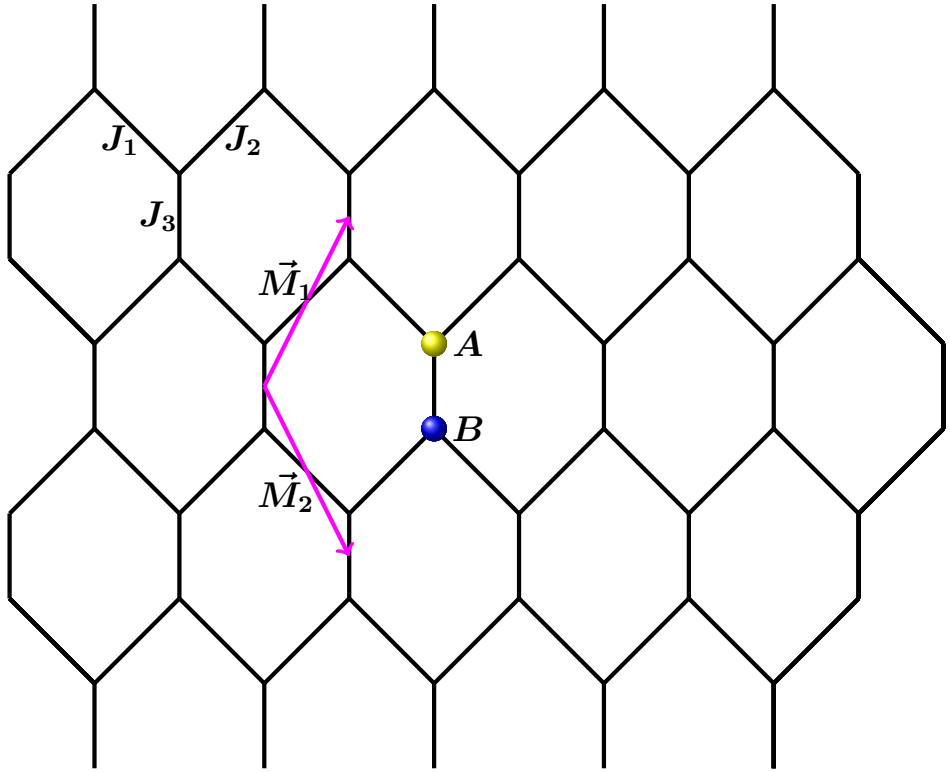
New topological invariant



Plot of b_0 and b_π as a function of ω. For each value of ω, the number of even and odd integers lying between b_0 and b_π gives the number of Majorana end modes with Floquet eigenvalues equal to $+1$ and -1 respectively

Thakurathi, Patel, Sen and Dutta, Phys. Rev. B 88, 155133 (2013)

Kitaev model

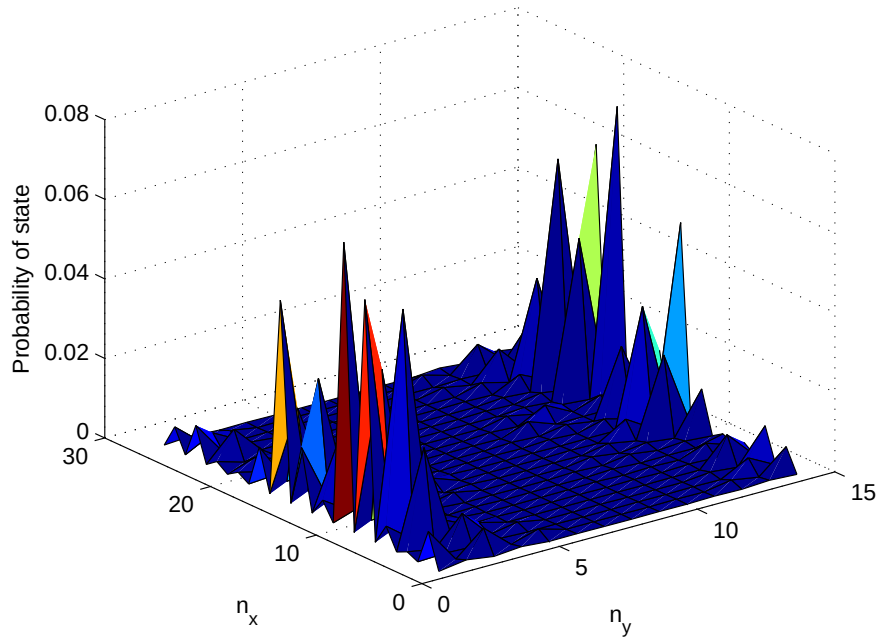


$$H = \sum_{j+l=even} (J_1 \, \sigma^x_{j,l}\sigma^x_{j+1,l} + J_2 \, \sigma^y_{j-1,l}\sigma^y_{j,l} + J_3 \, \sigma^z_{j,l}\sigma^z_{j,l+1})$$

A and B are the sublattices

Floquet Majorana edge modes

Give periodic kicks of the form $J_3 = J_0 + J_p \sum_{n=-\infty}^{\infty} \delta(t - nT)$



Zigzag edge states for a system with $N_x \times N_y = 27 \times 14$,
$J_1 = 0.7,\ J_2 = 0.15,\ J_0 = 0.15,\ J_p = 0.2$ and $\omega = 3$

There are also armchair edge states for the same parameters

Thakurathi, Sengupta and Sen, Phys. Rev. B 89, 235434 (2014)

Floquet Majorana edge modes

For a Majorana mode with momentum k along the edge, the system can be mapped to the transverse field Ising model with parameters which are functions of k

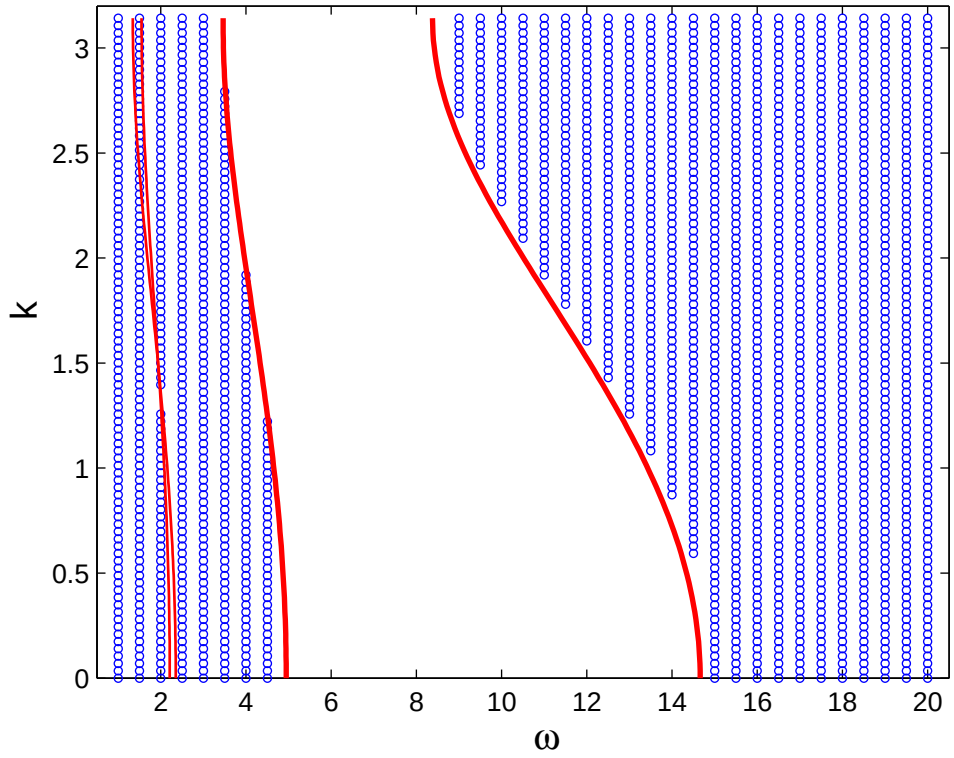
Majorana edge modes with Floquet eigenvalue $(-1)^n$ should appear or disappear at the frequency

$$\omega_k = \frac{4\pi \left[J_0 \pm J_k\right]}{n\pi - 2J_p}$$

$$J_k = \sqrt{J_1^2 + J_2^2 + 2J_1 J_2 \cos k}$$

where n is an integer

Floquet Majorana edge modes

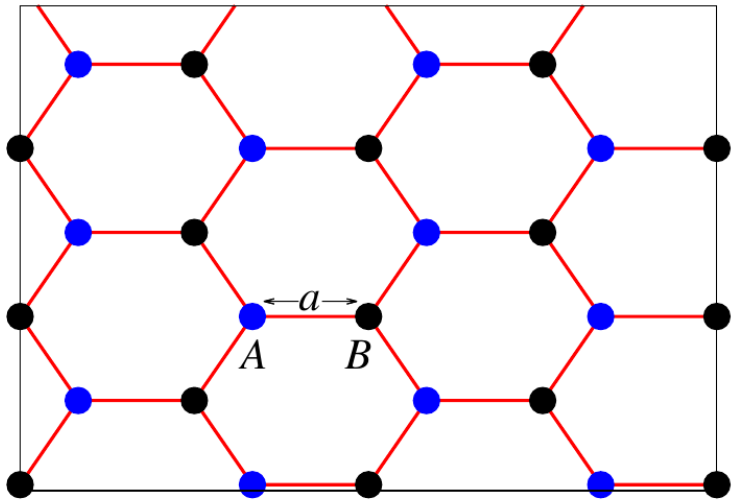


Regions where Majorana modes with momentum k exist on a zigzag edge.
The system has a width of 100 sites, and $J_1 = 0.7,\ J_2 = 0.15,\ J_0 = 0.15,\ J_p = 0.3$

The red lines show the analytical predictions for the values of ω where Majorana modes appear or disappear

Periodic driving in graphene

Periodic driving can be used to manipulate the energy-momentum dispersion in graphene



The Hamiltonian is

$$H_{\vec{k}} = -\gamma \begin{pmatrix} 0 & 1 + e^{i\vec{k}\cdot\vec{M}_1} + e^{i\vec{k}\cdot\vec{M}_2} \\ 1 + e^{-i\vec{k}\cdot\vec{M}_1} + e^{-i\vec{k}\cdot\vec{M}_2} & 0 \end{pmatrix}$$

where $\vec{M}_1 = (3a/2)(1, 1/\sqrt{3})$ and $\vec{M}_2 = (3a/2)(1, -1/\sqrt{3})$. The energies are given by $\pm E_{\vec{k}}$, where $E_{\vec{k}} = \gamma[3 + 2\cos(\sqrt{3}k_y a) + 4\cos(\frac{\sqrt{3}k_y a}{2})\cos(\frac{3k_x a}{2})]^{1/2}$

Periodic driving in graphene

The energy vanishes at two points called $\vec{K},\ \vec{K}'$ given by $\pm\,(0, 4\pi/(3\sqrt{3}a))$

We now apply δ-function kicks at all unit cells of the system,

$$H_{kick} \;=\; \sum_{\vec{n}} \left(\alpha_x \sigma^x + \alpha_y \sigma^y + \alpha_z \sigma^z\right) \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

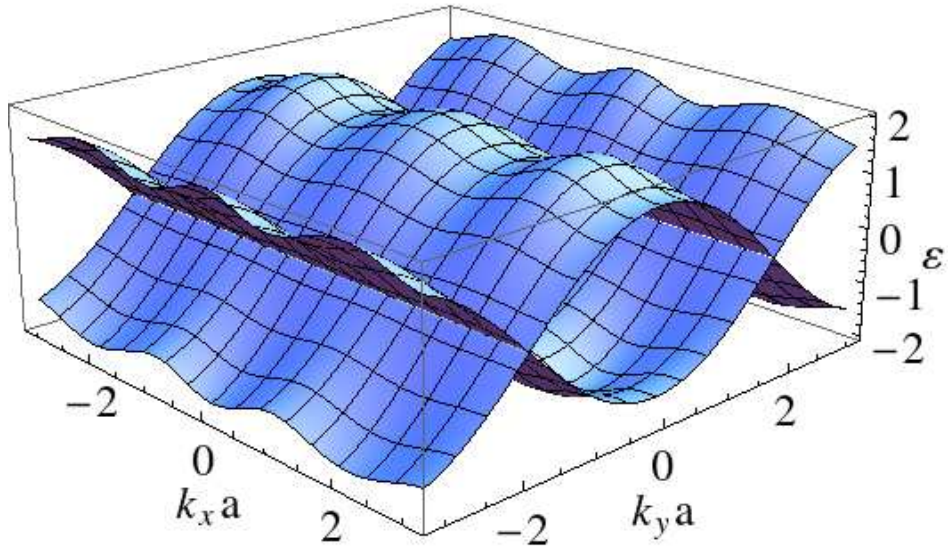
The Floquet operator is then $U_{\vec{k}} \;=\; e^{-i\vec{\alpha}\cdot\vec{\sigma}}\; e^{-iH_{\vec{k}}T}$. The eigenvalues of $U_{\vec{k}}$ equal to $e^{-i\epsilon_{\vec{k}}T}$ give the quasienergies $\epsilon_{\vec{k}}$. Depending on the values of $\alpha_x,\ \alpha_y, \alpha_z,$ the quasienergy dispersion can be modified in a variety of ways

To visualize the modified dispersion, we study the wave packet dynamics. We begin with a Gaussian wave packet with a momentum $\vec{k}_0 = (k_{0x}, k_{0y})$ and a width σ. Namely, $\Psi(\vec{r}, t = 0) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\; \exp(-\frac{r^2}{4\sigma^2})\; \exp(i\vec{k}_0 \cdot \vec{r}),$ whose Fourier transform is given by

$$\Psi(\vec{k}, t = 0) \;=\; \sigma\sqrt{8\pi}\exp[-\sigma^2\{(k_x - k_{0x})^2 + (k_y - k_{0y})^2\}]$$

We numerically evolve each momentum component in time using $U_{\vec{k}}$ and then superpose them to see the wave packet in real space

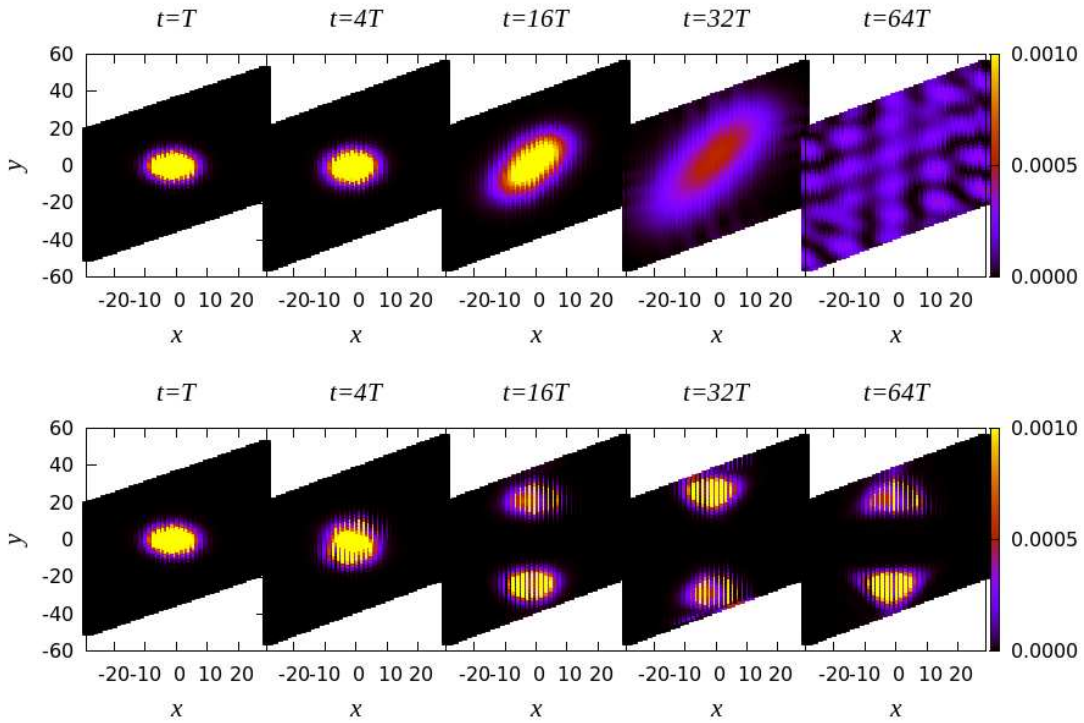
Flat dispersion in one direction



Quasienergy dispersion $\epsilon_{\vec{k}}$ (in units of $1/T$) when $\alpha_x = \gamma T = 1$ and $\alpha_y = \alpha_z = 0$.
There is a dispersionless line along the k_x direction when $k_y a = \pm\pi/\sqrt{3}$

A wave packet localized on this gapless line will move only in the y direction

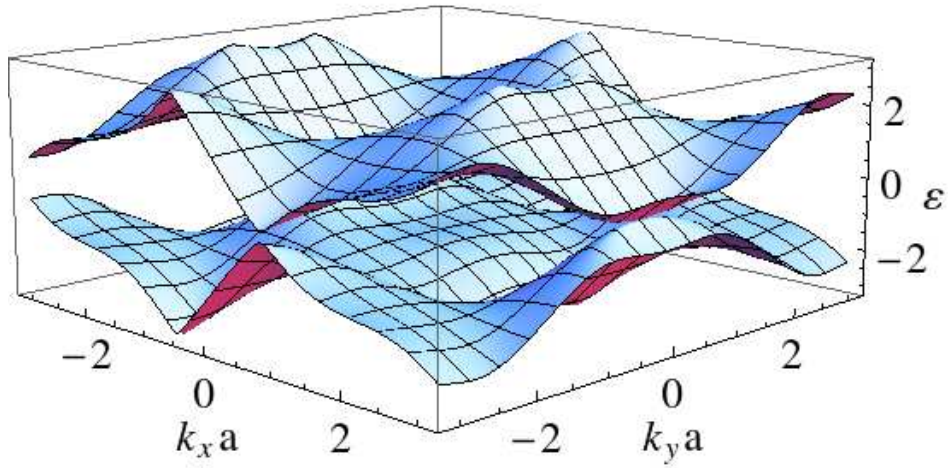
Wave packet moving in one direction



Time evolution of a wave packet initially centered at $\vec{r} = (0, 0),$ at $t = T,\ 4T,\ 16T,\ 32T,\ 64T$ for no kicking (upper panel) and for $\alpha_x = \gamma T = 1$ and $\alpha_y = \alpha_z = 0$ (lower panel), with $k_{ox}a = 1,\ k_{oy}a = \frac{\pi}{\sqrt{3}}$ and $\sigma = \frac{10}{2\sqrt{2}}a$

The upper panel shows that the wave packet spreads out in both x and y directions.
The lower panel shows that the wave packet only moves in the y direction

Semi-Dirac dispersion



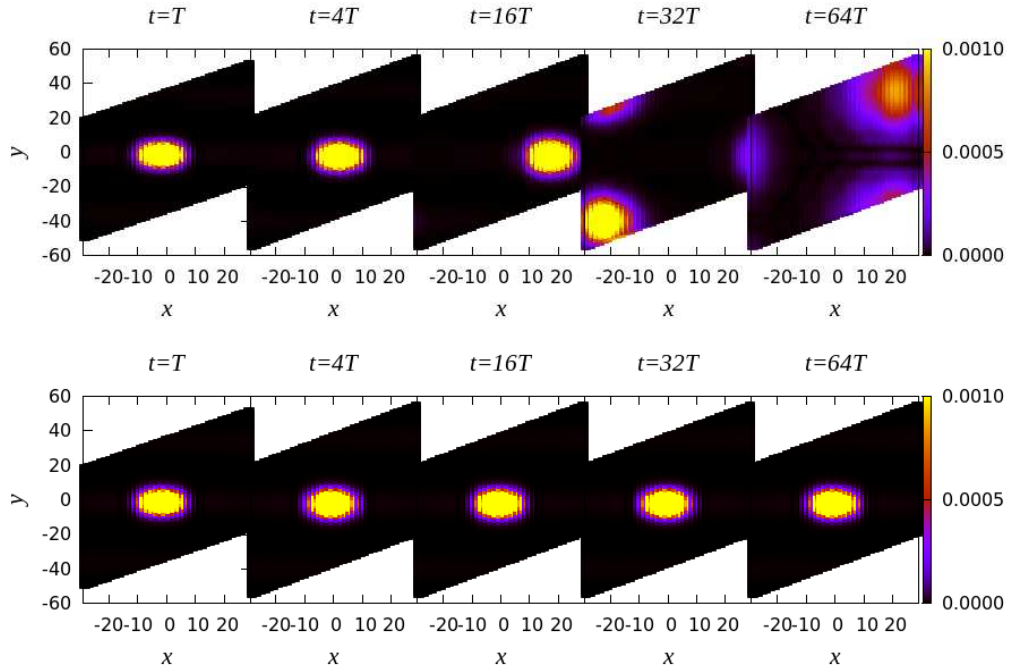
Quasienergy dispersion $\epsilon_{\vec{k}}$ (in units of $1/T$) when $\alpha_y = \sqrt{3},\ \gamma T = 1$ and $\alpha_x = \alpha_z = 0$

The dispersion is linear in the k_x direction and quadratic in the k_y direction near two gapless points lying at $k_x = 2\pi/9,\ k_y = \pm 2\pi/\sqrt{3}$

This is a semi-Dirac dispersion

Dynamical localization

For $\alpha_z = \pi/2$ and $\alpha_x = \alpha_y = 0,$ we find that the quasienergy is completely dispersionless. So a wave packet of any shape and size will not move at all ! We call this dynamical localization



Time evolution of a wave packet initially centered at $\vec{r} = (0, 0),$ for no kicking (upper panel) and for $\alpha_z = \pi/2,\ \gamma T = 1,\ \alpha_x = \alpha_y = 1$ (lower panel), with $k_{ox}a = 1$ and $k_{oy}a = 0$ and $\sigma = \frac{10}{2\sqrt{2}}a.$ The upper panel shows that the wave packet moves in the x direction. The lower panel clearly shows that the wave packet is localized

Agarwala, Bhattacharya, Dutta and Sen, Phys. Rev. B 93, 174301 (2016)

Other topics

We have not discussed the following aspects of quenching and Floquet dynamics

- Effects of disorder on defect production by quenching

  Caneva, Fazio and Santoro, Phys. Rev. B 76, 144427 (2007)

- Effects of interactions on Floquet dynamics

  Mikami, Kitamura, Yasuda, Tsuji, Oka and Aoki, Phys. Rev. B 93, 144307 (2016)

- Phase band crossings in Floquet dynamics of integrable systems

  Mukherjee, Sen, Sen and Sengupta, arXiv:1605.09178