CFCMR16 – ICTS, Bangalore

The Tenfold Way

Symmetry, Topology, . . . and Disorder

Vijay B. Shenoy

Department of Physics, Indian Institute of Science, Bangalore 560012
shenoy@physics.iisc.ernet.in

The Tenfold Way

- Thanks to Adhip A.  and Arijit H. 

The Tenfold Way

- Thanks to Adhip A.  and Arijit H. 

- "Classification" of fermionic systems
  - What are all the distinct types of ground states for fermionic systems?

The Tenfold Way

- Thanks to Adhip A.  and Arijit H. 

- "Classification" of fermionic systems
  - What are all the distinct types of ground states for fermionic systems?...hard problem...

The Tenfold Way

- Thanks to Adhip A.  and Arijit H. 

- "Classification" of fermionic systems
  - What are all the distinct types of ground states for fermionic systems?...hard problem...
  - Manageable...gapped ground states for non-interacting systems...

The Tenfold Way

- Thanks to Adhip A.  and Arijit H. 

- "Classification" of fermionic systems
  - What are all the distinct types of ground states for fermionic systems?...hard problem...
  - Manageable...gapped ground states for non-interacting systems...
  - With interactions...open question!

The Tenfold Way

- Thanks to Adhip A.  and Arijit H. 

- "Classification" of fermionic systems
  - What are all the distinct types of ground states for fermionic systems?...hard problem...
  - Manageable...gapped ground states for non-interacting systems...
  - With interactions...open question!
- References

1. Altland and Zirnbauer, Phys. Rev. B., **55**, 1142 (1997).
2. Ryu, Schnyder, Furusaki, and Ludwig, New Journal of Physics, **12**, 065010 (2010).
3. Chiu, Teo, Schnyder, Ryu, arXiv:1505.03535
4. Ludwig, arXiv:1512.08882
5. Agarwala, Haldar, et al., arXiv:1606.05483

What are we after?

TABLE - " Ten Fold Way" [`CARTAN Classes']

Name (Cartan)	T	C	S= T C	Time evolution operator $U(t) = \exp\{itH\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall* *Z2-Top.Ins.* *Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	yes	Singlet SC

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(Ludwig:1512:08882)

What are we after?

TABLE - " Ten Fold Way" [`CARTAN Classes']

Name (Cartan)	T	C	S= T C	Time evolution operator $U(t) = \exp\{itH\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems (Examples)
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall* *Z2-Top.Ins.* *Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	yes	Singlet SC

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(Ludwig:1512:08882)

- Goal: An (a qualitative) understanding how such tables arise, what the entries mean!

What are we after?

TABLE - “ Ten Fold Way” [`CARTAN Classes’]

Name (Cartan)	T	C	S= T C	Time evolution operator $U(t) = \exp\{itH\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems (Examples)
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall Z2-Top.Ins. Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	yes	Singlet SC

Cartan$\backslash d$	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(Ludwig:1512:08882)

- Goal: An (a qualitative) understanding how such tables arise, what the entries mean!
- Prerequisites: Second quantization (essential), some basic knowledge of homotopy (math), quantum field theory (physics) may be useful...

What are we after?

TABLE - "Ten Fold Way" [`CARTAN Classes']

Name (Cartan)	T	C	S= T C	Time evolution operator $U(t) = \exp\{itH\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall Z2-Top.Ins. Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(N)	Sp(2n)	yes	Singlet SC

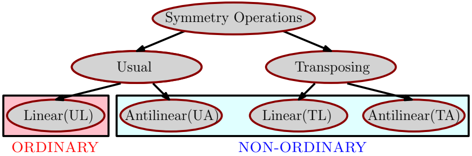
Examples

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

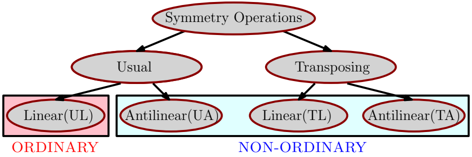
(Ludwig:1512:08882)

- Goal: An (a qualitative) understanding how such tables arise, what the entries mean!
- Prerequisites: Second quantization (essential), some basic knowledge of homotopy (math), quantum field theory (physics) may be useful...
- **Disclaimer:** Speaker is *not* an expert...

Summary of Different Types of Symmetries



Summary of Different Types of Symmetries



Symmetry Operation	$\mathscr{U}\Psi^\dagger\mathscr{U}^{-1}$	Symmetry Condition
LU	$\Psi^\dagger \mapsto \Psi^\dagger \mathbf{U}$	$\mathbf{UHU}^\dagger = \mathbf{H}$
AU	$\Psi^\dagger \mapsto \Psi^\dagger \mathbf{U}$	$\mathbf{UH}^*\mathbf{U}^\dagger = \mathbf{H}$
LT	$\Psi^\dagger \mapsto \Psi^T \mathbf{U}^*$	$-\mathbf{UH}^*\mathbf{U}^\dagger = \mathbf{H}, \quad \mathrm{tr}\mathbf{H} = 0$
AT	$\Psi^\dagger \mapsto \Psi^T \mathbf{U}^*$	$-\mathbf{UHU}^\dagger = \mathbf{H}, \quad \mathrm{tr}\mathbf{H} = 0$

Symmetry-Type Multiplication Table

$G_1 \downarrow \mid G_2 \rightarrow$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$
$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$
$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$
$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$
$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$

Symmetry-Type Multiplication Table

$G_1 \downarrow \mid G_2 \rightarrow$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$
$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$
$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$
$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$
$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$

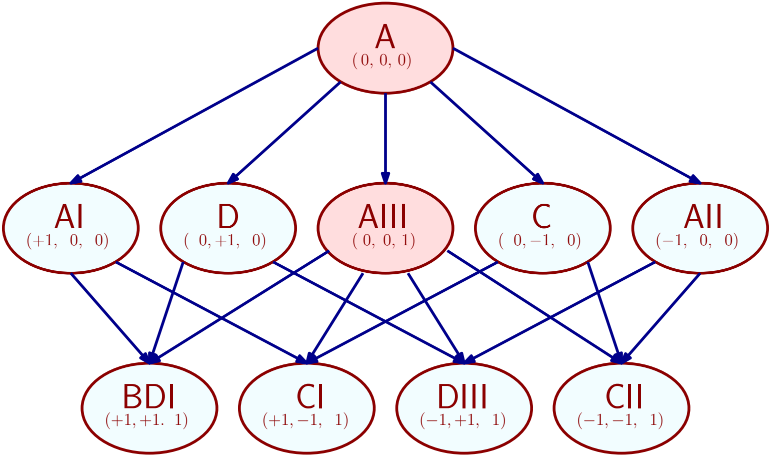
$G_{\mathcal{V}}/G_{\mathcal{V}}^{\mathrm{UL}}$

Symmetry-Type Multiplication Table

$G_1 \downarrow \mid G_2 \rightarrow$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$
$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$
$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$
$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$
$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TA}}$	$G_{\mathcal{V}}^{\mathrm{TL}}$	$G_{\mathcal{V}}^{\mathrm{UA}}$	$G_{\mathcal{V}}^{\mathrm{UL}}$

$$G_{\mathcal{V}}/G_{\mathcal{V}}^{\mathrm{UL}} = \mathcal{K}_4$$

Ten Symmetry Classes



Canonical Representations of Symmetry Operations

Class	T	C	S	L	$\mathbf{U}_\mathrm{T}$	$\mathbf{U}_\mathrm{C}$	$\mathbf{U}_\mathrm{S}$
A	0	0	0	L	–	–	–
AI	+1	0	0	L	$\mathbf{1}$	–	–
AII	−1	0	0	$L = 2M$	$\mathbf{J}$	–	–
D	0	+1	0	L	–	$\mathbf{1}$	–
C	0	−1	0	$L = 2M$	–	$\mathbf{J}$	–
AIII	0	0	1	$L = p + q$	–	–	$\mathbf{1}_{p,q}$
BDI	+1	+1	1	$L = p + q$	$\mathbf{1}$	$\mathbf{1}_{p,q}$	$\mathbf{1}_{p,q}$
CII	−1	−1	1	$L = p + q$ $p = 2r;\ q = 2s$	$\begin{pmatrix} \mathbf{J}_{pp} & \mathbf{0}_{pq} \\ \mathbf{0}_{qp} & \mathbf{J}_{qq} \end{pmatrix}$	$\begin{pmatrix} -\mathbf{J}_{pp} & \mathbf{0}_{pq} \\ \mathbf{0}_{qp} & \mathbf{J}_{qq} \end{pmatrix}$	$\mathbf{1}_{p,q}$
CI	+1	−1	1	$L = 2M$	$\mathbf{F}$	$-\mathbf{J}$	$\mathbf{1}_{M,M}$
DIII	−1	+1	1	$L = 2M$	$\mathbf{J}$	$\mathbf{F}$	$\mathbf{1}_{M,M}$

$\mathcal{K}_4$ Multiplier Table

$\mathcal{K}_4$	I	Θ	Ξ	Σ
I	I	Θ	Ξ	Σ
Θ	Θ	I	Σ	Ξ
Ξ	Ξ	Σ	I	Θ
Σ	Σ	Ξ	Θ	I

$\omega(g_1 \downarrow, g_2 \rightarrow)$	I	Θ	Ξ	Σ
I	1	1	1	1
Θ	1	T	1	T
Ξ	1	TC	C	T
Σ	1	C	C	1

Tenfold Way: Noninteracting Hamiltonians

Class	L	$\mathbf{H}^{(1)}$	$\dim i\mathcal{H}^{(1)}$	$i\mathcal{H}^{(1)}$	$\mathbb{U}_{\text{Schröd}}(t)$
A(0, 0, 0)	L	$\mathbf{H}^{(1)} = [\mathbf{H}^{(1)}]^\dagger$	L^2	$\mathfrak{u}(L)$	$U(L)$
AI(+1, 0, 0)	L	$\mathbf{H}^{(1)} = [\mathbf{H}^{(1)}]^*$	$L(L+1)/2$	$\mathfrak{u}(L) \setminus \mathfrak{o}(L)$	$U(L)/O(L)$
AII(−1, 0, 0)	$L = 2M$	$\begin{pmatrix} \mathbf{h}_{aa} & \mathbf{h}_{ab} \\ -\mathbf{h}^*_{ab} & \mathbf{h}^*_{aa} \end{pmatrix}$	$M(2M-1)$	$\mathfrak{u}(2M) \setminus \mathfrak{usp}(2M)$	$U(2M)/USp(2M)$
D(0, +1, 0)	L	$\mathbf{H}^{(1)} = -[\mathbf{H}^{(1)}]^*$	$L(L-1)/2$	$\mathfrak{o}(L)$	$O(L)$
C(0, −1, 0)	$L = 2M$	$\begin{pmatrix} \mathbf{h}_{aa} & \mathbf{h}_{ab} \\ \mathbf{h}^*_{ab} & -\mathbf{h}^*_{aa} \end{pmatrix}$	$M(2M+1)$	$\mathfrak{usp}(2M)$	$USp(2M)$
AIII(0, 0, 1)	$L = p + q$	$\begin{pmatrix} \mathbf{0}_{pp} & \mathbf{h}_{pq} \\ \mathbf{h}^\dagger_{pq} & \mathbf{0}_{qq} \end{pmatrix}$	$2pq$	$\mathfrak{u}(p+q) \setminus (\mathfrak{u}(p) \oplus \mathfrak{u}(q))$	$U(p+q)/(U(p) \times U(q))$
BDI(+1, +1, 1)	$L = p + q$	$\begin{pmatrix} \mathbf{0}_{pp} & \mathbf{h}_{pq} \\ \mathbf{h}^T_{pq} & \mathbf{0}_{qq} \end{pmatrix}, \mathbf{h}^*_{pq} = \mathbf{h}_{pq}$	pq	$\mathfrak{o}(p+q) \setminus (\mathfrak{o}(p) \oplus \mathfrak{o}(q))$	$O(p+q)/(O(p) \times O(q))$
CII(−1, −1, 1)	$L = p + q$, $p = 2r, q = 2s$	$\left(\begin{array}{c\|c} \mathbf{0}_{pp} & \begin{matrix} \mathbf{h}_{rr} & \mathbf{h}_{rs} \\ -\mathbf{h}^*_{rs} & \mathbf{h}^*_{rr} \end{matrix} \\ \hline \text{h.c.} & \mathbf{0}_{qq} \end{array}\right)$	$4rs$	$\mathfrak{usp}(p+q) \setminus (\mathfrak{usp}(p) \oplus \mathfrak{usp}(q))$	$USp(2(r+s))/(USp(2r) \times USp(2s))$
CI(+1, -1, 1)	$L = 2M$	$\begin{pmatrix} \mathbf{0}_{MM} & \mathbf{h}_{MM} \\ \mathbf{h}^*_{MM} & \mathbf{0}_{MM} \end{pmatrix}, \mathbf{h}^T_{MM} = \mathbf{h}_{MM}$	$M(M+1)$	$\mathfrak{usp}(2M) \setminus \mathfrak{u}(M)$	$USp(2M)/U(M)$
DIII(-1, +1, 1)	$L = 2M$	$\begin{pmatrix} \mathbf{0}_{MM} & \mathbf{h}_{MM} \\ -\mathbf{h}^*_{MM} & \mathbf{0}_{MM} \end{pmatrix}, \mathbf{h}^T_{MM} = -\mathbf{h}_{MM}$	$M(M-1)$	$\mathfrak{o}(2M) \setminus \mathfrak{u}(M)$	$O(2M)/U(M)$

Tenfold Way: N-Body Hamiltonians (N even)

Class	L	P	Q	$\mathbf{H}^{(N)}$	$\dim i\mathcal{H}^{(N)}$	$i\mathcal{H}^N_+$
A (0,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^\dagger$	P^2	$\mathfrak{u}(P)$
AI (+1,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^*$	$P(P+1)/2$	$\mathfrak{u}(P) \setminus \mathfrak{o}(P)$
AII (-1,0,0)	$L = 2M$	$\frac{1}{2}\left\{\binom{L}{N} + \binom{M}{N/2}\right\}$	$\frac{1}{2}\left\{\binom{L}{N} - \binom{M}{N/2}\right\}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{h}^{(N)}_{PQ} \\ \left[\mathbf{h}^{(N)}_{PQ}\right]^\dagger & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^*$ $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^*$ $\mathbf{h}^{(N)}_{PQ} = -\left[\mathbf{h}^{(N)}_{PQ}\right]^*$	$\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$	$\mathfrak{u}(P+Q)$ $\setminus \mathfrak{o}(P+Q)$
D (0,+1,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^*$	$P(P+1)/2$	$\mathfrak{u}(P) \setminus \mathfrak{o}(P)$
C (0,-1,0)	$L = 2M$	$\frac{1}{2}\left\{\binom{L}{N} + \binom{M}{N/2}\right\}$	$\frac{1}{2}\left\{\binom{L}{N} - \binom{M}{N/2}\right\}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{h}^{(N)}_{PQ} \\ \left[\mathbf{h}^{(N)}_{PQ}\right]^\dagger & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^*$ $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^*$ $\mathbf{h}^{(N)}_{PQ} = -\left[\mathbf{h}^{(N)}_{PQ}\right]^*$	$\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$	$\mathfrak{u}(P+Q)$ $\setminus \mathfrak{o}(P+Q)$
AIII (0,0,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{0}_{PQ} \\ \mathbf{0}_{QP} & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^\dagger$ $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^\dagger$	$P^2 + Q^2$	$\mathfrak{u}(P) \oplus \mathfrak{u}(Q)$
BDI (+1,+1,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{0}_{PQ} \\ \mathbf{0}_{QP} & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^*$ $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^*$	$\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2}$	$(\mathfrak{u}(P) \setminus \mathfrak{o}(P))$ $\oplus$ $(\mathfrak{u}(Q) \setminus \mathfrak{o}(Q))$
CII (-1,-1,1)	$L = p + q$ $p = 2r \quad q = 2s$	$P = \sum_{a=1,3,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$ $A(B) = P/2$	$Q = \sum_{a=0,2,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$ $C(D) = \frac{Q}{2} \pm \sum_{a=0,2,\ldots}^{m} \frac{1}{2}\binom{r}{a/2}\binom{s}{(m-a)/2}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} & & \\ \left[\mathbf{h}^{(N)}_{AB}\right]^\dagger & \mathbf{h}^{(N)}_{BB} & \multicolumn{2}{c}{\mathbf{0}_{PQ}} \\ \multicolumn{2}{c}{\mathbf{0}_{QP}} & \mathbf{h}^{(N)}_{CC} & \mathbf{h}^{(N)}_{CD} \\ & & \left[\mathbf{h}^{(N)}_{CD}\right]^\dagger & \mathbf{h}^{(N)}_{DD} \end{pmatrix}$ $\mathbf{h}^{(N)}_{AA(CC)} = \left[\mathbf{h}^{(N)}_{AA(CC)}\right]^*$ $\mathbf{h}^{(N)}_{BB(DD)} = \left[\mathbf{h}^{(N)}_{BB(DD)}\right]^*$ $\mathbf{h}^{(N)}_{AB(CD)} = -\left[\mathbf{h}^{(N)}_{AB(CD)}\right]^*$	$\frac{A(A+1)}{2} + \frac{B(B+1)}{2} + AB$ $+ \frac{C(C+1)}{2} + \frac{D(D+1)}{2} + CD$	$(\mathfrak{u}(A+B)$ $\setminus \mathfrak{o}(A+B))$ $\oplus$ $(\mathfrak{u}(C+D)$ $\setminus \mathfrak{o}(C+D))$
CI (+1,-1,1)	$L = 2M$	$P = \sum_{a=1,3,\ldots}^{N-1} \binom{M}{a}\binom{M}{N-a}$ $A(B) = \begin{cases} P/2 & ; N/2 \text{ even} \\ \frac{P}{2} \mp \binom{M}{N/2} & ; N/2 \text{ odd} \end{cases}$	$Q = \sum_{a=0,2,\ldots}^{N} \binom{M}{a}\binom{M}{N-a}$ $C(D) = \begin{cases} \frac{Q}{2} \pm \binom{M}{N/2} & ; N/2 \text{ even} \\ Q/2 & ; N/2 \text{ odd} \end{cases}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} & & \\ \left[\mathbf{h}^{(N)}_{AB}\right]^\dagger & \mathbf{h}^{(N)}_{BB} & \multicolumn{2}{c}{\mathbf{0}_{PQ}} \\ \multicolumn{2}{c}{\mathbf{0}_{QP}} & \mathbf{h}^{(N)}_{CC} & \mathbf{h}^{(N)}_{CD} \\ & & \left[\mathbf{h}^{(N)}_{CD}\right]^\dagger & \mathbf{h}^{(N)}_{DD} \end{pmatrix}$ $\mathbf{h}^{(N)}_{AA(CC)} = \left[\mathbf{h}^{(N)}_{AA(CC)}\right]^*$ $\mathbf{h}^{(N)}_{BB(DD)} = \left[\mathbf{h}^{(N)}_{BB(DD)}\right]^*$ $\mathbf{h}^{(N)}_{AB(CD)} = -\left[\mathbf{h}^{(N)}_{AB(CD)}\right]^*$	$\frac{A(A+1)}{2} + \frac{B(B+1)}{2} + AB$ $+ \frac{C(C+1)}{2} + \frac{D(D+1)}{2} + CD$	-do-
DIII (-1,+1,1)	$L = 2M$	$P = \sum_{a=1,3,\ldots}^{N-1} \binom{M}{a}\binom{M}{N-a}$ $A(B) = \begin{cases} P/2 & ; N/2 \text{ even} \\ \frac{P}{2} \pm \binom{M}{N/2} & ; N/2 \text{ odd} \end{cases}$	$Q = \sum_{a=0,2,\ldots}^{N} \binom{M}{a}\binom{M}{N-a}$ $C(D) = \begin{cases} \frac{Q}{2} \pm \binom{M}{N/2} & ; N/2 \text{ even} \\ Q/2 & ; N/2 \text{ odd} \end{cases}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} & & \\ \left[\mathbf{h}^{(N)}_{AB}\right]^\dagger & \mathbf{h}^{(N)}_{BB} & \multicolumn{2}{c}{\mathbf{0}_{PQ}} \\ \multicolumn{2}{c}{\mathbf{0}_{QP}} & \mathbf{h}^{(N)}_{CC} & \mathbf{h}^{(N)}_{CD} \\ & & \left[\mathbf{h}^{(N)}_{CD}\right]^\dagger & \mathbf{h}^{(N)}_{DD} \end{pmatrix}$ $\mathbf{h}^{(N)}_{AA(CC)} = \left[\mathbf{h}^{(N)}_{AA(CC)}\right]^*$ $\mathbf{h}^{(N)}_{BB(DD)} = \left[\mathbf{h}^{(N)}_{BB(DD)}\right]^*$ $\mathbf{h}^{(N)}_{AB(CD)} = -\left[\mathbf{h}^{(N)}_{AB(CD)}\right]^*$	$\frac{A(A+1)}{2} + \frac{B(B+1)}{2} + AB$ $+ \frac{C(C+1)}{2} + \frac{D(D+1)}{2} + CD$	-do-

(Agarwala, Haldar, et al., 1606.05483)

Tenfold Way: N-Body Hamiltonians (N odd)

Class	L	P	Q	$\mathbf{H}^{(N)}$	$\dim \mathfrak{i}\mathcal{H}^{(N)}$	$\mathfrak{i}\mathcal{H}_{+}^{(N)}$
A (0,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^{\dagger}$	P^2	$\mathfrak{u}(P)$
AI (+1,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = [\mathbf{H}^{(N)}]^{*}$	$P(P+1)/2$	$\mathfrak{u}(P) \setminus \mathfrak{o}(P)$
AII (-1,0,0)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{h}_{PP}^{(N)} & \mathbf{h}_{PQ}^{(N)} \\ -\left[\mathbf{h}_{PQ}^{(N)}\right]^{*} & \left[\mathbf{h}_{PP}^{(N)}\right]^{*} \end{pmatrix}$	$P^2 + 2 \times \frac{P(P-1)}{2}$	$\mathfrak{u}(2P) \setminus \mathfrak{usp}(2P)$
D (0,+1,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = -\left[\mathbf{H}^{(N)}\right]^{*}$	$P(P-1)/2$	$\mathfrak{o}(P)$
C (0,-1,0)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{h}_{PP}^{(N)} & \mathbf{h}_{PQ}^{(N)} \\ \left[\mathbf{h}_{PQ}^{(N)}\right]^{*} & -\left[\mathbf{h}_{PP}^{(N)}\right]^{*} \end{pmatrix}$	$P^2 + 2 \times \frac{P(P+1)}{2}$	$\mathfrak{usp}(2P)$
AIII (0,0,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ \left[\mathbf{h}_{PQ}^{(N)}\right]^{\dagger} & \mathbf{0}_{QQ} \end{pmatrix}$	$2PQ$	$\mathfrak{u}(P+Q) \setminus (\mathfrak{u}(P) \oplus \mathfrak{u}(Q))$
BDI (+1,+1,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ \left[\mathbf{h}_{PQ}^{(N)}\right]^{T} & \mathbf{0}_{QQ} \end{pmatrix}$	PQ	$\mathfrak{o}(P+Q) \setminus (\mathfrak{o}(P) \oplus \mathfrak{o}(Q))$
CII (-1,-1,1)	$L = p + q$ $p = 2r \;\; q = 2s$	$P = \sum_{a=1,3,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$ $A(B) = P/2$	$Q = \sum_{a=0,2,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$ $C(D) = \frac{Q}{2}$	$\left(\begin{array}{c\|cc} \mathbf{0}_{PP} & \mathbf{h}_{AC}^{(N)} & \mathbf{h}_{AD}^{(N)} \\ & -\left[\mathbf{h}_{AD}^{(N)}\right]^{*} & \left[\mathbf{h}_{AC}^{(N)}\right]^{*} \\ \hline \text{h.c.} & \multicolumn{2}{c}{\mathbf{0}_{QQ}} \end{array}\right)$	PQ	$\mathfrak{usp}(P+Q) \setminus (\mathfrak{usp}(P) \oplus \mathfrak{usp}(Q))$
CI (+1,-1,1)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ \left[\mathbf{h}_{PQ}^{(N)}\right]^{*} & \mathbf{0}_{QQ} \end{pmatrix}$	$P(P+1)$	$\mathfrak{usp}(2P) \setminus \mathfrak{u}(P)$
DIII (-1,+1,1)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ -\left[\mathbf{h}_{PQ}^{(N)}\right]^{*} & \mathbf{0}_{QQ} \end{pmatrix}$	$P(P-1)$	$\mathfrak{o}(2P) \setminus \mathfrak{u}(P)$

(Agarwala, Haldar, et al., 1606.05483)