

# Turbulent Mixing of Tracers: A Perspective

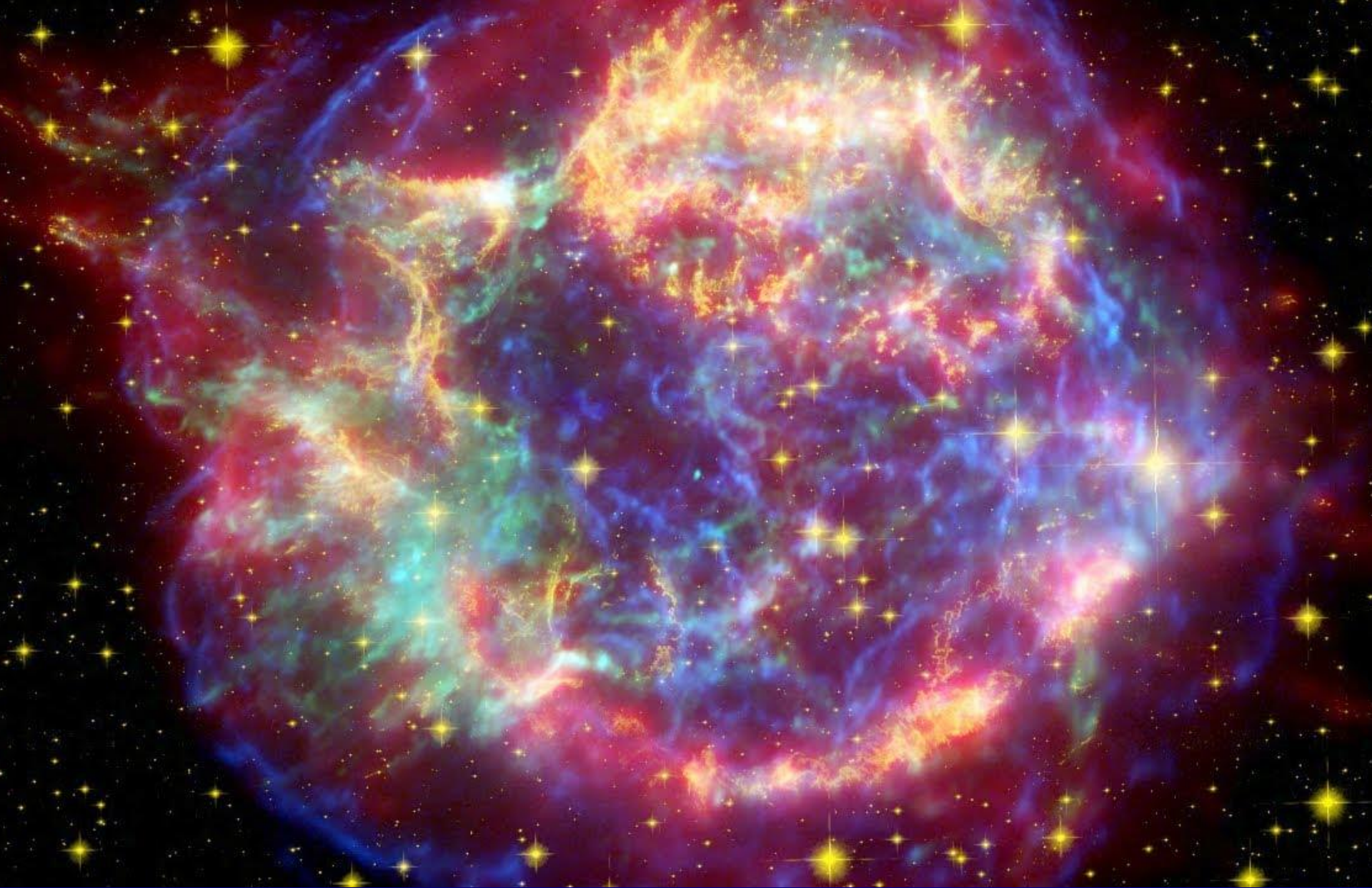
K.R. Sreenivasan

ICTS Program on  
Turbulence from Angstroms to Light Years:  
Chandrasekhar Lecture -- II



NEW YORK UNIVERSITY



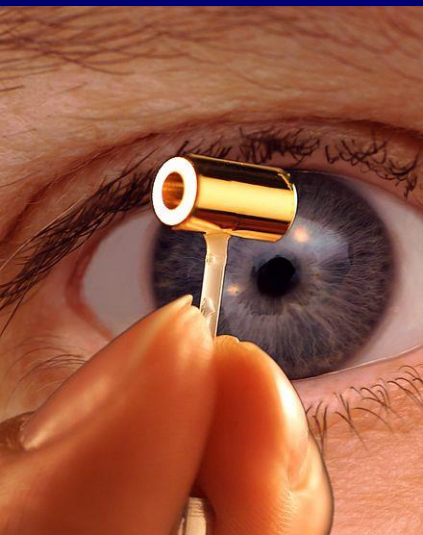
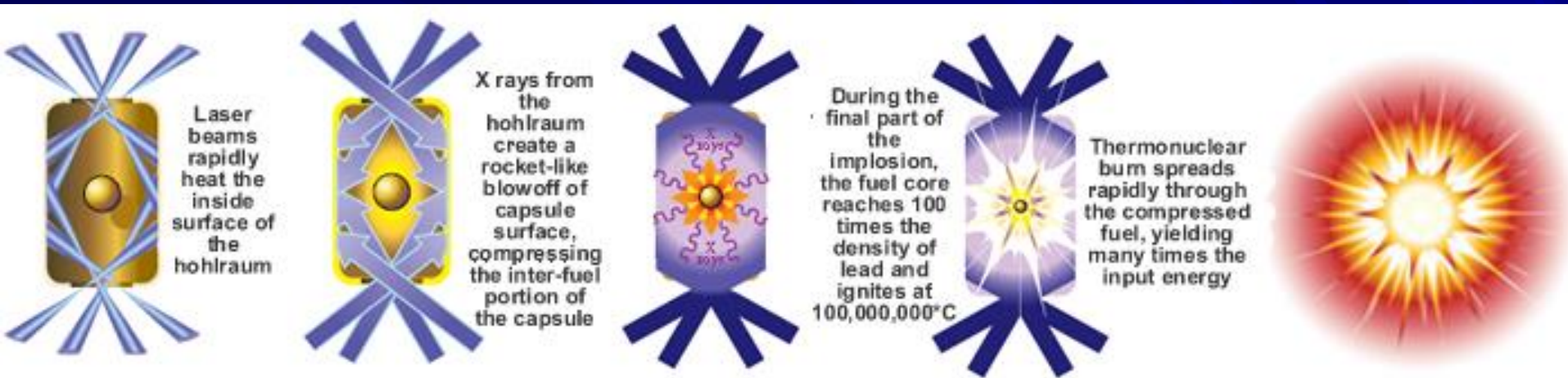


Supernova Cassiopeia A: This image from the orbiting Chandra x-ray observatory shows the youngest supernova remnant in the Milky Way. Courtesy of NASA



# Fluid dynamics of the national ignition facility

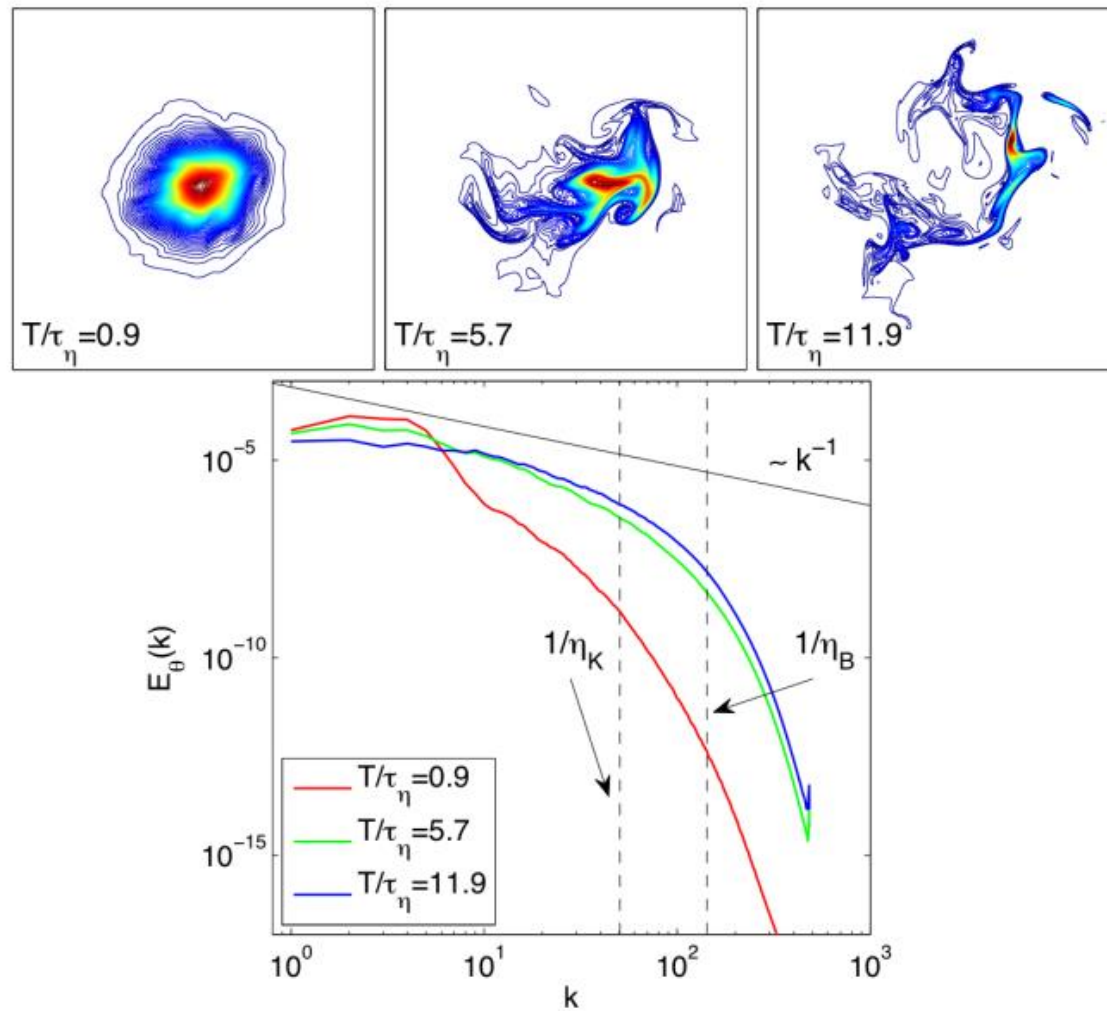
A few milligrams of fusion fuel, typically a mix of deuterium and tritium



500 terawatts, shining on the fuel pellet for a few picoseconds

- Rayleigh-Taylor instability, Kelvin-Helmholtz instability are quite common in astrophysical contexts, where rotation, magnetic field, density changes are felt simultaneously. Chandra studied these problems and also thermal convection. Several speakers at this meeting have worked in these areas.
- Chandra codified these effects on a grand scale. He wrote some 50 technically sophisticated papers on stability, reworked and condensed them in his book, *Hydrodynamic and Hydromagnetic Stability*, Oxford (1961) and Dover.
- For more discussion of such complex cases, see Abarzhi & Sreenivasan (2010) and Abarzhi et al. (2013)---Trans. Roy. Soc. Lond.
- All these instabilities are precursors to turbulent mixing, which is the main topic to be covered in this lecture. PERSPECTIVE

Will consider the simplest case of mixing of a passive tracer by an incompressible flow.



Turbulent mixing of tracers

## Advection diffusion equation

$$\partial\theta/\partial t + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta$$

$\theta(\mathbf{x};t)$ , the additive;  $\kappa$ , its diffusivity (usually small);  $\mathbf{u}(\mathbf{x};t)$ , the advection velocity which solves NS=0; no source terms

The equation is linear with respect to  $\theta$ .

BCs (perhaps mixed) are almost always linear as well.

Linearity holds for each realization but the equation is statistically nonlinear because of  $\langle \mathbf{u} \cdot \nabla \theta \rangle$ , etc.

## Langevin equation

$$d\mathbf{X} = \mathbf{u}[\mathbf{X}(t);t] dt + (2\kappa)^{1/2} d\chi(t), \mathbf{X}(t=0) = \mathbf{x}_0$$

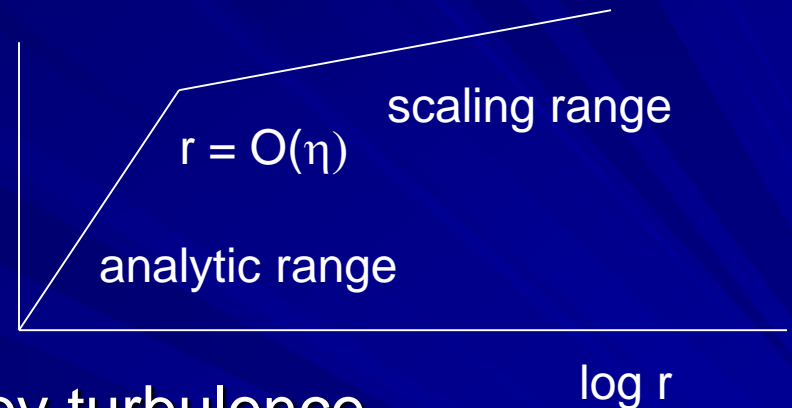
$\chi(t)$  = vectorial Brownian motion, statistically independent in its three components

What aspects of the NS solutions makes the problem difficult?

The turbulent velocity field is analytic for  $r < O(\eta)$ , and is

Hölder continuous, or “rough,” in the scaling range ( $\Delta_r u \sim r^h$ ,  $h < 1$ ).

a quantity such as a structure function (log)



$h = 1/3$  for Kolmogorov turbulence  
but has a distribution in practice.

“multiscaling”

Parisi & Frisch (1985); Chen et al. *JFM* **533**, 183-192 (2005)

If  $\Delta_r u \sim r^h$  ( $h < 1$ ), we get  $r(t) \sim t^{1/(1-h)}$ , and Lagrangian paths separate explosively and are not unique; this introduces various complexities.

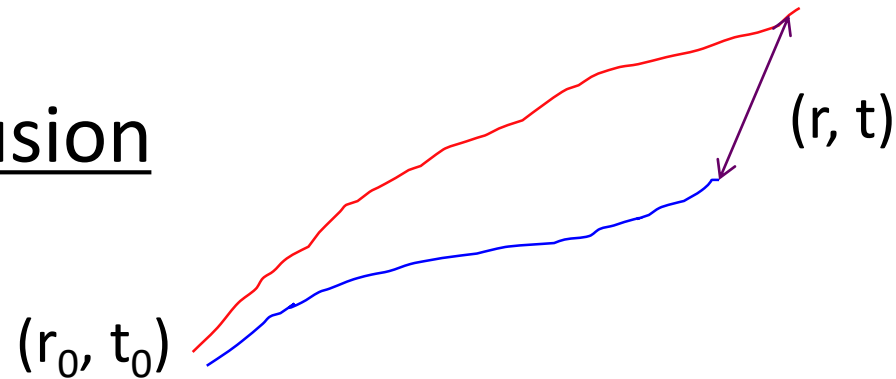


- $\Delta u_r \sim r^h$
- $p_h(r) \sim r^{F(h)}$
- $\langle (\Delta u_r)^n \rangle \sim \int dh r^{nh+F(h)}$
- Define  $\zeta_n = \min_h [nh + F(h)]$
- Perform the integral using saddle-point method to yield
- $\langle \Delta u_r^n \rangle \sim r^{\zeta_n}$



## Richardson's law of diffusion

$$\langle r^2 \rangle = C_R \varepsilon (t - t_0)^3$$



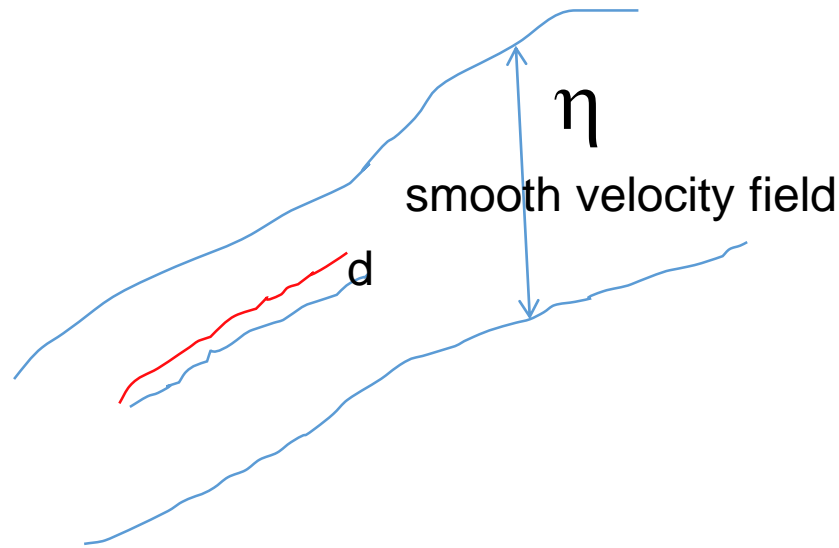
Not present in the formula is  $r_0$ : Two particles that start exactly at the same location at  $t_0$  can separate by a finite distance for any time  $t > t_0$ .

***Non-uniqueness: “Spontaneous stochasticity”  
e.g., Bernard (2000), Eyink & Drivas (2015)***

Note 1: The explosive separation in Richardson's law (non-unique trajectories) does not violate short-time uniqueness of solutions for initial value problems because the velocity is “rough”.

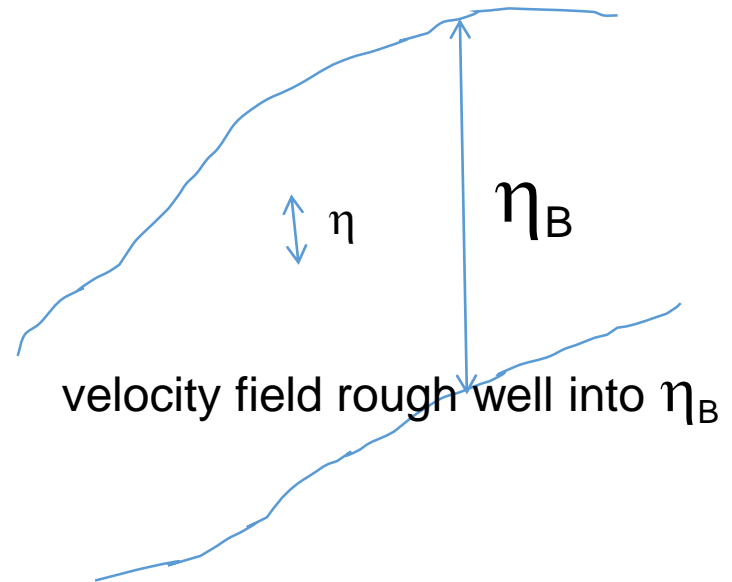
Note 2: Contrast with chaos:  $r(t) = r_0 \exp\{\lambda(t - t_0)\}$

$Re \rightarrow (Sc \gg 1)$



Anomaly begins to hold  
only after  $t > (v/\varepsilon)^{1/2} \log Sc$ .

$Re \rightarrow (Sc \ll 1)$



Spread of particles does not  
depend on  $\kappa$ : anomalous

Massive parallelism, with  $O(10^5 - 10^6)$  CPU cores, has made simulation to require a different type of expertise and support

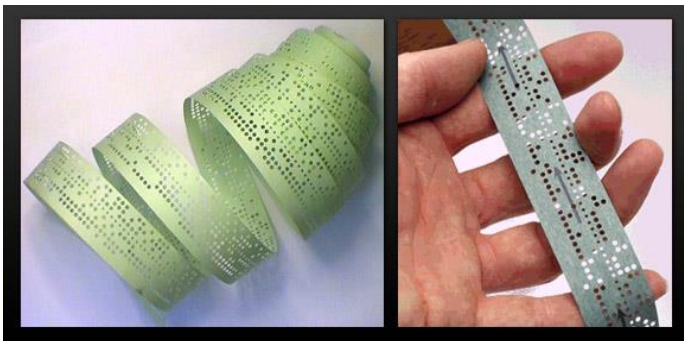
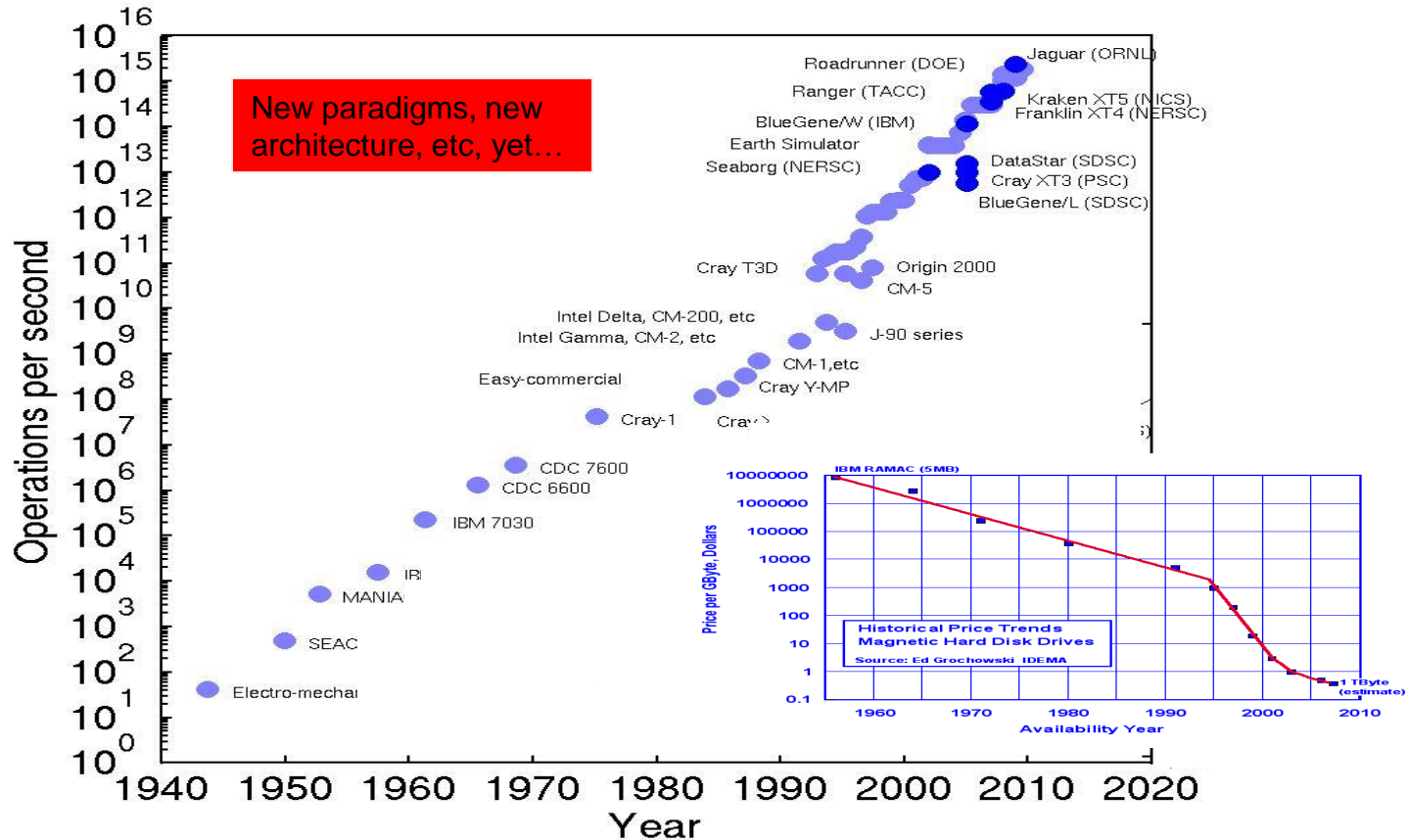
DNS

P.K. Yeung  
Diego Donzis

$8 < R_\lambda < 650$   
 $1/512 < Sc < 1024$

Toshi Gotoh  
Jörg Schumacher

Different forcing  
schemes



> 10 Petaflops  
with Exaflop machines by ~2020 (20 MW?)



Classical phenomenology  
and the “universality” in  
second-order statistical  
quantities

$\log E_\theta(k)$

*Inertial-convective range* (Obukhov, Corrsin)

$$k^{-5/3}$$

$\Pi_\theta$

From Gotoh and Yeung (2012)

*Viscous-convective range*

$$k^{-1}$$

$\Pi_\theta$

Batchelor  
Kraichnan

$$k^{-17/3}$$

*Inertial-diffusive range*

(Batchelor, Townsend & Howell)

$\bar{\chi}_{in}$

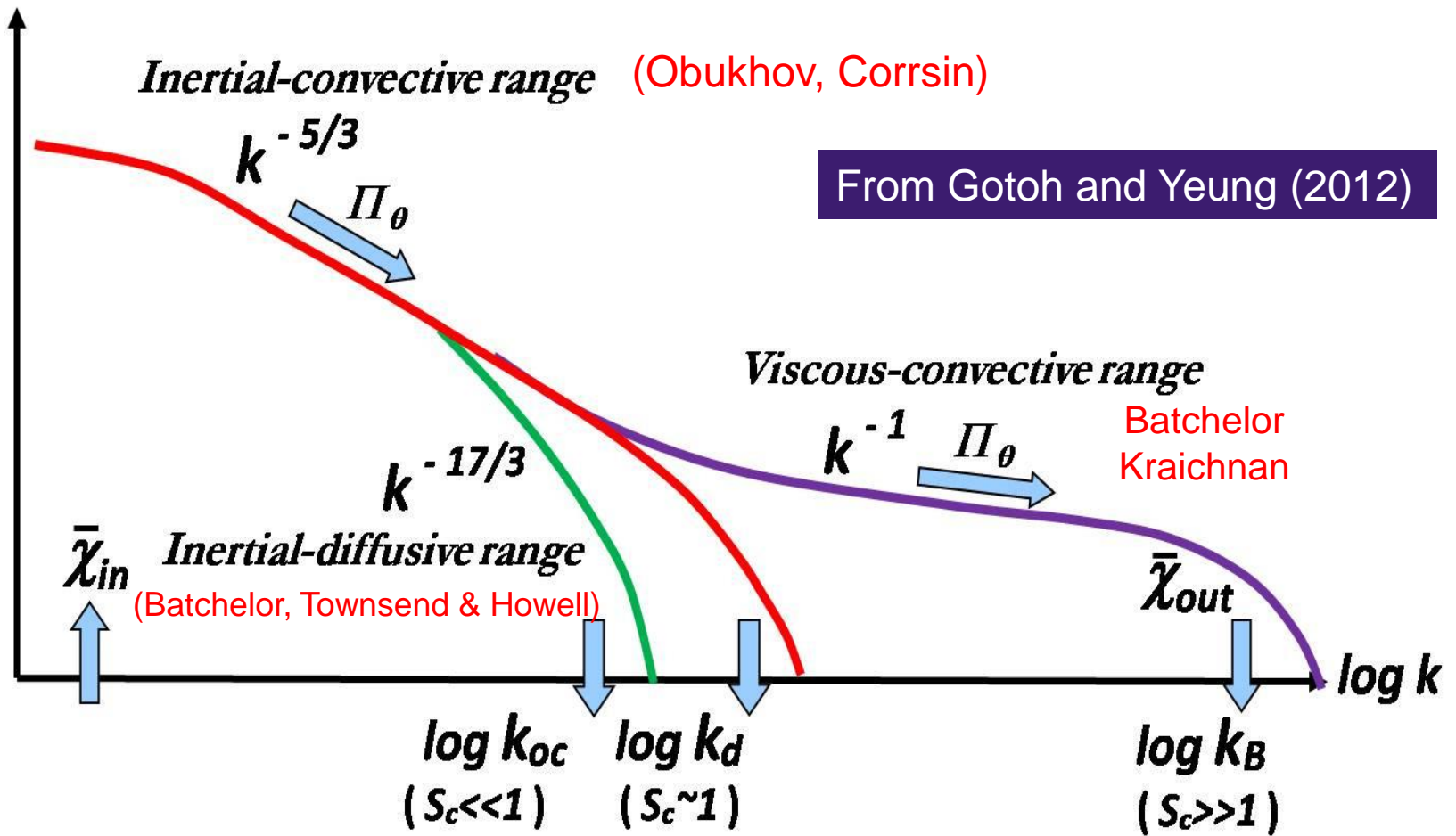
$\bar{\chi}_{out}$

$\log k_{oc}$   
( $Sc \ll 1$ )

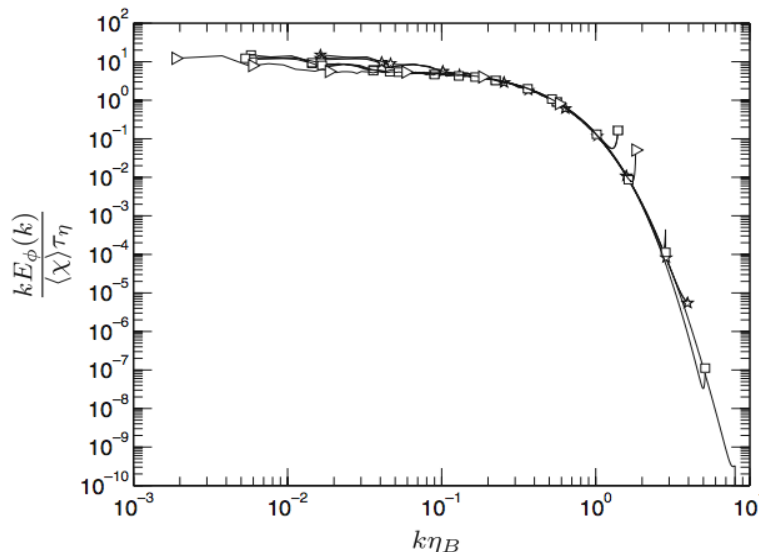
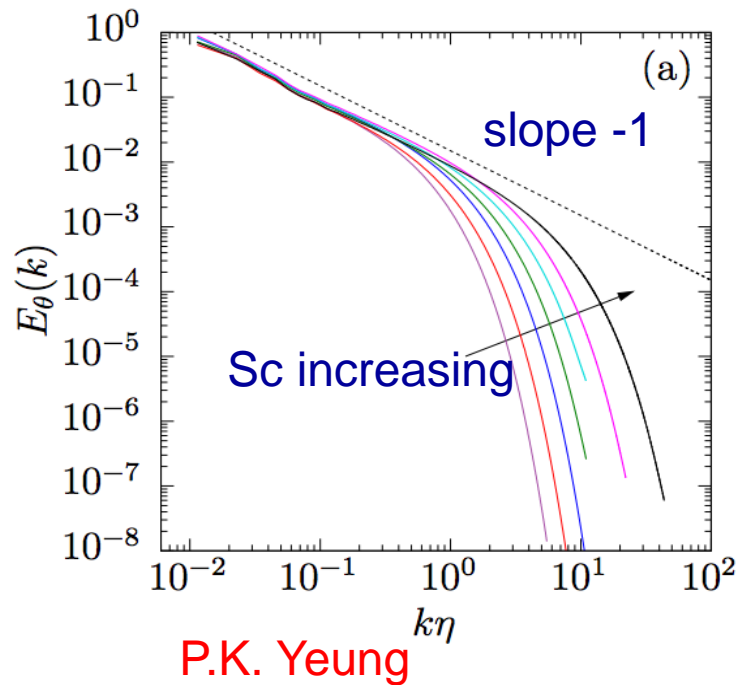
$\log k_d$   
( $Sc \sim 1$ )

$\log k_B$   
( $Sc \gg 1$ )

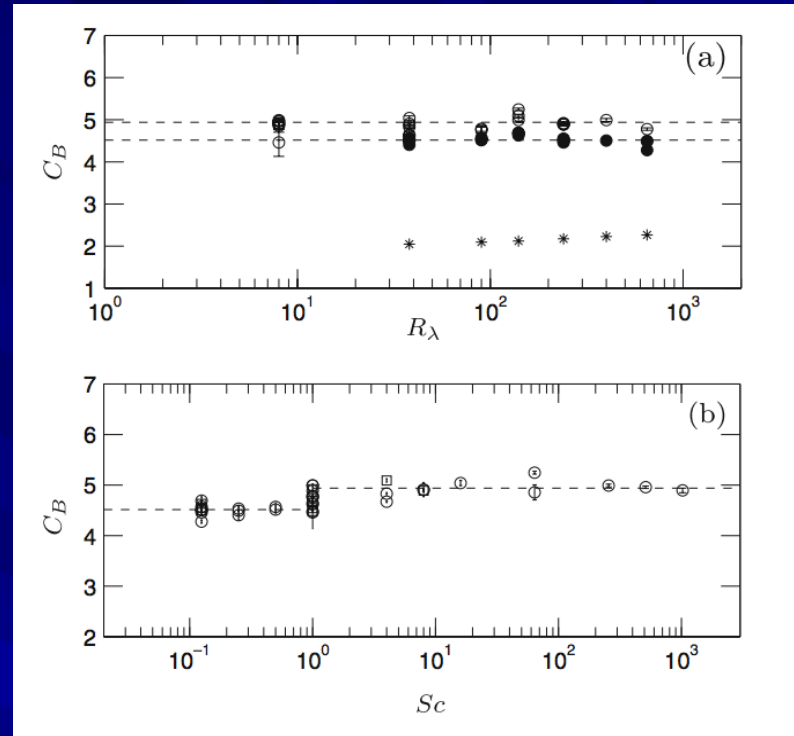
$\log k$



## The Batchelor regime



Reynolds number:  $Re \gg 1$   
 Schmidt number,  $Sc = \nu/\kappa \gg 1$



Batchelor (1956)

$$E_\theta(k) = C_B \kappa(\nu/\varepsilon)^{1/2} k^{-1} \exp[-q(k\eta_B)^2]$$

Kraichnan (1968)

$$E_\theta(k) = C_B \kappa(\nu/\varepsilon)^{1/2} k^{-1} [1 + (6q)^{1/2} k\eta_B \exp(-(6q)^{1/2} k\eta_B)]$$



# The Yaglom relation (1949)

$$\langle \Delta_r u (\Delta_r \theta)^2 \rangle = -(2/3) \langle \chi \rangle r$$

## Refined similarity hypothesis

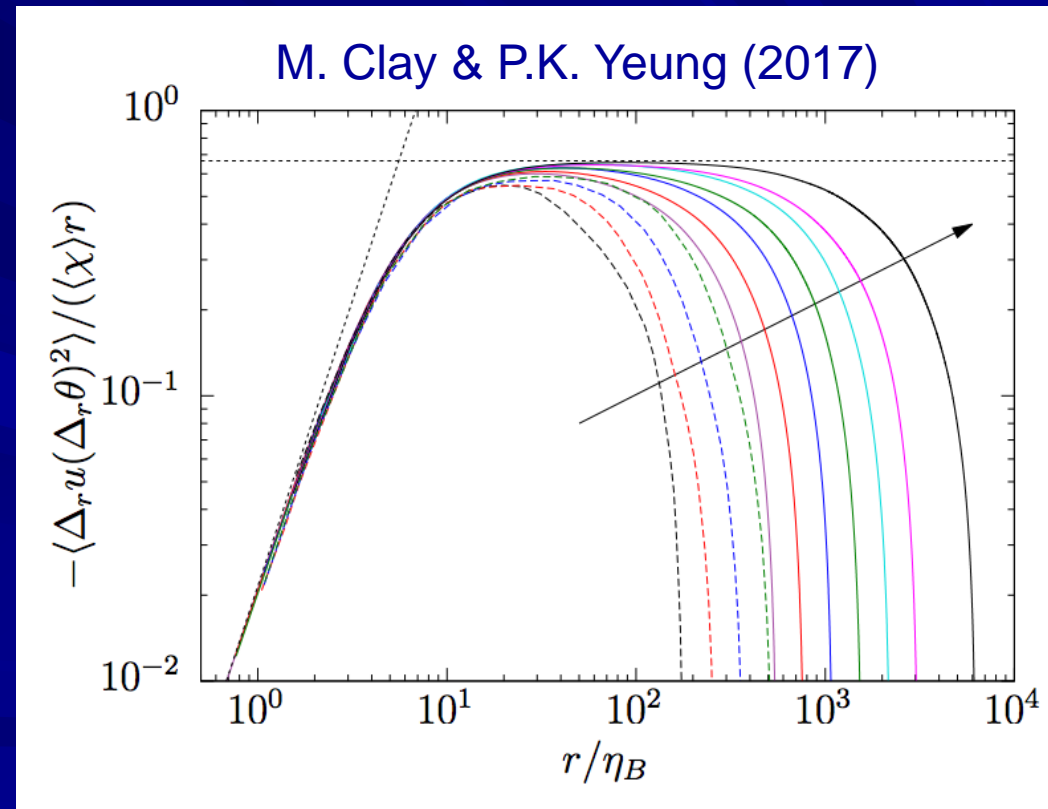
- G. Stolovitzky, P. Kailasnath & KRS, JFM 297, 275 (1995)

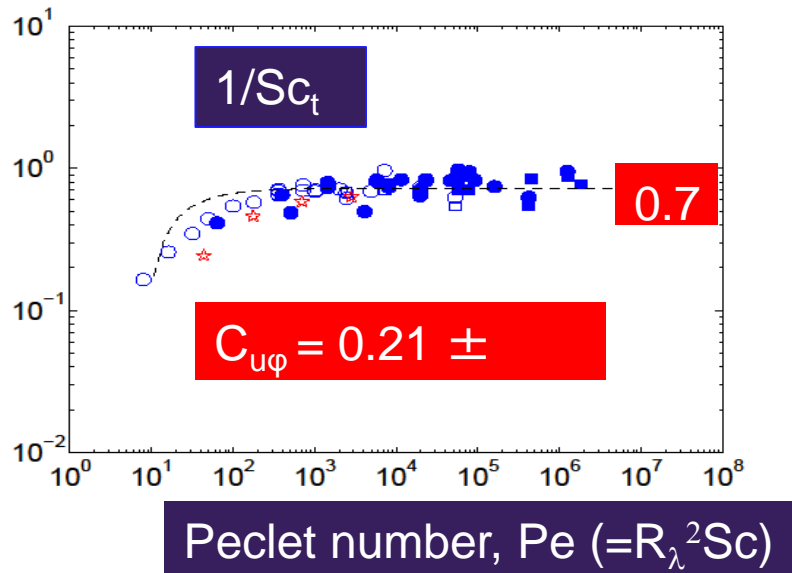
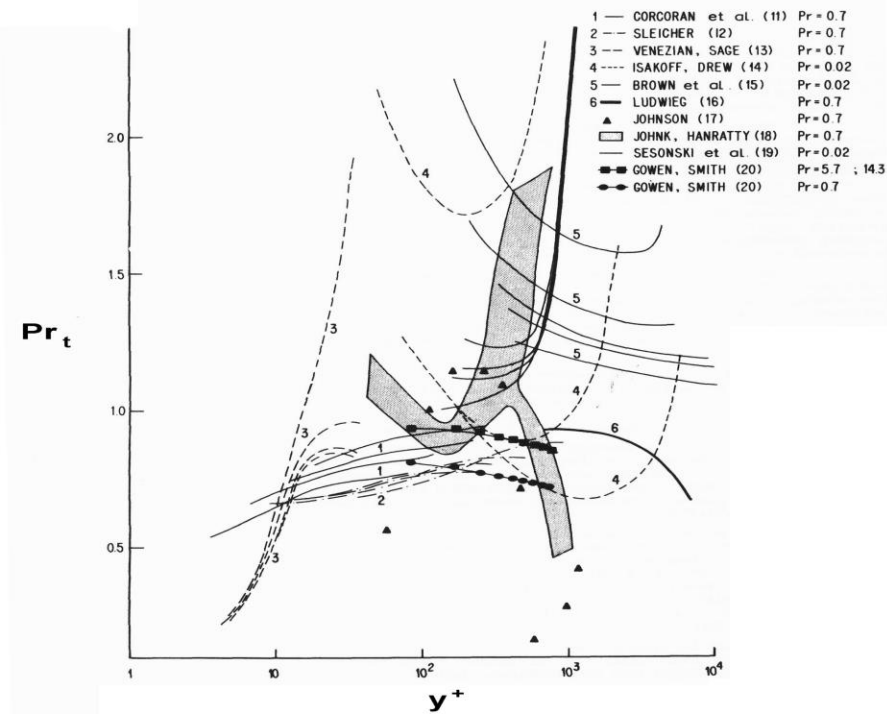
## Extension to non-stationary forcing conditions

- L. Danaila, F. Anselmet, T. Zhou & R.A. Antonia, JFM 391, 359 (1999)
- P. Orlandi & R.A. Antonia, JFM 451, 99 (2002): DNS

## Experiment

- G. Stolovitzky, P. Kailasnath & KRS, JFM 297, 275 (1995)
- L. Midlarsky, JFM 475, 173 (2003): Experiment





## Experiment

Homogeneous shear flows  
Boundary layers, Jets, Wakes

Direct Numerical Simulations  
(D. Donzis et al., 2014)

$$8 < R_\lambda < 650$$

$$1/512 < Sc < 1024$$

## Dimensional Theory

Flux spectrum

$$E_{u\varphi}(k) = C_{u\varphi} G_{<\varepsilon>}^{1/3} k^{-7/3}$$

in the inertial convection range  
(following Lumely 1964)

Using  $\langle u\varphi \rangle = -\int E_{u\varphi}(k) dk$  (with appropriate limits),

we get

$$1/Sc_t = (10/3) C_{u\varphi} (1 - 1/Pe)$$

# Anomalous behaviors



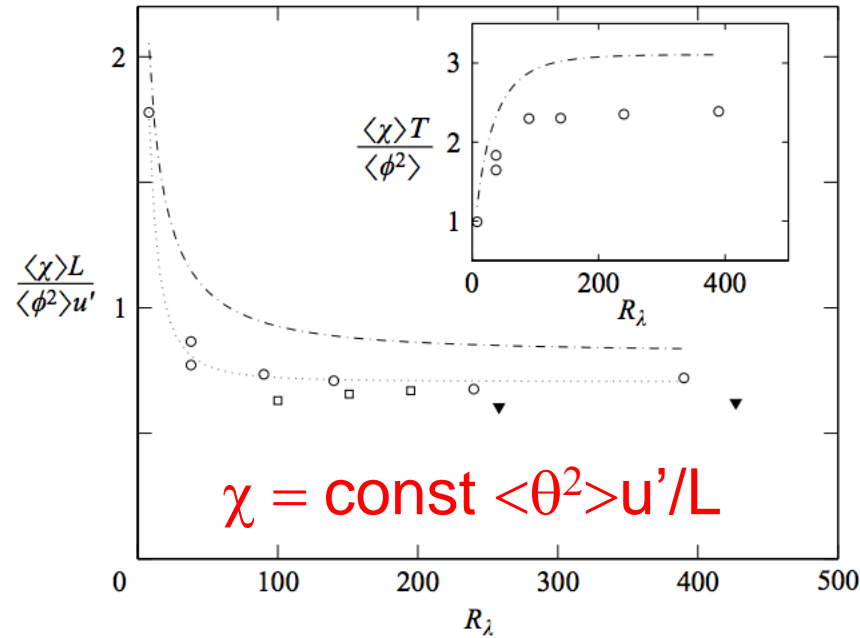


FIGURE 2. Scalar dissipation rate normalized with  $L/u'$  for  $Sc=1$ .  $\circ$ , present data;  $\blacktriangledown$ , Wang *et al.* (1999);  $\square$ , Watanabe & Gotoh (2004). Dotted line: equation (3.1) as the best fit for the present data. Dash-dotted line: theoretical prediction of (3.12), which will be described towards the end of §3.2. Inset shows the present data using the normalization of  $T$  instead of  $L/u'$ , as well as (3.12). While the asymptotic constancy holds for both normalizations, the direction of approach of this constancy is different.

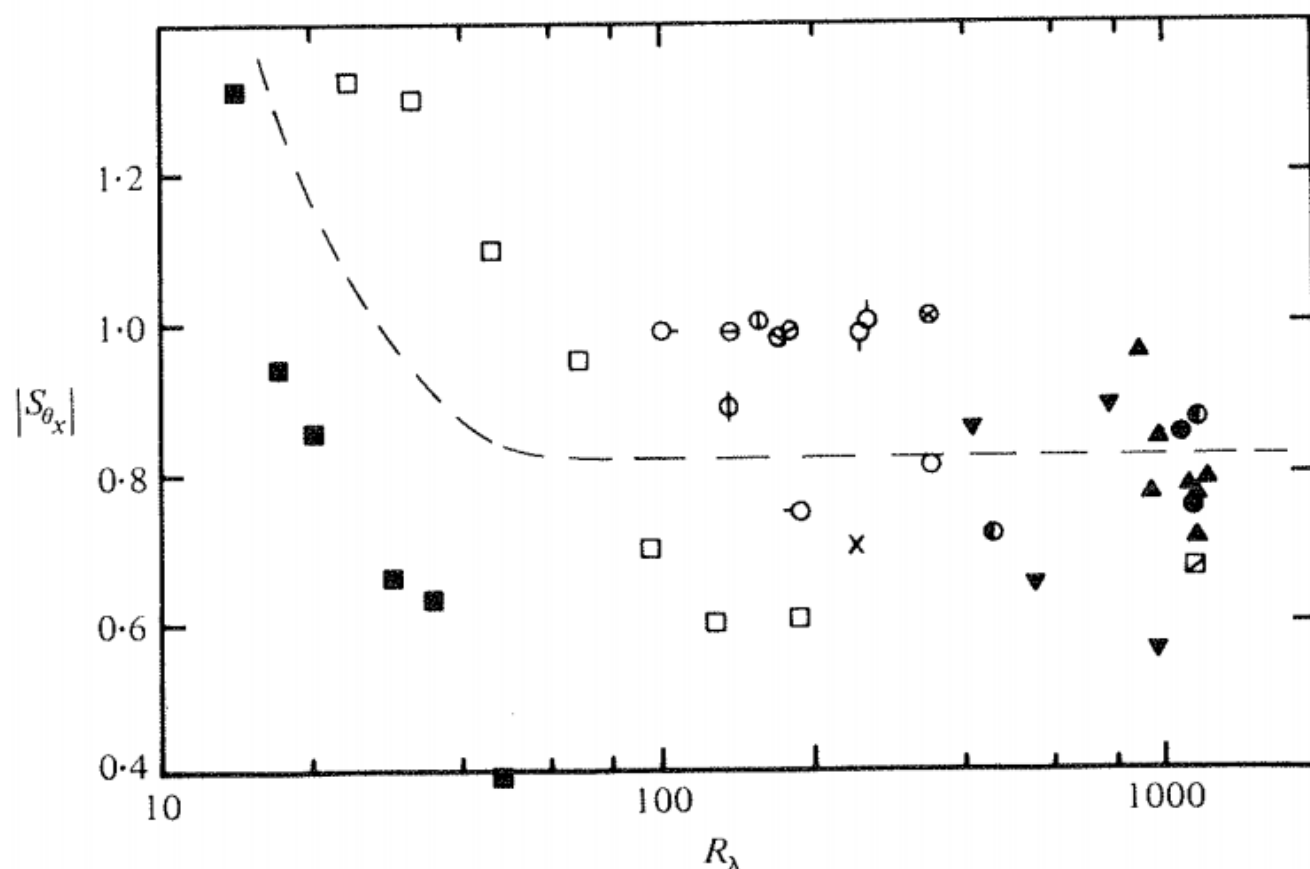
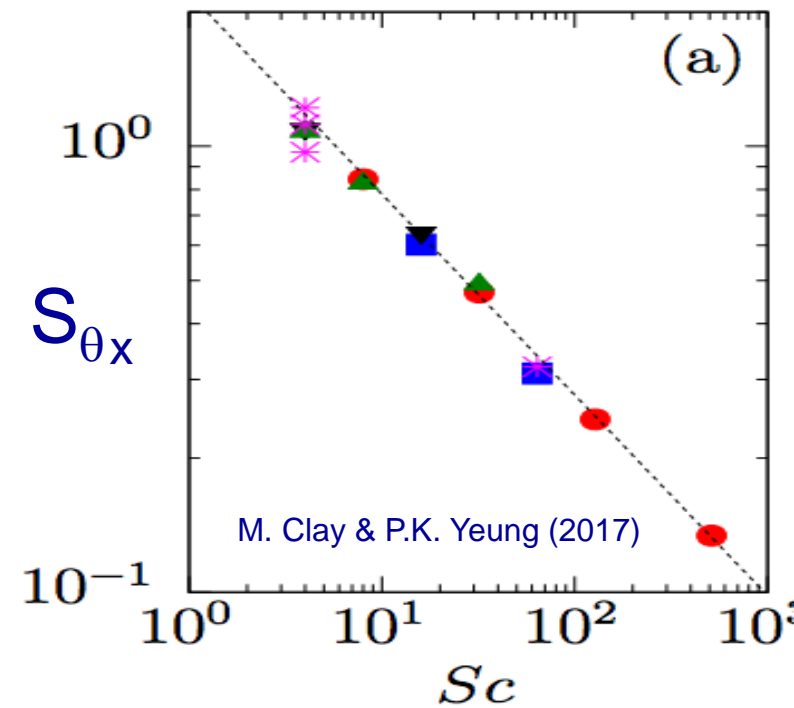
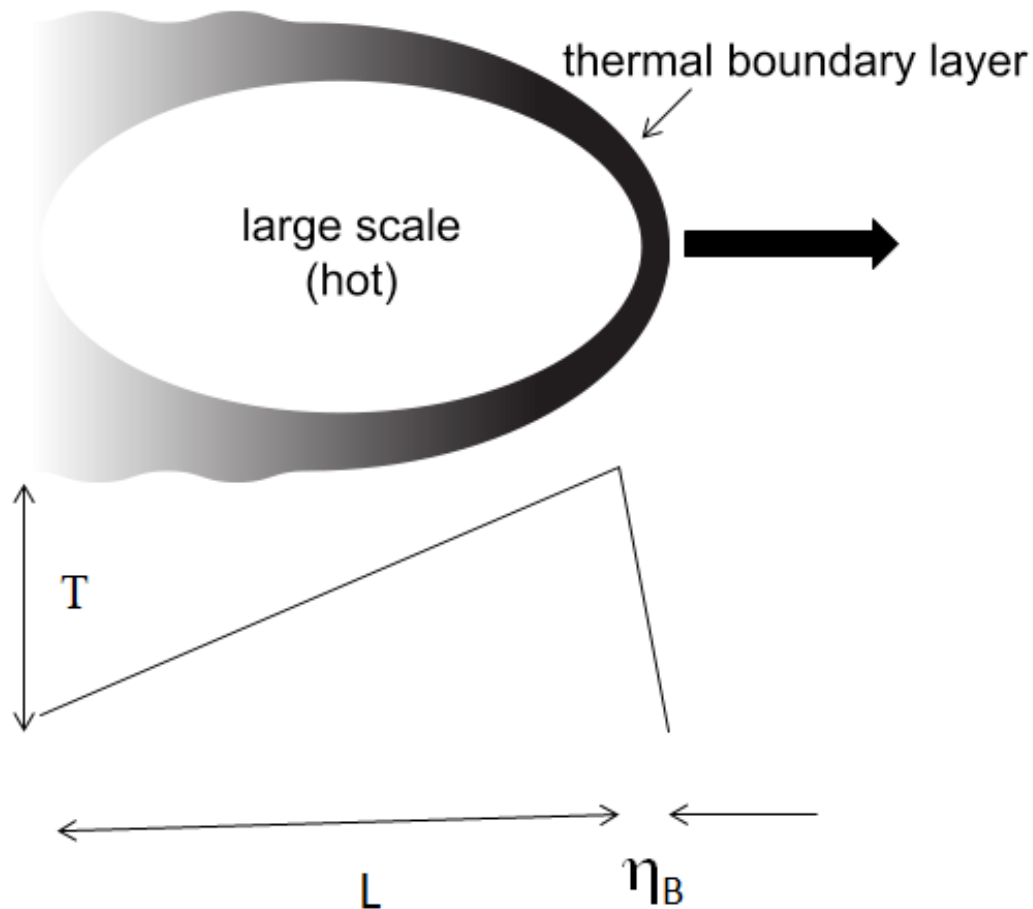
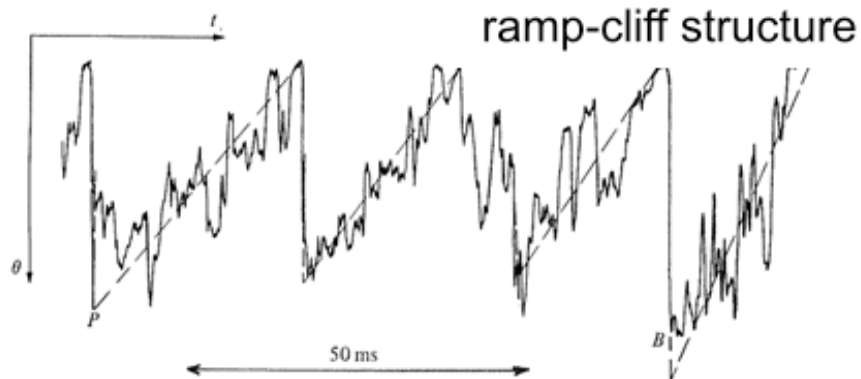


FIGURE 1. Variation of  $|S_{\theta_x}|$  with  $R_\lambda$ . Gibson *et al.* (1977):  $\circ$ , atmospheric boundary layer;  $\bullet$ , heated jet;  $\bullet$ , heated wake;  $\otimes$ , cooled wake,  $\blacktriangle$ , Mestayer *et al.* (1976), heated boundary layer.  $\blacktriangledown$ , Gibson *et al.* (1970), atmospheric boundary layer (corrected for velocity sensitivity);  $\times$ , Antonia & Van Atta (1975), heated jet. Freymuth & Uberoi (1971, 1973):  $\square$ , heated two-dimensional wake;  $\blacksquare$ , heated axisymmetric wake.  $\ominus$ ,  $\oplus$ ,  $\otimes$ ,  $\oslash$ , Sreenivasan, Antonia & Danh (1977), heated boundary layer.  $\boxdot$ , our unpublished data, atmospheric surface layer.  $\circ$ ,  $\odot$ ,  $\phi$ ,  $\ominus$ ,  $\ominus$ , present data, axisymmetric heated jet,  $\eta = 0, 0.89, 1.15, 1.48$  and  $1.63$  respectively. ---, suggested mean trend.



$$S_{\theta_x} = \frac{(T/\eta_B)^3 \times \eta_B/L}{\langle \theta_x^2 \rangle^{3/2}}$$

Taking  $\langle \theta_x^2 \rangle$  from the anomalous dissipation plot, we can show that

$$S_{\theta_x} = \text{const } Re^0 Sc^{1/2}$$



## Inertial-convective region

$$\langle \Delta_r \theta^2 \rangle \sim r^{\xi_2}$$

Standard “theory” gets  $\xi_2$  by assuming that the structure functions obey the same symmetries as the equations. They (dimensional arguments) yield  $\xi_2 = 2/3$

Two questions arise:

1.  $\ln \langle \Delta_r \theta^4 \rangle \sim r^{\xi_4}$

the same argument yields  $\xi_4 = 2\xi_2$  (in general,  $\xi_{2n} = n\xi_2$ ).

This normal scaling is  $\xi_{2n} = 2n/3$ .

Measurements have shown that  $\xi_4 < 2\xi_2$  (or generally  $\xi_{2n} < n\xi_2$ )

What is the source of these “anomalous exponents”?

The exponent for each order order has to be determined on its own merit.

2. We have  $\langle \Delta_r u (\Delta_r \theta)^2 \rangle = -(2/3) \langle \chi \rangle r$

# Kraichnan model

(motivated by small-scale intermittency)

- R.H. Kraichnan, *Phys. Fluids* **11**, 945 (1968); *Phys. Rev. Lett.* **72**, 1016 (1994)
- Review: G. Falkovich, K. Gawedzki & M. Vergassola, *Rev. Mod. Phys.* **73**, 913 (2001)

## Surrogate Gaussian velocity field

$$\langle u_i(\mathbf{x};t)u_j(\mathbf{y};t') \rangle = |\mathbf{x}-\mathbf{y}|^{2-\gamma} \delta(t-t')$$

$\gamma = 2/3$  recovers Richardson's diffusion

Forcing for stationarity:

$$\langle f_\theta(\mathbf{x};t)f_\theta(\mathbf{y};t') \rangle = C(r/L)\delta(t-t')$$

$C(r/L)$  is non-zero only on the large scale, decays rapidly to zero for smaller scale.

## OUTSTANDING CHALLENGES Turbulence nears a final answer

From **Uriel Frisch** at the Observatoire de la Côte d'Azur, Nice, France

zero modes, shape geometry, etc.

examining the turbulent wakes created behind obstacles placed in the path of a fluid, he found that there are three key stages to turbulent flow. Turbulence is first generated near an obstacle. Long-lived "eddies" – beautiful whirls of fluid – are then formed. Finally, the smallest-scale eddies are formed, which have shown to be the most difficult to understand.

However, it was not until the early 19th century that Claude Navier was able to write down the equations that govern the motion of a fluid. Navier realized that the earlier equations of Leonhard Euler for ideal flow had to be modified to account for the viscosity of a fluid. A few decades later, Adhémar de Saint-Venant noticed that turbulent flow – for example, the flow of water in a pipe or the flow of air over a wing – was more complicated than the flow found, for instance, in a capillary. It turns out that the turbulent transport of momentum and energy is much more efficient than the predictions based on Maxwell's kinetic theory for transport in laminar flow. Indeed, if it were not for turbulence, pollution in our cities would linger for millennia, the heat generated by nuclear reactions deep inside stars would not be able to escape within an acceptable time, and the weather could be predicted far into the future.

Modelling turbulent transport thus became – and remains to this day – a major challenge. The first attempt goes back to a student of Saint-Venant called Joseph Boussinesq, who introduced what is now known as the "eddy viscosity" approach. He argued that the turbulent transport proceeds as a random walk, and the typical step is the size of eddies.

Since then, vastly improved models that can deal with increasingly complex flows of the type found in nature have been developed by Ludwig Prandtl, Lev Landau, and Andrei Kolmogorov, and many others.

This progress has depended on an ever-increasing theoretical and experimental understanding of the physics of turbulence, and I can do no more than point to the crucial contributions of Lord Kelvin, Osborne Reynolds, Geoffrey Ingram Taylor, Jean Leray, Theodor von Kármán and many others. I will thus turn to one of the major challenges in the field, which is to understand what is known as fully developed turbulence (FDT) in the case of a high Reynolds number flow. The key question is: what are the statistical properties of the flow? The answer is: the flow is highly irregular, with eddies of many sizes of the fluid's internal and viscous forces. The molecular viscosity then acts to dissipate the energy of the flow, and those at the smallest scale are the most difficult to understand.

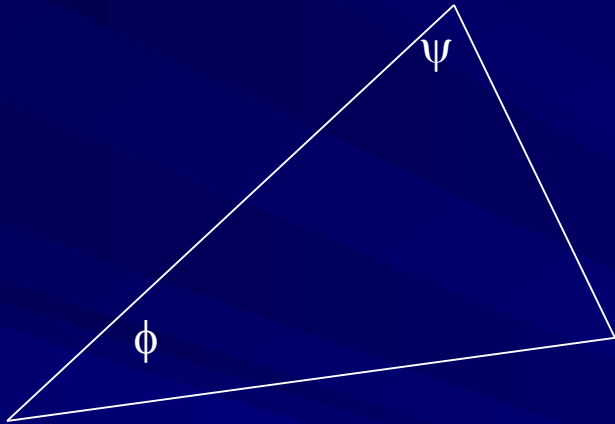
A coherent picture of FDT first emerged around the middle of this century, thanks to work by Kolmogorov, Lewis Fry Richardson, Lars Onsager and, again, many others. In the language of modern physics, it was postulated that the Navier-Stokes equations – which describe the hydrodynamic properties of a fluid – have solutions that would display the same scale invariance as the equations themselves, but in a statistical sense. For example, the average of the velocity difference across a certain distance raised to a certain power would be proportional to that distance raised to an exponent proportional to the power. The actual exponents can then be obtained by a simple dimensional argument, rather than having to solve the Navier-Stokes equations themselves.

A range of increasingly accurate experiments have been carried out to study FDT. These started with work by George Batchelor and Alan Townsend in the 1930s, right through to new table-top facilities that use low-temperature helium flowing between counter-rotating disks. New data-processing techniques that can measure scaling exponents with good accuracy have also been developed, as have advanced numerical simulations, the importance of which was first perceived by the mathematician John von Neumann.

The evidence is that the assumed scale invariance is actually broken and that fully developed turbulence is "intermittent". In other words, the flow is not statistically self-similar, and the dissipation of energy has fractal properties – in other words energy is dissipated in a cascade of energy transfers to smaller and smaller scales. Roberto Benzi, Benoît Mandelbrot, Steven Orszag, Patrick Saffman and many others have been instrumental in developing this work.

For many years, only models that were rather loosely connected with the traditional picture of turbulence were available. The first models that were truly developed were developed by Kolmogorov and colleagues in the 1960s, while in the 1980s the Russian physicist Vladimir Yakovlevich Geurts developed a model that successfully predicted that intermittency and anomalous scaling exponents were present in the flow. A few years ago Robert Kraichnan proposed that intermittency and anomalous scaling exponents are actually due to the presence of "zero modes" in the flow. These are linear – namely for a passive scalar, such as a pollutant advected by a scale-invariant turbulent velocity, which is not intermittent itself. This phenomenon can be studied by numerical simulations (see figure).

Methods borrowed in part from modern field theory have recently led to a real breakthrough for Kraichnan's model. For the first time we have a theoretical prediction derived from first principles that can predict the values of the anomalous exponents. The anomalous corrections would be predicted to arise through the presence of "null space" in the evolution of the flow. This is a new exponent that characterizes the roughness of the prescribed velocity (as Krzysztof Gawedzki and Antti Kupiainen have done) or using the inverse of the space dimension (with the work of Mikhail Chertkov, Gregory Falkovich, Vladimir L'vov, and Andrei Kolokolov). Non-perturbative methods have been used to study the extension of this model to the case of a linear problem of intermittency, which is being actively pursued. Optimism that fully developed turbulence has been understood in a few years' time. But many more years may be needed to truly understand all of the complexity of turbulent flow – a problem that has been challenging physicists, mathematicians and engineers for at least half a millennium.

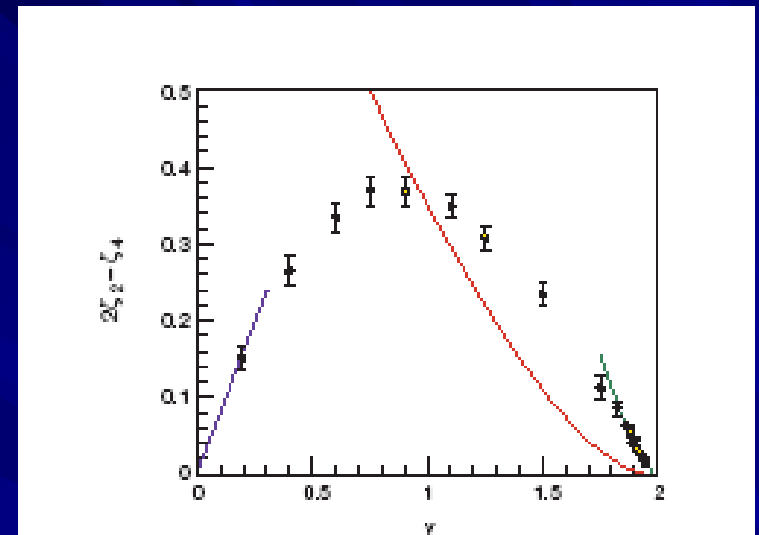
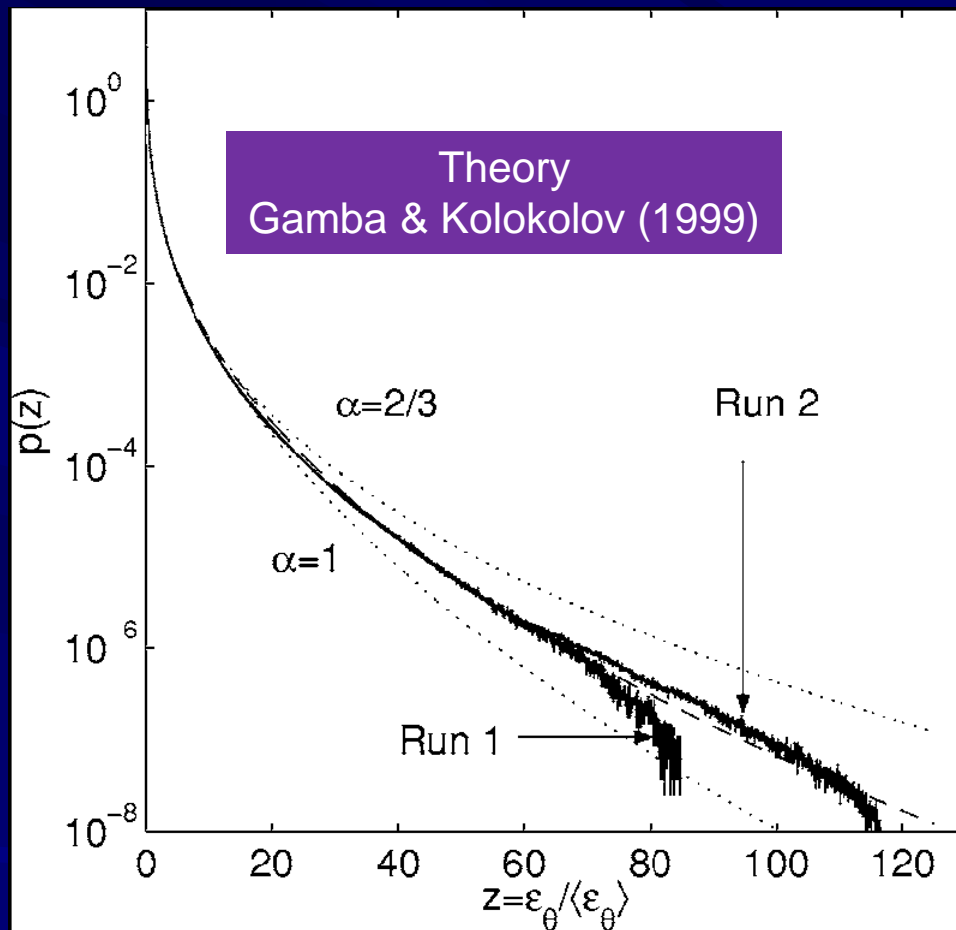


- Advection changes the shape and size
- $R$  = geometric mean of the lengths, say
- After rescaling by  $R$ , we can get an idea of shape fluctuations looking  $f(\psi, \phi)$ .
- For the Kraichnan model, the three point statistics are given by those trajectories for which
- $R^{\zeta_3} f(\psi, \phi) = \text{constant}$ .

The important qualitative lesson from the work on the Kraichnan model is that certain types of Lagrangian characteristics, conserved only on the average, determine the statistical scaling.

Breaking of symmetry yields the conservation of flux condition. Are there are other statistical conservation laws whose symmetry breaking provides the basis for determining the exponents of higher orders?

$$2\zeta_2 - \zeta_4$$



A measure of anomalous scaling,  $2\zeta_2 - \zeta_4$ , versus the index  $\gamma$ , for the Kraichnan model. The circles are obtained from Lagrangian Monte Carlo simulations. The results are compared with analytic perturbation theories (blue, green) and an ansatz due to Kraichnan (red).

Mixing process itself imprints features independent of the velocity field!



# One consequences of fluctuations

## Traditional definitions

$$\langle \eta \rangle = (\nu^3 / \langle \varepsilon \rangle)^{1/4}$$

$$\langle \eta_B \rangle = \langle \eta \rangle / Sc^{1/2}$$

$$\langle \tau_d \rangle = \langle \eta_B \rangle^2 / \kappa$$

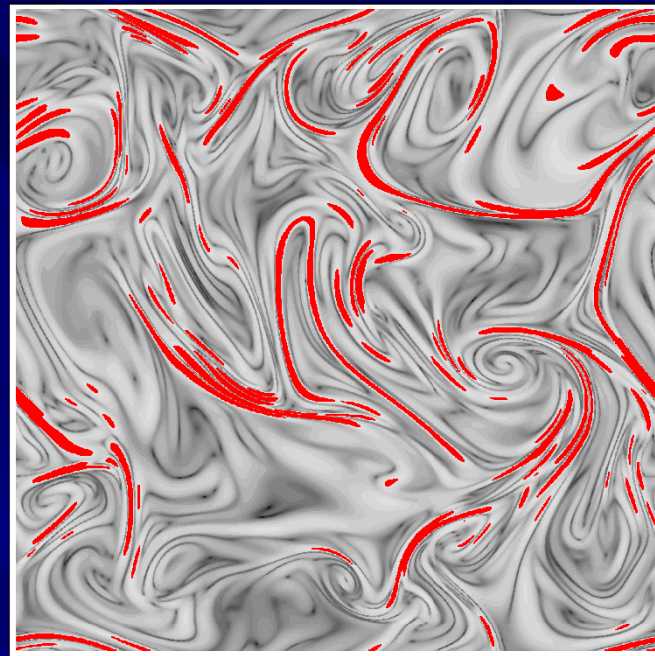
## Local scales

$\eta = (\nu^3 / \varepsilon)^{1/4}$ , or define  $\eta$  through  $\eta \Delta_\eta u / \nu = 1$

$$\eta_B = \eta / Sc^{1/2}, \quad \tau_d = \eta_B^2 / \kappa$$

Distribution of time scales is similar. In particular, we have  $\langle \tau_d \rangle = \langle \eta_B^2 \rangle / \kappa \approx 10 \langle \eta_B \rangle^2 / \kappa$

Eddy diffusive time/molecular diffusive time  $\approx Re^{1/2}/100$ ; exceeds unity only for  $Re \approx 10^4$  (mixing transition?)



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*J. Schumacher, K. R. Sreenivasan and P. K. Yeung*

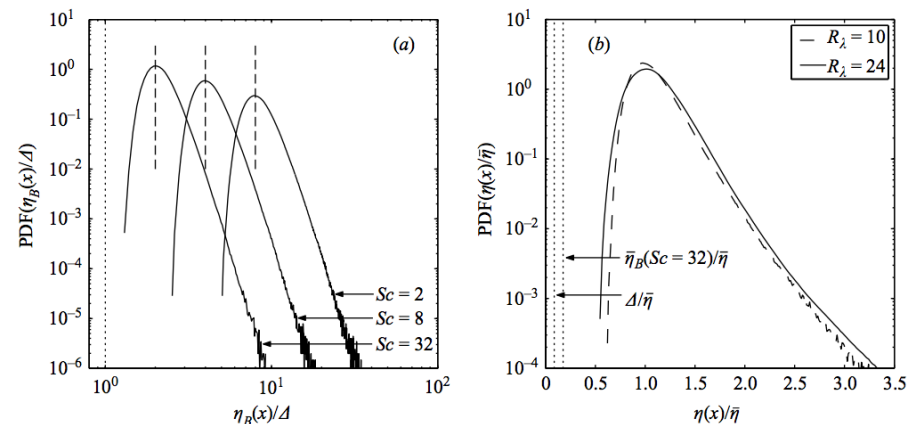
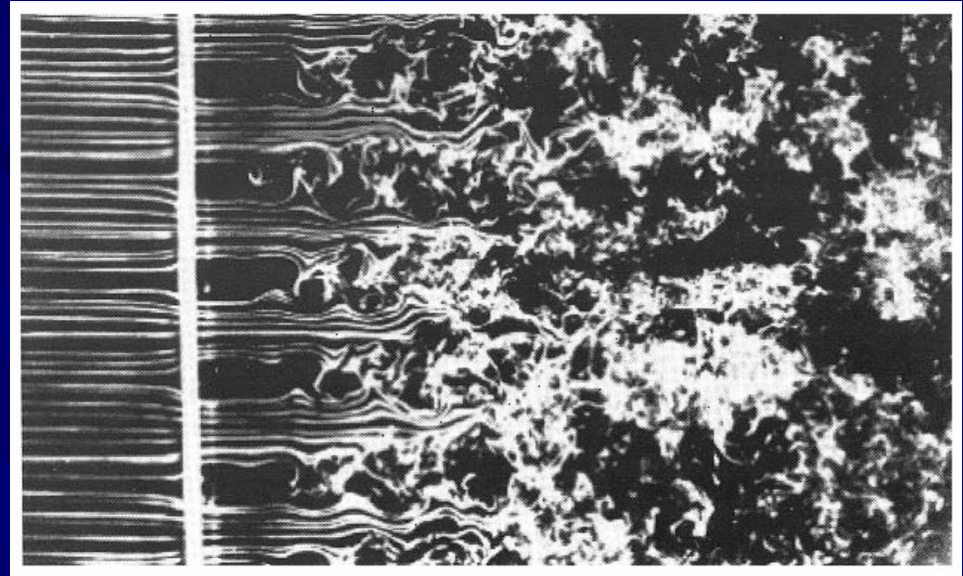
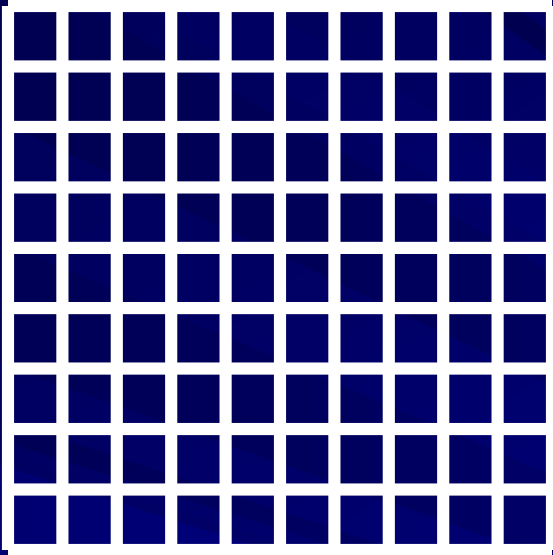


FIGURE 5. (a): Probability density function (PDF) of local fluctuations of the Batchelor scale in the mixing problem. The Taylor microscale Reynolds number is 10 and the Schmidt numbers are 2, 8, and 32. The computational domain is  $512\Delta$  on the side, where  $\Delta$  is the grid spacing. The vertical dashed line close to the maximum of each PDF indicates the average Batchelor scale  $\bar{\eta}_B$ . The dotted vertical line corresponds to  $\Delta$ . (b): Reynolds number dependence of the fluctuations of the local dissipation scale,  $\eta(\mathbf{x})$ . The Batchelor scale for  $Sc = 32$  and the grid spacing  $\Delta$  are indicated by dotted lines. Since all lengths are rescaled by  $\bar{\eta}$ , both these parameters collapse for the two Reynolds numbers.

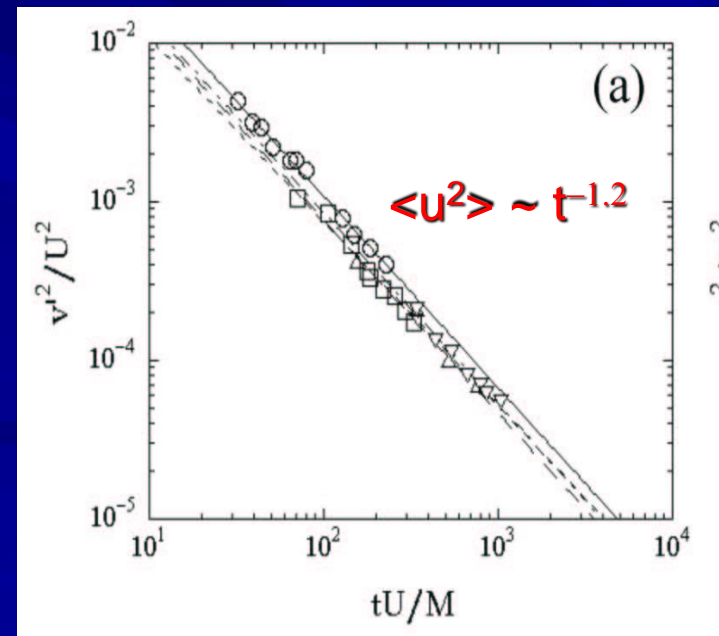
Some large scale features



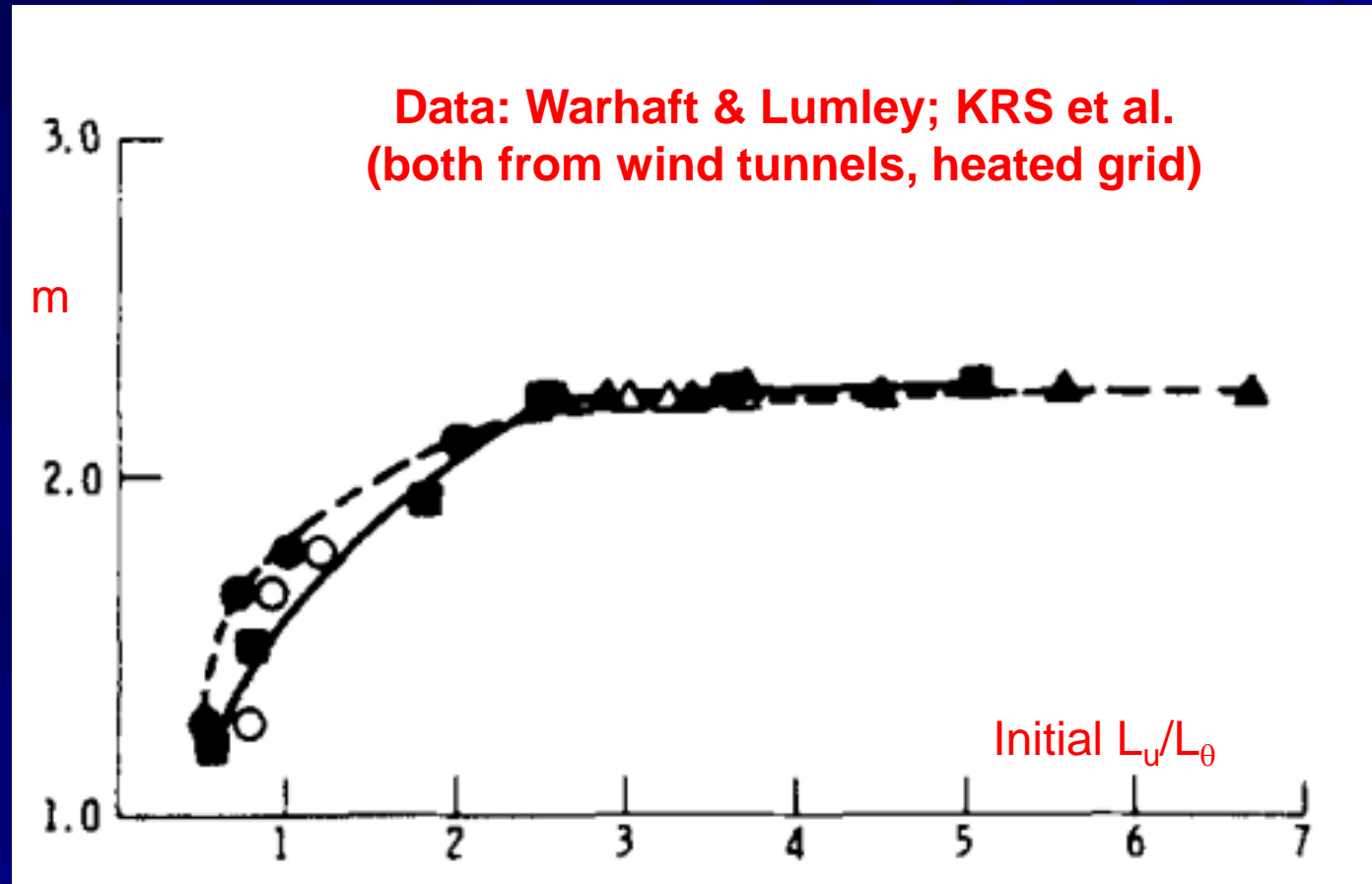
- $L_u$  is set by the mesh size
- $L_\theta$  can be set independently and the ratio  $L_u/L_\theta$  can be varied

$$\langle \theta^2 \rangle \sim t^{-m} \text{ (variable } m)$$

On what does  $m_0$  depend?  
Conflicting experimental  
data in the early days



Non-uniqueness of the exponent is not difficult to understand.



P.A. Durbin, Phys. Fluids 25:1326 (1982)

# Effect of length-scale ratio: PDF of $\theta$ in stationary turbulence

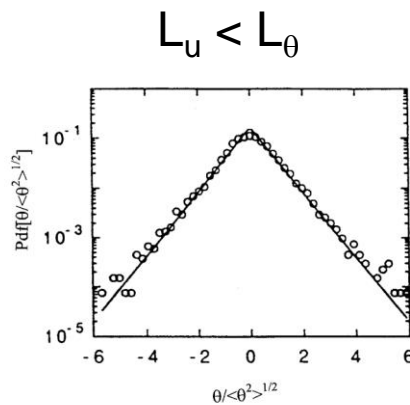
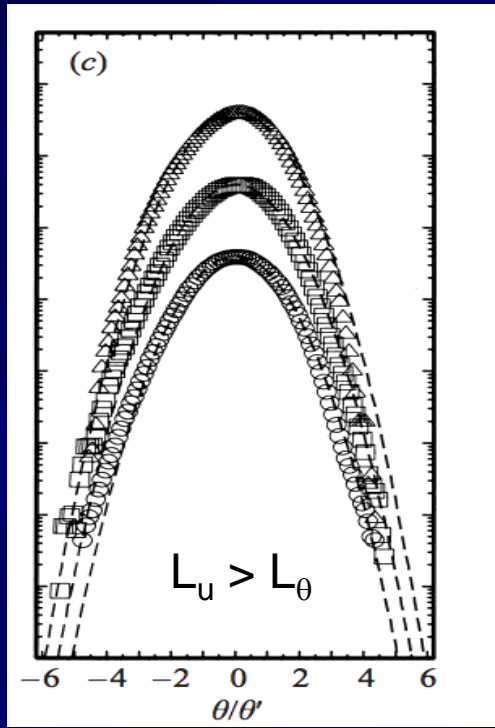
Both PDFs are for stationary velocity and scalar fields, under comparable Reynolds and Schmidt numbers.

Passive scalars in homogeneous flows most often have Gaussian tails, but long tails are observed for column-integrated tracer distributions in horizontally homogeneous atmospheres.

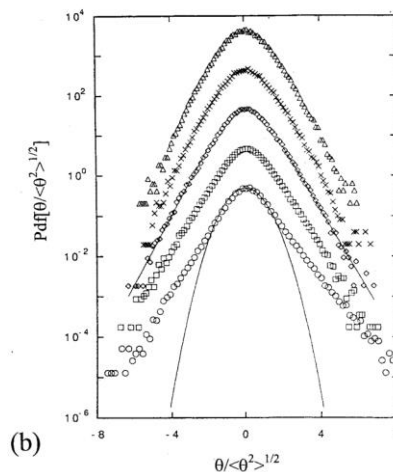
Models of Bourlioux & Majda, *Phys. Fluids* **14**, 881 (2002), closely connected with models studied by Avellaneda & Majda

**Probability density function of the passive scalar**

Top: Ferchichi & Tavoularis (2002)  
Bottom: Warhaft (2000)



(a)



(b)



## Model studies

- Assume some artificial velocity field satisfying  $\text{div } \mathbf{u} = 0$  (see A.J. Majda & P.R. Kramer, *Phys. Rep.* **314**, 239, 1999)

### **Broad-brush summary of “large-scale, long-time” results**

1. For smooth velocity fields (e.g., periodic and deterministic), homogenization is possible. That is,  
$$\langle \mathbf{u}(\mathbf{x};t) \cdot \nabla(\theta) \rangle = \nabla(\kappa_T \cdot \nabla(\theta(\mathbf{x};t)))$$
where  $\kappa_T$  is an effective diffusivity (Varadhan, Papanicolaou, Majda, and others)
2. Velocity is a homogeneous random field, but a scale separation exists:  $L_u/L_\theta \ll 1$ . Homogenization is possible here as well.
3. Velocity is a homogeneous random field but delta correlated in time,  $L_u/L_\theta = O(1)$ ; eddy diffusivity can be computed.
4. For the special case of shearing velocity (with and without transverse drift), the problem can be solved essentially completely: eddy diffusivity, anomalous diffusion, etc., can be calculated without any scale separation. See, e.g., G. Glimm, B. Lundquist, F. Pereira, R. Peierls, *Math. Appl. Comp.* **11**, 187 (1992); M. Avellaneda & A.J. Majda, *Phil. Trans. Roy. Soc. Lond. A* **346**, 205 (1994); G. Ben Arous & H. Owhadi, *Comp. Math. Phys.* **237**, 281 (2002)

My perspective:

While we may not be able to explain everything, we seem to have reached a state at which we can string a plausible story.

