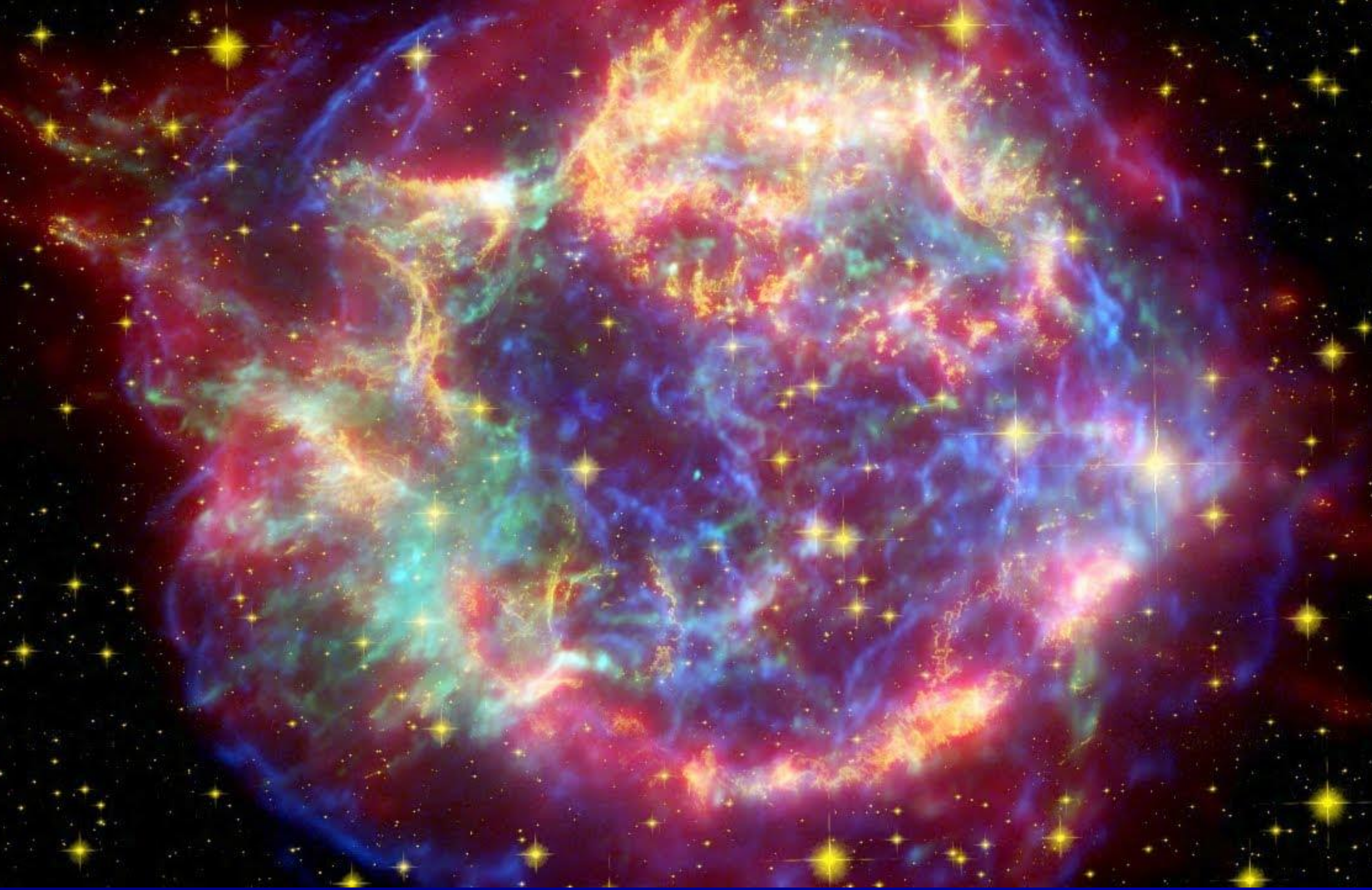
Turbulent Mixing of Tracers: A Perspective

K.R. Sreenivasan

ICTS Program on
Turbulence from Angstroms to Light Years:
Chandrasekhar Lecture -- II

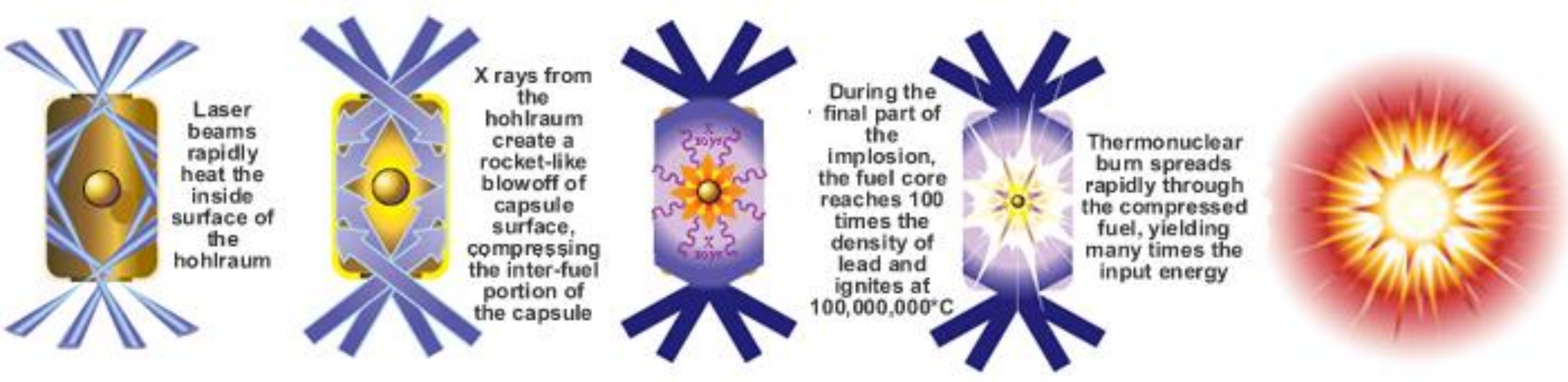
NEW YORK UNIVERSITY

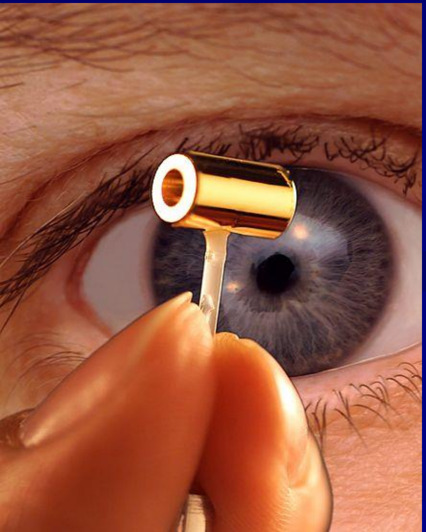


Supernova Cassiopeia A: This image from the orbiting Chandra x-ray observatory shows the youngest supernova remnant in the Milky Way. Courtesy of NASA

Fluid dynamics of the national ignition facility

A few milligrams of fusion fuel, typically a mix of deuterium and tritium

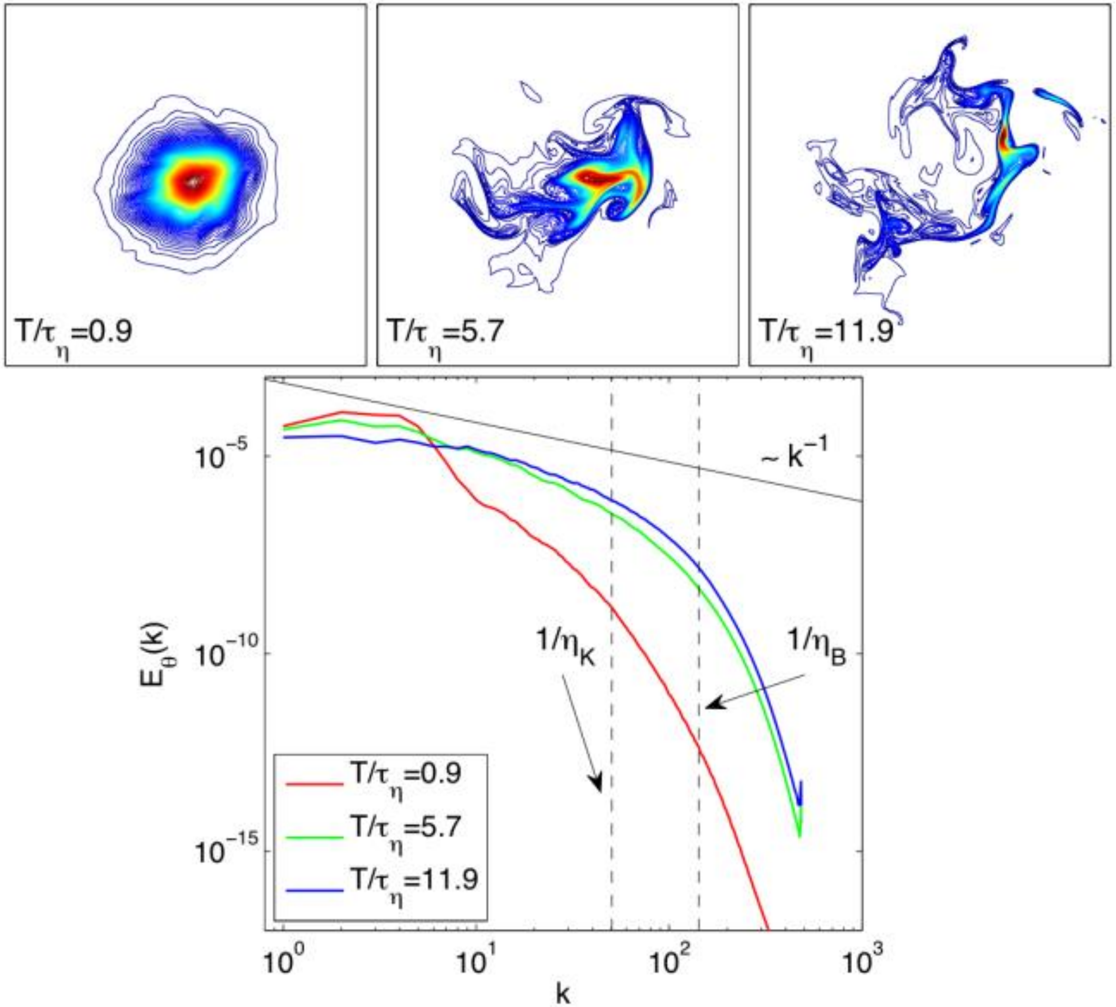




500 terawatts, shining on the fuel pellet for a few picoseconds

- Rayleigh-Taylor instability, Kelvin-Helmholtz instability are quite common in astrophysical contexts, where rotation, magnetic field, density changes are felt simultaneously. Chandra studied these problems and also thermal convection. Several speakers at this meeting have worked in these areas.
- Chandra codified these effects on a grand scale. He wrote some 50 technically sophisticated papers on stability, reworked and condensed them in his book, *Hydrodynamic and Hydromagnetic Stability,* Oxford (1961) and Dover.
- For more discussion of such complex cases, see Abarzhi & Sreenivasan (2010) and Abarzhi et al. (2013)---Trans. Roy. Soc. Lond.
- All these instabilities are precursors to turbulent mixing, which is the main topic to be covered in this lecture. PERSPECTIVE

Will consider the simplest case of mixing of a passive tracer by an incompressible flow.



Turbulent mixing of tracers

Advection diffusion equation

$$\partial\theta/\partial t + \mathbf{u}.\nabla\theta = \kappa\nabla^2\theta$$

$\theta(\mathbf{x};t)$, the additive; κ, its diffusivity (usually small); $\mathbf{u}(\mathbf{x};t)$, the advection velocity which solves NS=0; no source terms

The equation is linear with respect to θ.

BCs (perhaps mixed) are almost always linear as well.

Linearity holds for each realization but the equation is statistically nonlinear because of $<\mathbf{u}.\nabla\theta>$, etc.

Langevin equation

$$d\mathbf{X} = \mathbf{u}[\mathbf{X}(t);t]\,dt + (2\kappa)^{1/2}\,d\chi(t),\ \mathbf{X}(t=0) = \mathbf{x}_0$$

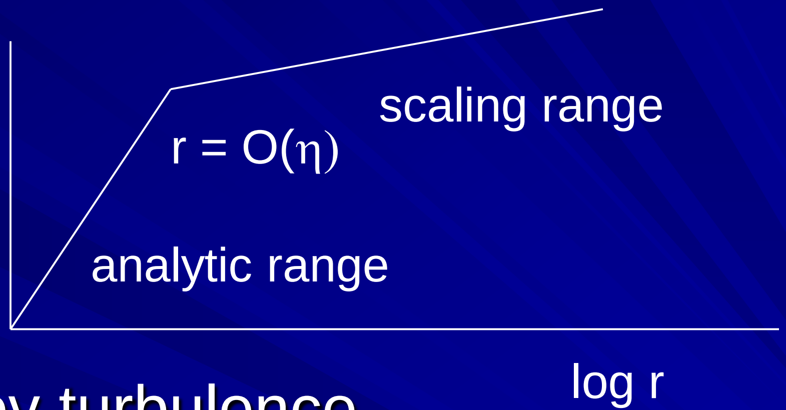
$\chi(t)$ = vectorial Brownian motion, statistically independent in its three components

What aspects of the NS solutions makes the problem difficult?

The turbulent velocity field is analytic for $r < O(\eta)$, and is

Hölder continuous, or “rough,” in the scaling range ($\Delta_r u \sim r^h$, $h < 1$).

a quantity such as a structure function (log)



$h = 1/3$ for Kolmogorov turbulence

but has a distribution in practice.

“multiscaling”

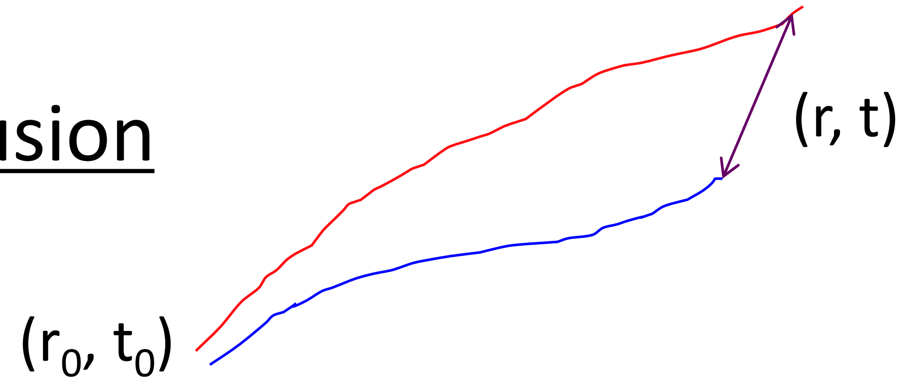
Parisi & Frisch (1985); Chen et al. *JFM* **533**, 183-192 (2005)

If $\Delta_r u \sim r^h$ ($h < 1$), we get $r(t) \sim t^{1/(1-h)}$, and Lagrangian paths separate explosively and are not unique; this introduces various complexities.

- $\Delta u_r \sim r^h$
- $p_h(r) \sim r^{F(h)}$
- $<(\Delta u_r)^n> \sim$ \int dh $r^{nh+F(h)}$
- Define $\zeta_n = \min_h [nh + F(h)]$
- Perform the integral using saddle-point method to yield
- $<\Delta u_r^n> \sim r^{\zeta_n}$

Richardson's law of diffusion

$$\langle r^2 \rangle = C_R \, \varepsilon \, (t-t_0)^3$$



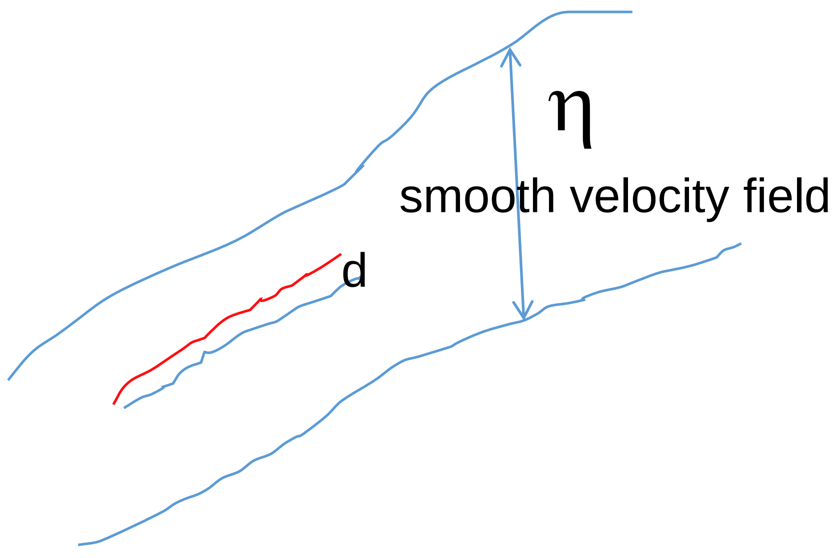
Not present in the formula is r_0: Two particles that start exactly at the same location at t_0 can separate by a finite distance for any time $t > t_0$.

Non-uniqueness: "Spontaneous stochasticity"
e.g., Bernard (2000), Eyink & Drivas (2015)

Note 1: The explosive separation in Richardson's law (non-unique trajectories) does not violate short-time uniqueness of solutions for initial value problems because the velocity is "rough".

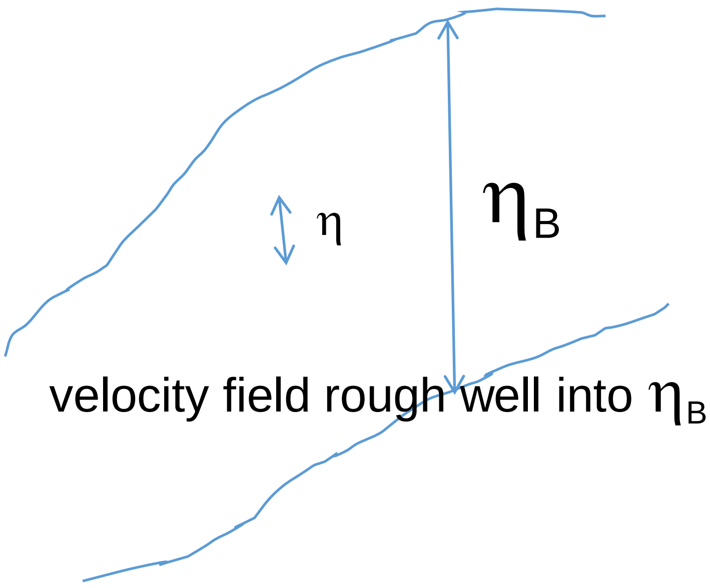
Note 2: Contrast with chaos: $r(t) = r_0 \exp\{\lambda(t-t_0)\}$

Re → (Sc >> 1)



Anomaly begins to hold only after $t > (\nu/\varepsilon)^{1/2} \log Sc$.

Re → (Sc << 1)



Spread of particles does not depend on κ: anomalous

Massive parallelism, with O(10^5 -10^6) CPU cores, has made simulation to require a different type of expertise and support

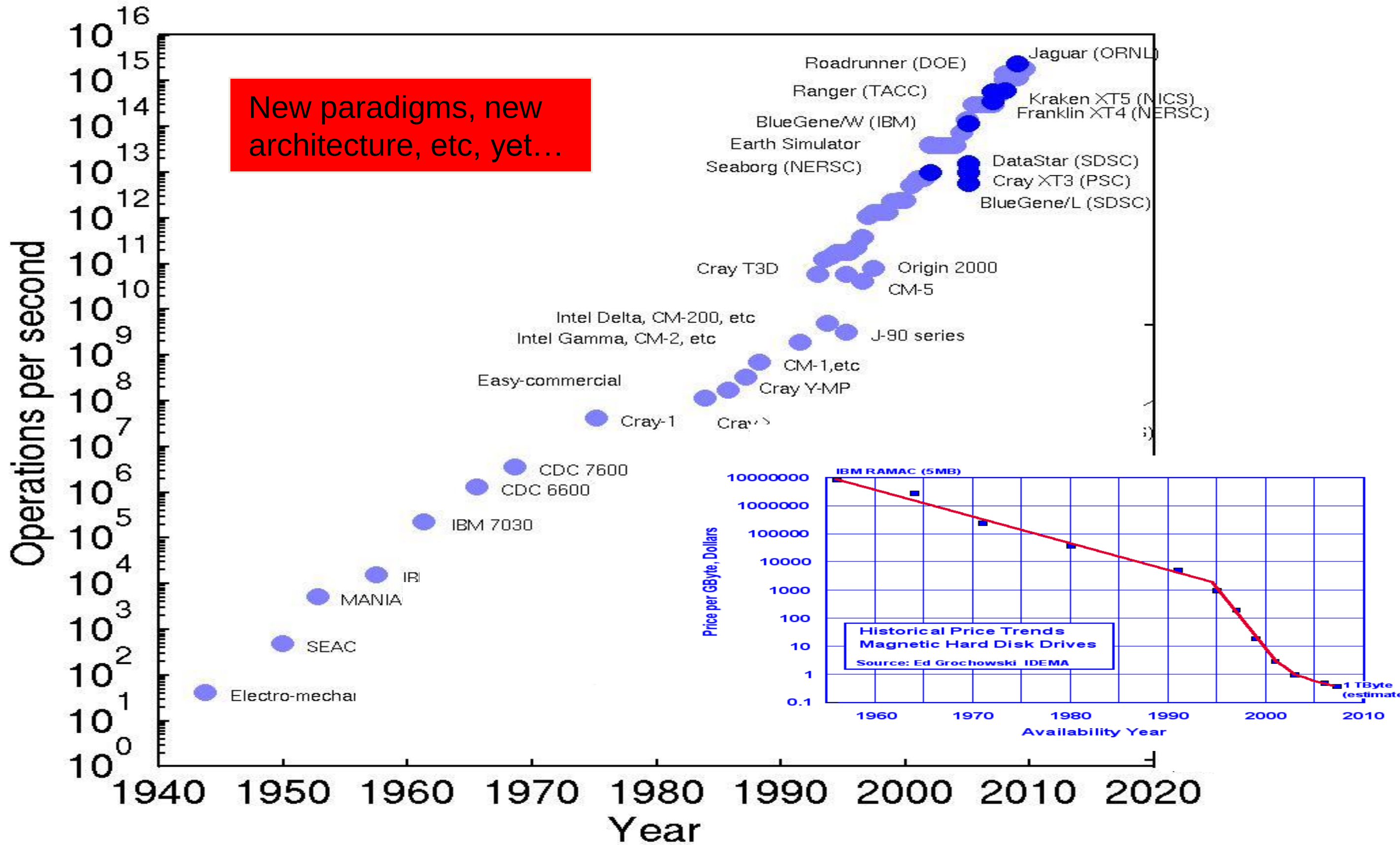
DNS

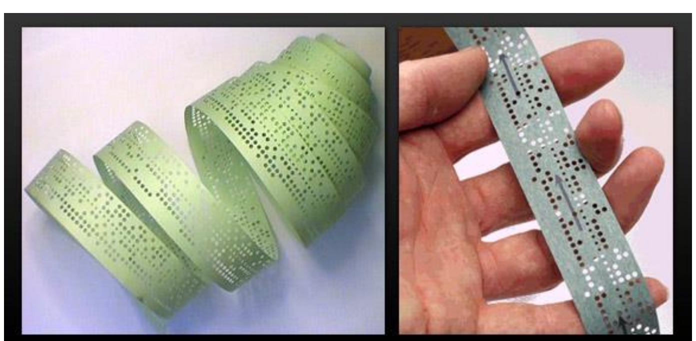
P.K. Yeung
Diego Donzis

$8 < R_\lambda < 650$
$1/512 < Sc < 1024$

Toshi Gotoh
Jörg Schumacher

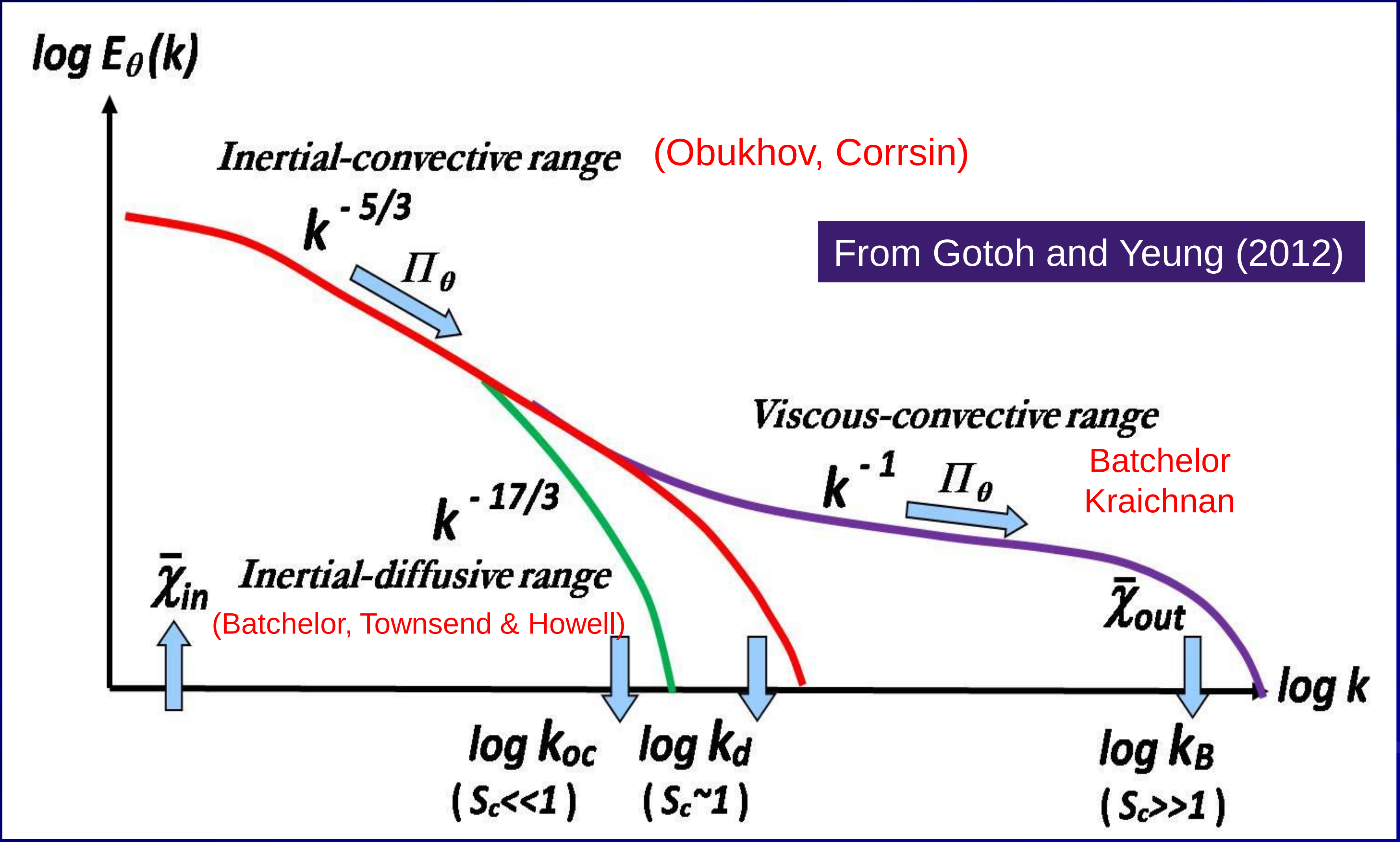
Different forcing schemes





> 10 Petaflops
with Exaflop machines by ~2020 (20 MW?)

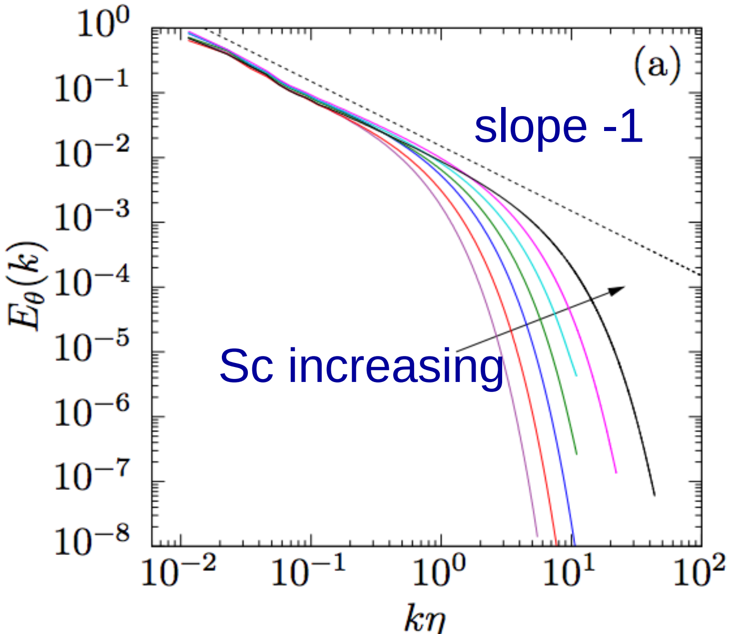
Classical phenomenology and the “universality” in second-order statistical quantities



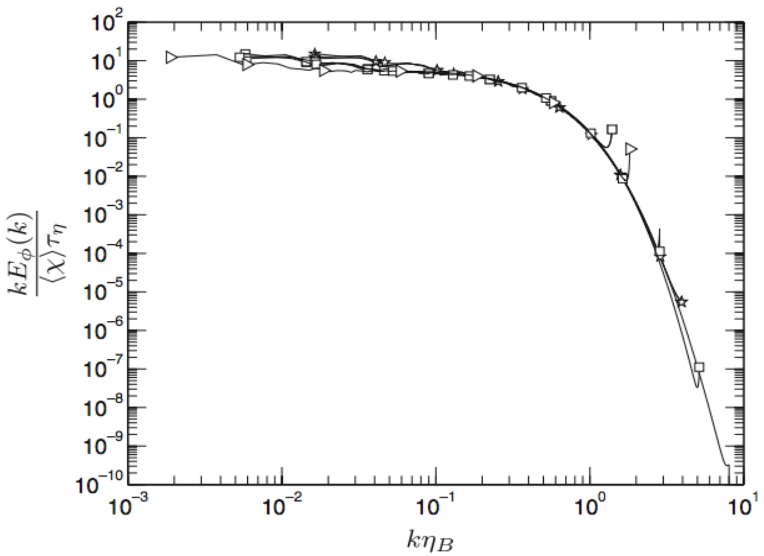
log $E_\theta(k)$
Inertial-convective range
(Obukhov, Corrsin)
$k^{-5/3}$
Π_θ
From Gotoh and Yeung (2012)
Viscous-convective range
Batchelor
Kraichnan
k^{-1}
Π_θ
$k^{-17/3}$
$\bar{\chi}_{in}$
Inertial-diffusive range
(Batchelor, Townsend & Howell)
$\bar{\chi}_{out}$
log k
log k_{oc}
(S_c<<1)
log k_d
(S_c~1)
log k_B
(S_c>>1)

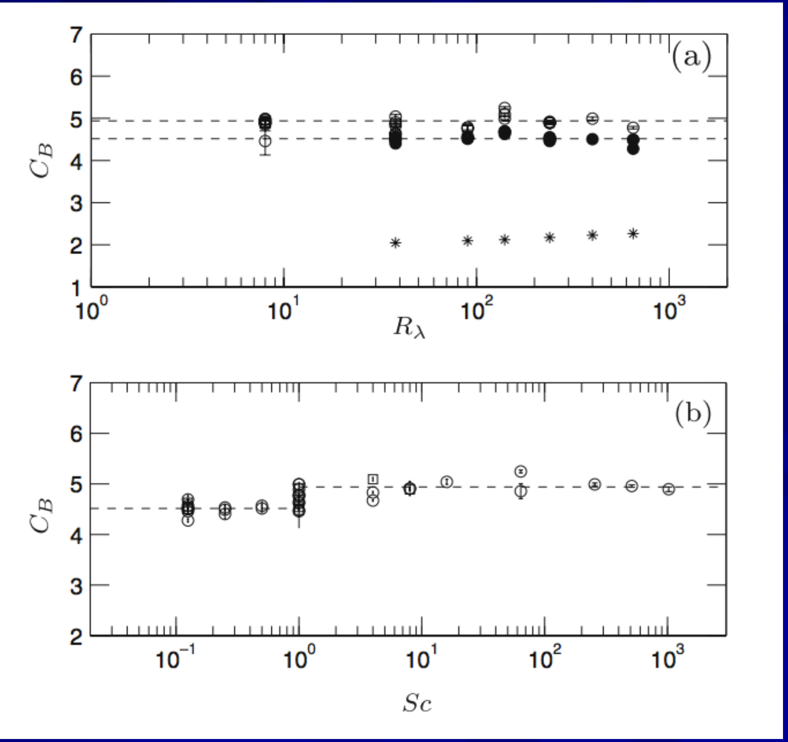
The Batchelor regime

Reynolds number: Re >>1
Schmidt number, Sc = ν/κ >>1



P.K. Yeung





Batchelor (1956)

$$E_\theta(k) = C_B\, \kappa(\nu/\varepsilon)^{1/2}k^{-1}\exp[-q(k\eta_B)^2]$$

Kraichnan (1968)

$$E_\theta(k) = C_B\, \kappa(\nu/\varepsilon)^{1/2}k^{-1}\,[1+(6q)^{1/2}k\eta_B \times \exp(-(6q)^{1/2}k\eta_B)]$$

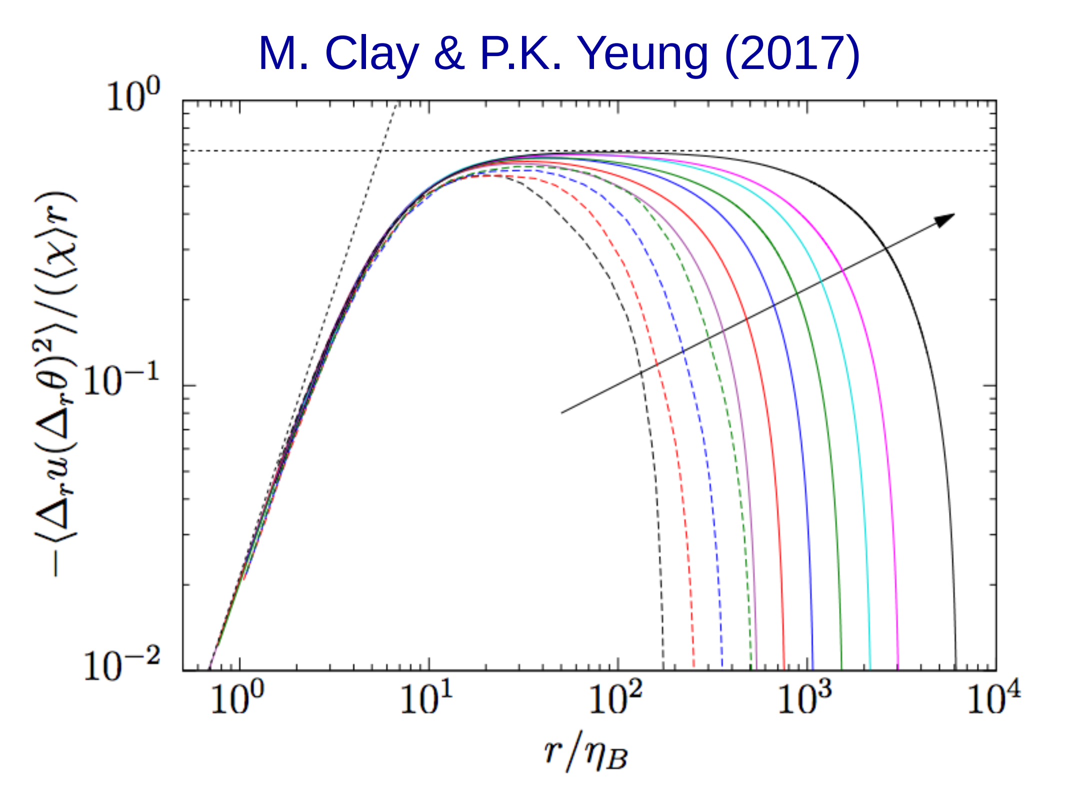
The Yaglom relation (1949)

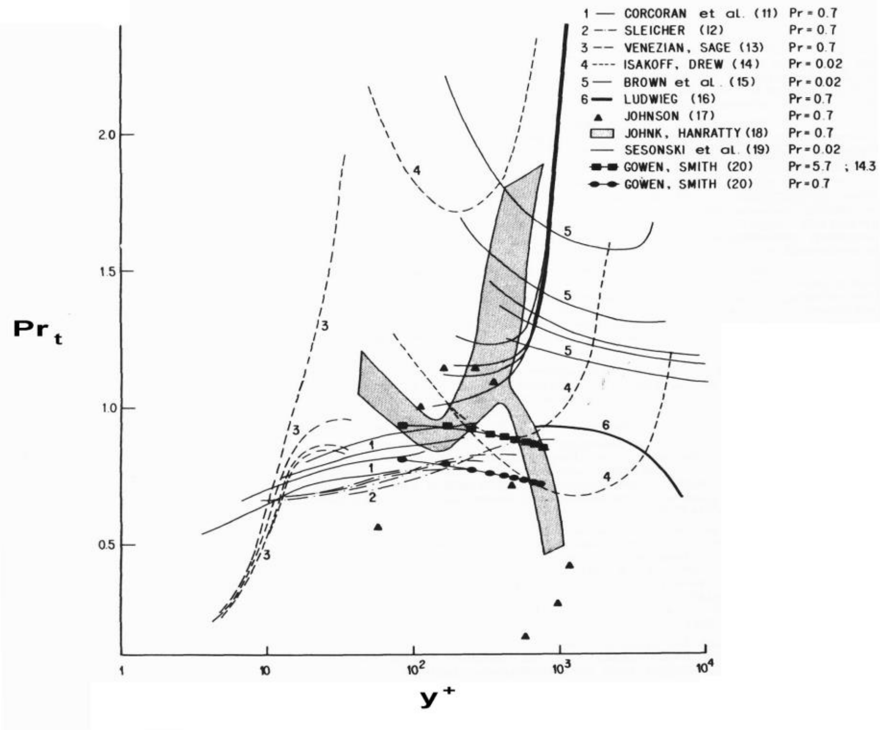
$$\langle \Delta_r u \, (\Delta_r \theta)^2 \rangle = -(2/3)\langle \chi \rangle r$$

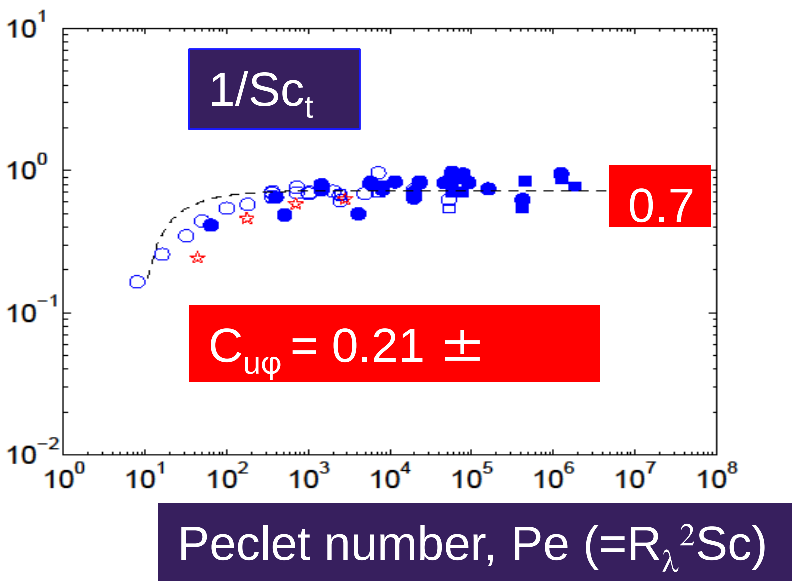
Refined similarity hypothesis
- G. Stolovitzky, P. Kailasnath & KRS, JFM 297, 275 (1995)

Extension to non-stationary forcing conditions
- L. Danaila, F. Anselmet, T. Zhou & R.A. Antonia, JFM 391, 359 (1999)
- P. Orlandi & R.A. Antonia, JFM 451, 99 (2002): DNS

Experiment
- G. Stolovitzky, P. Kailasnath & KRS, JFM 297, 275 (1995)
- L. Midlarsky, JFM 475, 173 (2003): Experiment







Experiment

Homogeneous shear flows
Boundary layers, Jets, Wakes

Direct Numerical Simulations
(D. Donzis et al., 2014)

$8 < R_\lambda < 650$
$1/512 < Sc < 1024$

<u>Dimensional Theory</u>

Flux spectrum

$E_{u\varphi}(k) = C_{u\varphi} G \langle\varepsilon\rangle^{1/3} k^{-7/3}$

in the inertial convection range
(following Lumely 1964)

Using $\langle u\varphi\rangle = -\int E_{u\varphi}(k)\, dk$ (with appropriate limits),

we get

$1/Sc_t = (10/3)\, C_{u\varphi}\, (1 - 1/Pe)$

Anomalous behaviors

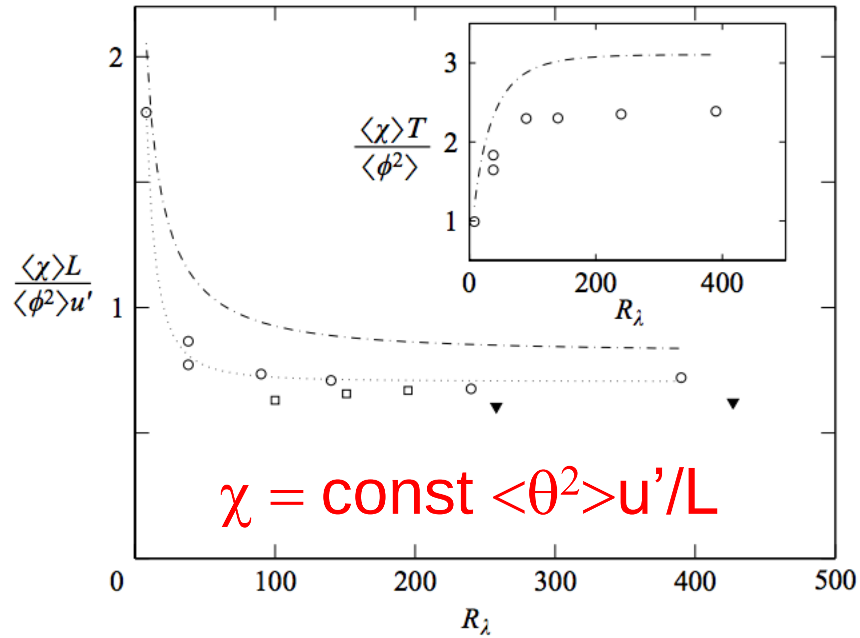


FIGURE 2. Scalar dissipation rate normalized with L/u' for $Sc = 1$. ○, present data; ▼, Wang *et al.* (1999); □, Watanabe & Gotoh (2004). Dotted line: equation (3.1) as the best fit for the present data. Dash-dotted line: theoretical prediction of (3.12), which will be described towards the end of § 3.2. Inset shows the present data using the normalization of T instead of L/u', as well as (3.12). While the asymptotic constancy holds for both normalizations, the direction of approach of this constancy is different.

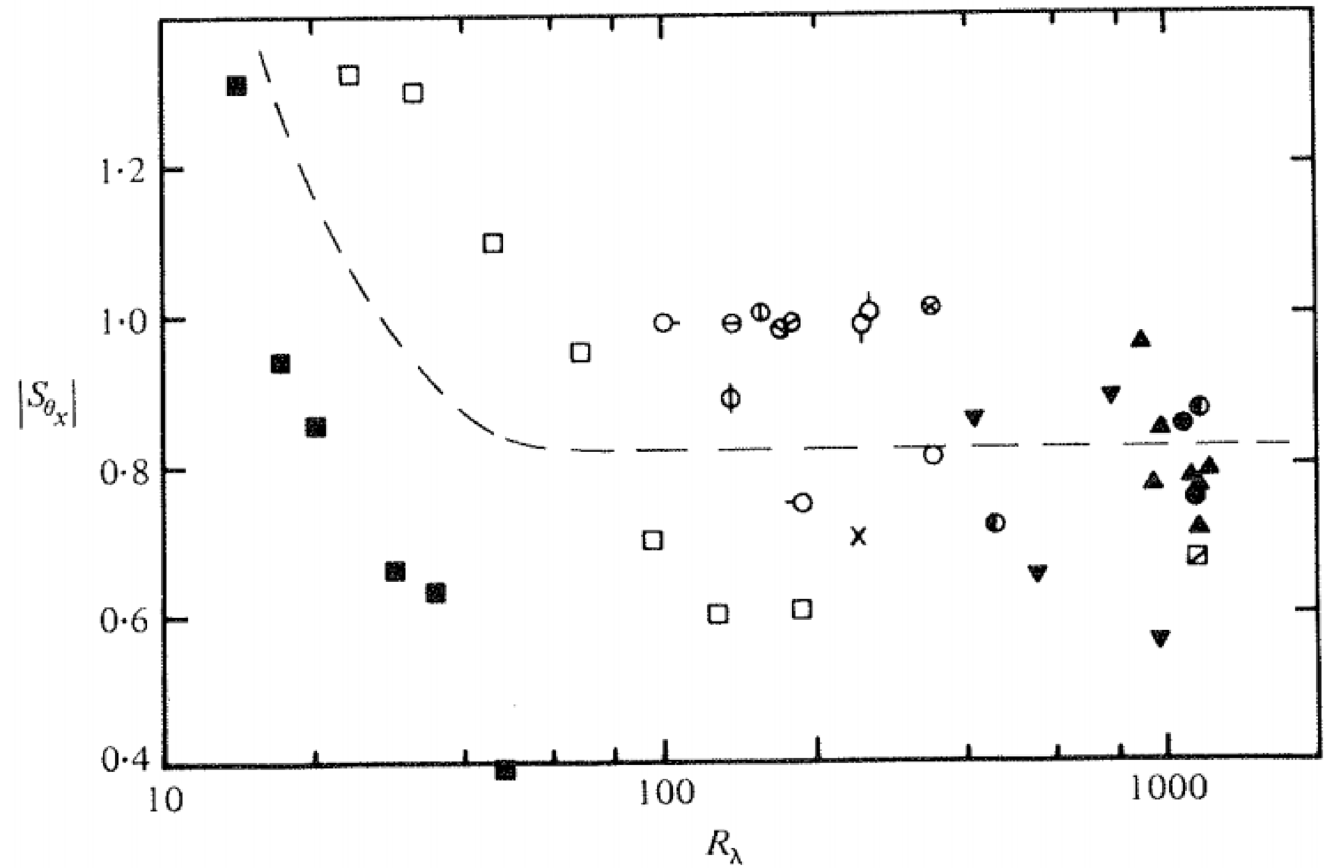
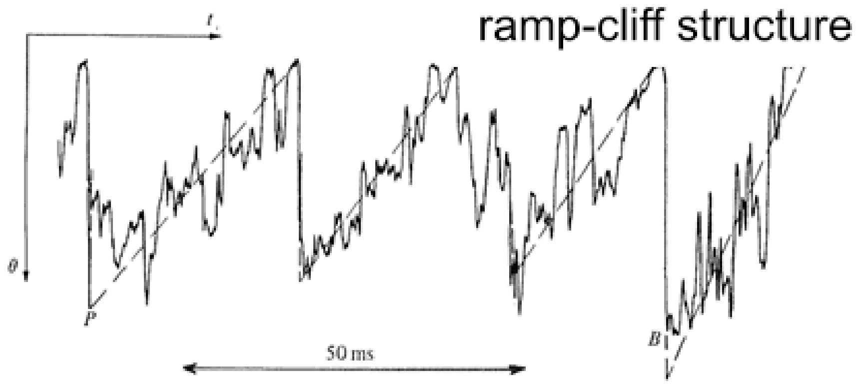
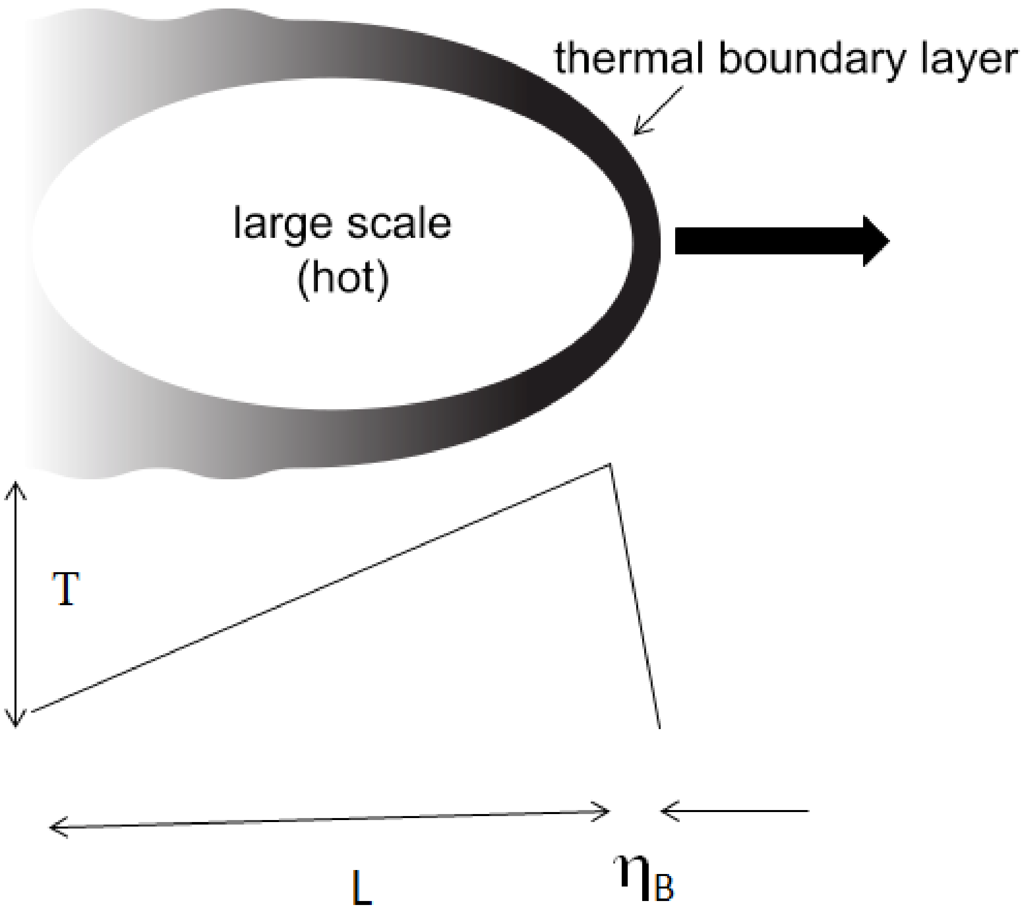
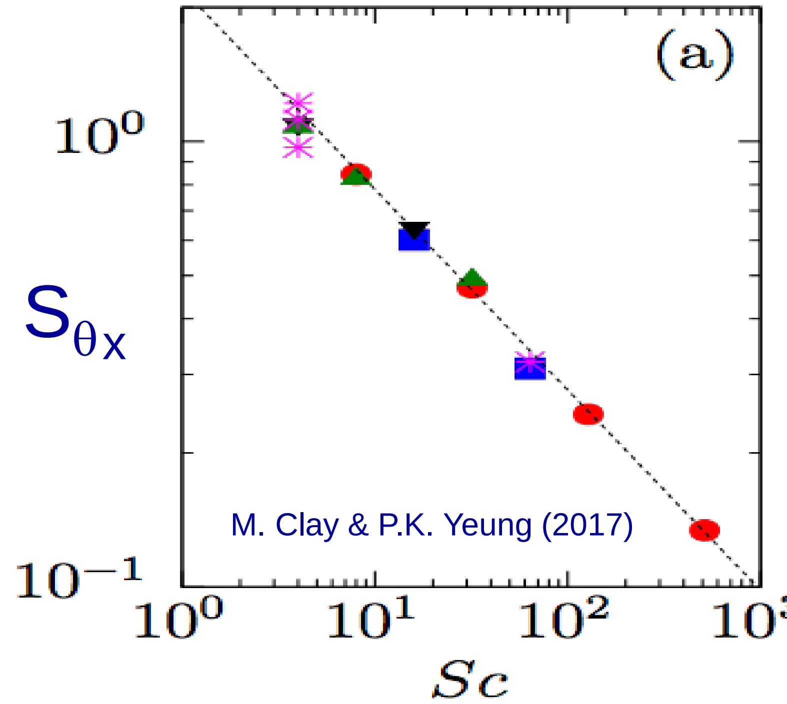


FIGURE 1. Variation of $|S_{\theta_x}|$ with R_λ. Gibson *et al.* (1977): ○, atmospheric boundary layer; ◐, heated jet; ●, heated wake; ⊗, cooled wake, ▲, Mestayer *et al.* (1976), heated boundary layer. ▼, Gibson *et al.* (1970), atmospheric boundary layer (corrected for velocity sensitivity); ×, Antonia & Van Atta (1975), heated jet. Freymuth & Uberoi (1971, 1973): □, heated two-dimensional wake; ■, heated axisymmetric wake. ⊖, ⦶, ⊘, ⦸, Sreenivasan, Antonia & Danh (1977), heated boundary layer. ⧄, our unpublished data, atmospheric surface layer. -○, ○̍, ɸ, ♀, ○-, present data, axisymmetric heated jet, $\eta = 0$, 0·89, 1·15, 1·48 and 1·63 respectively. ---, suggested mean trend.







$$S_{\theta_x} = \frac{(T/\eta_B)^3 \text{ X } \eta_B/L}{\langle \theta_x{}^2 \rangle^{3/2}}$$

Taking $\langle\theta_x{}^2\rangle$ from the anomalous dissipation plot, we can show that

$$S_{\theta_x} = \text{const } Re^0 \; Sc^{1/2}$$

Inertial-convective region

$$<\Delta_r\theta^2> \sim r^{\xi_2}$$

Standard "theory" gets ξ_2 by assuming that the structure functions obey the same symmetries as the equations. They (dimensional arguments) yield $\xi_2 = 2/3$

Two questions arise:

1. In $<\Delta_r\theta^4> \sim r^{\xi_4}$

the same argument yields $\xi_4 = 2\xi_2$ (in general, $\xi_{2n} = n\xi_2$).

This normal scaling is $\xi_{2n} = 2n/3$.

Measurements have shown that $\xi_4 < 2\xi_2$ (or generally $\xi_{2n} < n\xi_2$)

What is the source of these "anomalous exponents"?

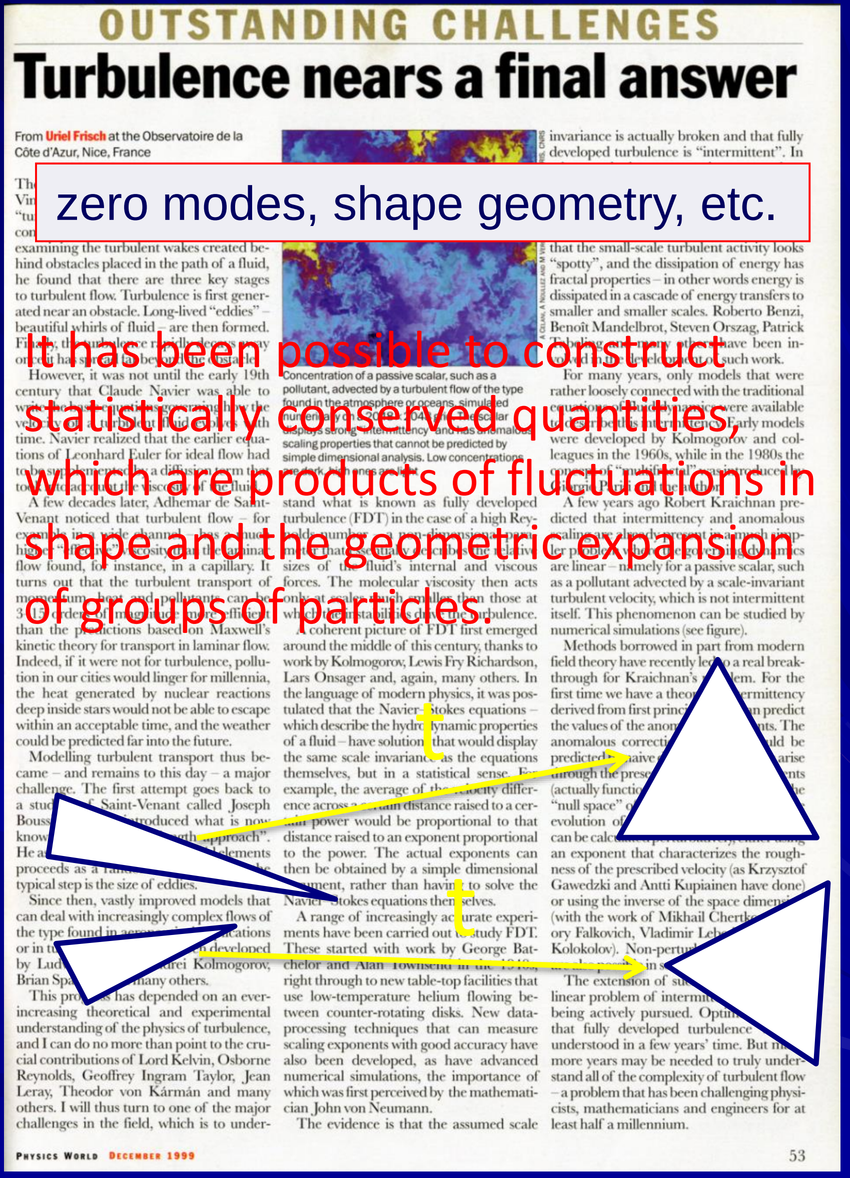
The exponent for each order order has to be determined on its own merit.

2. We have $<\Delta_r u\,(\Delta_r\theta)^2> = -(2/3)<\chi>r$

Kraichnan model

(motivated by small-scale intermittency)

- R.H. Kraichnan, *Phys. Fluids* **11**, 945 (1968); *Phys. Rev. Lett.* **72,** 1016 (1994)
- Review: G. Falkovich, K. Gawedzki & M. Vergassola, *Rev. Mod. Phys.* **73**, 913 (2001)



Surrogate Gaussian velocity field

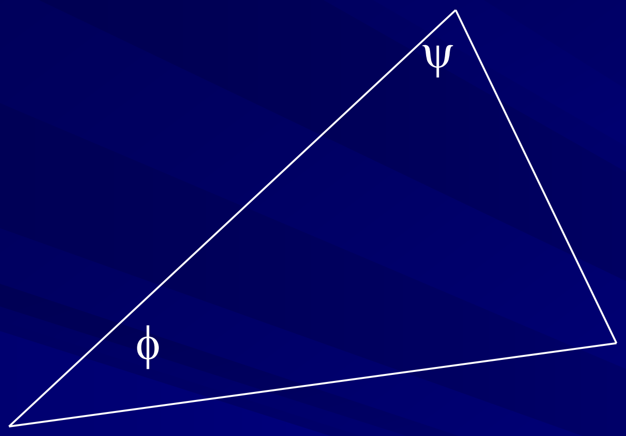
$\langle u_i(\mathbf{x};t)u_j(\mathbf{y};t')\rangle = |\mathbf{x}-\mathbf{y}|^{2-\gamma}\,\delta(t-t')$

$\gamma = 2/3$ recovers Richardson's diffusion

Forcing for stationarity:

$\langle f_\theta(\mathbf{x};t)f_\theta(\mathbf{y};t')\rangle = C(r/L)\delta(t-t')$

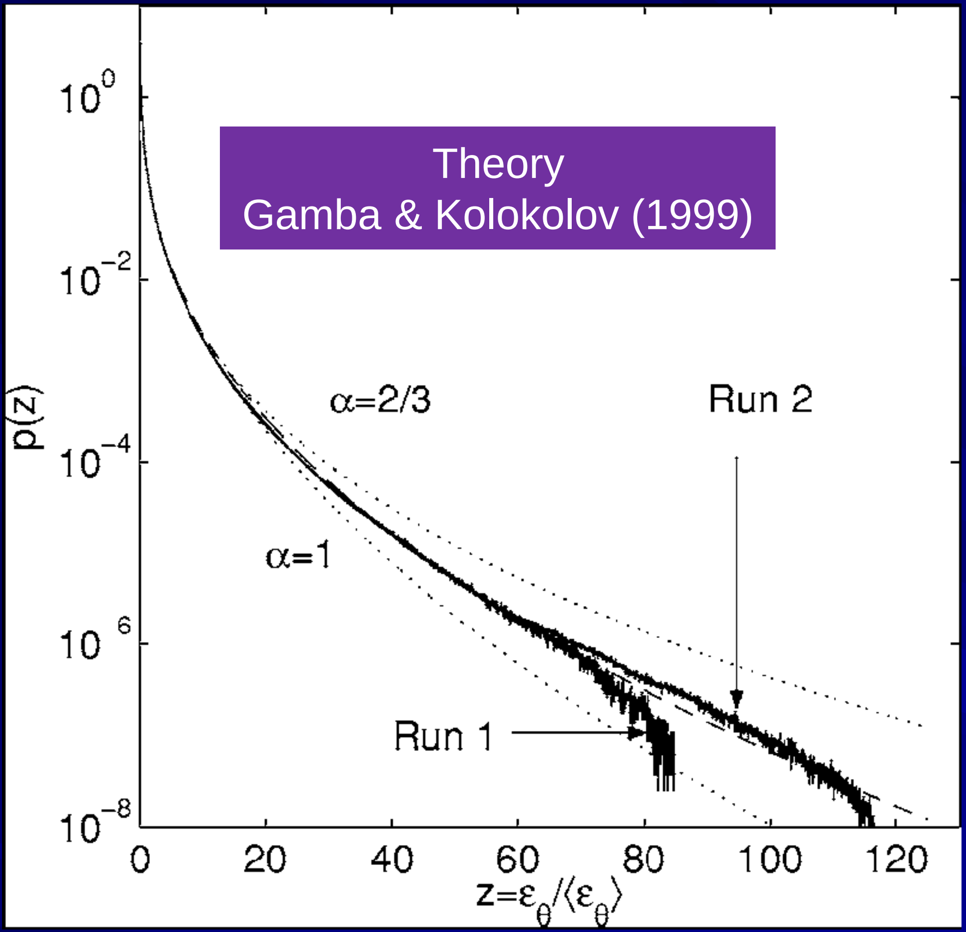
C(r/L) is non-zero only on the large scale, decays rapidly to zero for smaller scale.

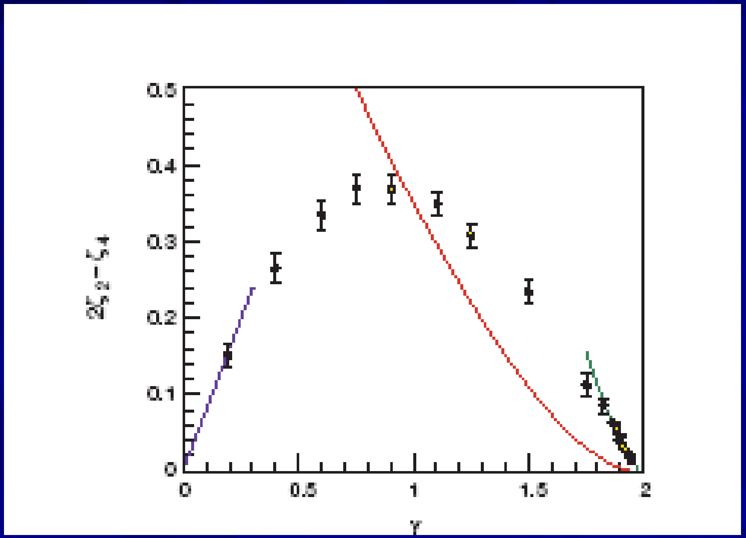


- Advection changes the shape and size
- R = geometric mean of the lengths, say
- After rescaling by R, we can get an idea of shape fluctuations looking $f(\psi, \phi)$.
- For the Kraichnan model, the three point statistics are given by those trajectories for which
- $R^{\zeta_3} f(\psi, \phi) = \text{constant}$.

The important qualitative lesson from the work on the Kraichnan model is that certain types of Lagrangian characteristics, conserved only on the average, determine the statistical scaling.
Breaking of symmetry yields the conservation of flux condition. Are there are other statistical conservation laws whose symmetry breaking provides the basis for determining the exponents of higher orders?

$2\zeta_2 - \zeta_4$





A measure of anomalous scaling, $2\zeta_2 - \zeta_4$, versus the index γ, for the Kraichnan model. The circles are obtained from Lagrangian Monte Carlo simulations. The results are compared with analytic perturbation theories (blue, green) and an ansatz due to Kraichnan (red).

Mixing process itself imprints features independent of the velocity field!

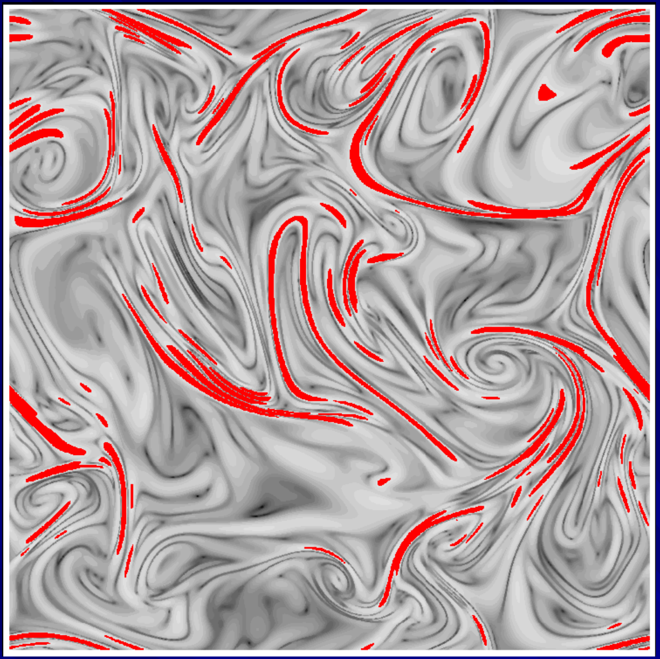
One consequences of fluctuations

Traditional definitions
$\langle\eta\rangle = (\nu^3/\langle\varepsilon\rangle)^{1/4}$
$\langle\eta_B\rangle = \langle\eta\rangle/Sc^{1/2}$
$\langle\tau_d\rangle = \langle\eta_B\rangle^2/\kappa$

Local scales
$\eta = (\nu^3/\varepsilon)^{1/4}$, or define η through $\eta\Delta_\eta u/\nu = 1$
$\eta_B = \eta/Sc^{1/2}$, $\tau_d = \eta_B{}^2/\kappa$

Distribution of time scales is similar. In particular, we have $\langle\tau_d\rangle = \langle\eta_B{}^2\rangle/\kappa \approx 10\,\langle\eta_B\rangle^2/\kappa$

Eddy diffusive time/molecular diffusive time $\approx Re^{1/2}/100$; exceeds unity only for $Re \approx 10^4$ (mixing transition?)



118 *J. Schumacher, K. R. Sreenivasan and P. K. Yeung*

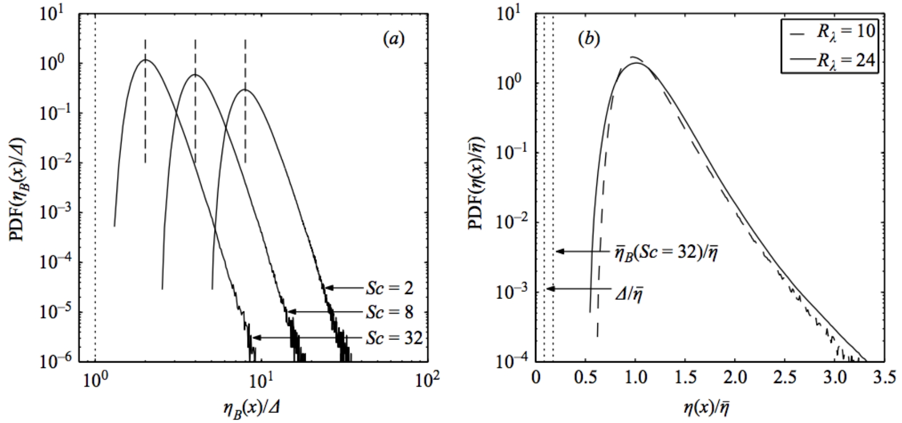
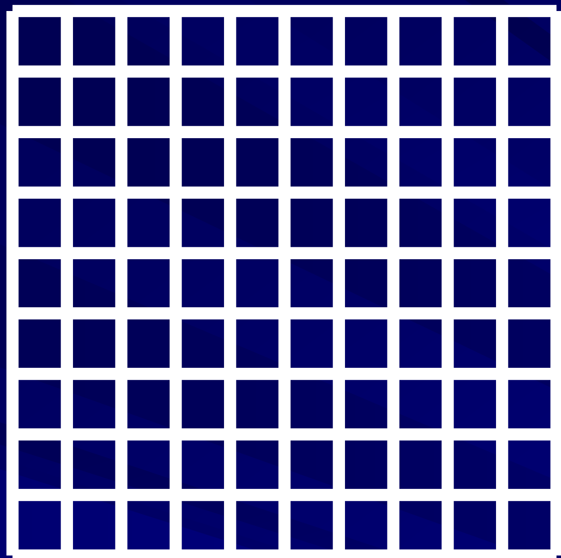
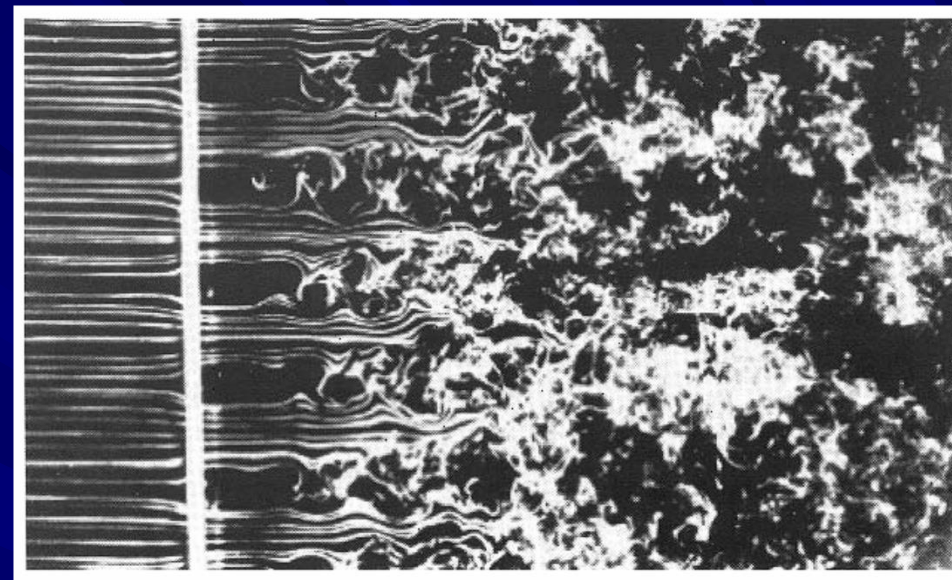


FIGURE 5. (*a*): Probability density function (PDF) of local fluctuations of the Batchelor scale in the mixing problem. The Taylor microscale Reynolds number is 10 and the Schmidt numbers are 2, 8, and 32. The computational domain is 512Δ on the side, where Δ is the grid spacing. The vertical dashed line close to the maximum of each PDF indicates the average Batchelor scale $\bar{\eta}_B$. The dotted vertical line corresponds to Δ. (*b*): Reynolds number dependence of the fluctuations of the local dissipation scale, $\eta(\boldsymbol{x})$. The Batchelor scale for $Sc = 32$ and the grid spacing Δ are indicated by dotted lines. Since all lengths are rescaled by $\bar{\eta}$, both these parameters collapse for the two Reynolds numbers.

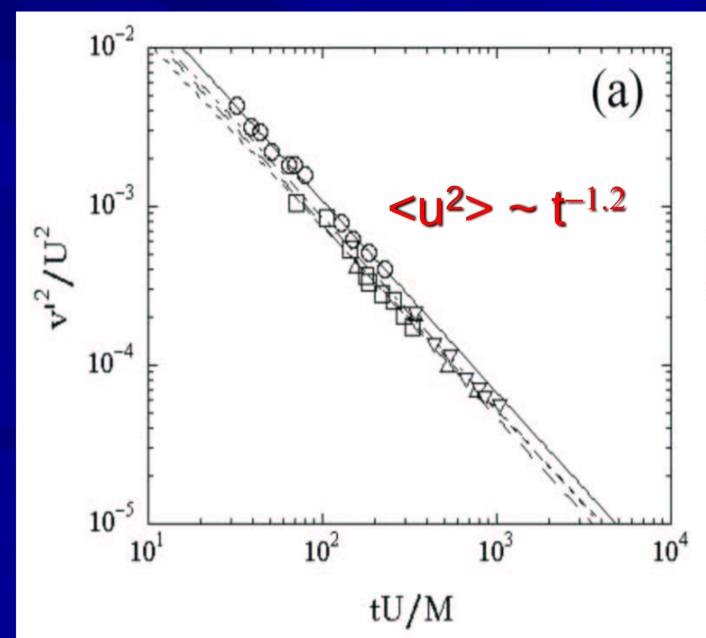
Some large scale features

Decaying fields of turbulence and scalar





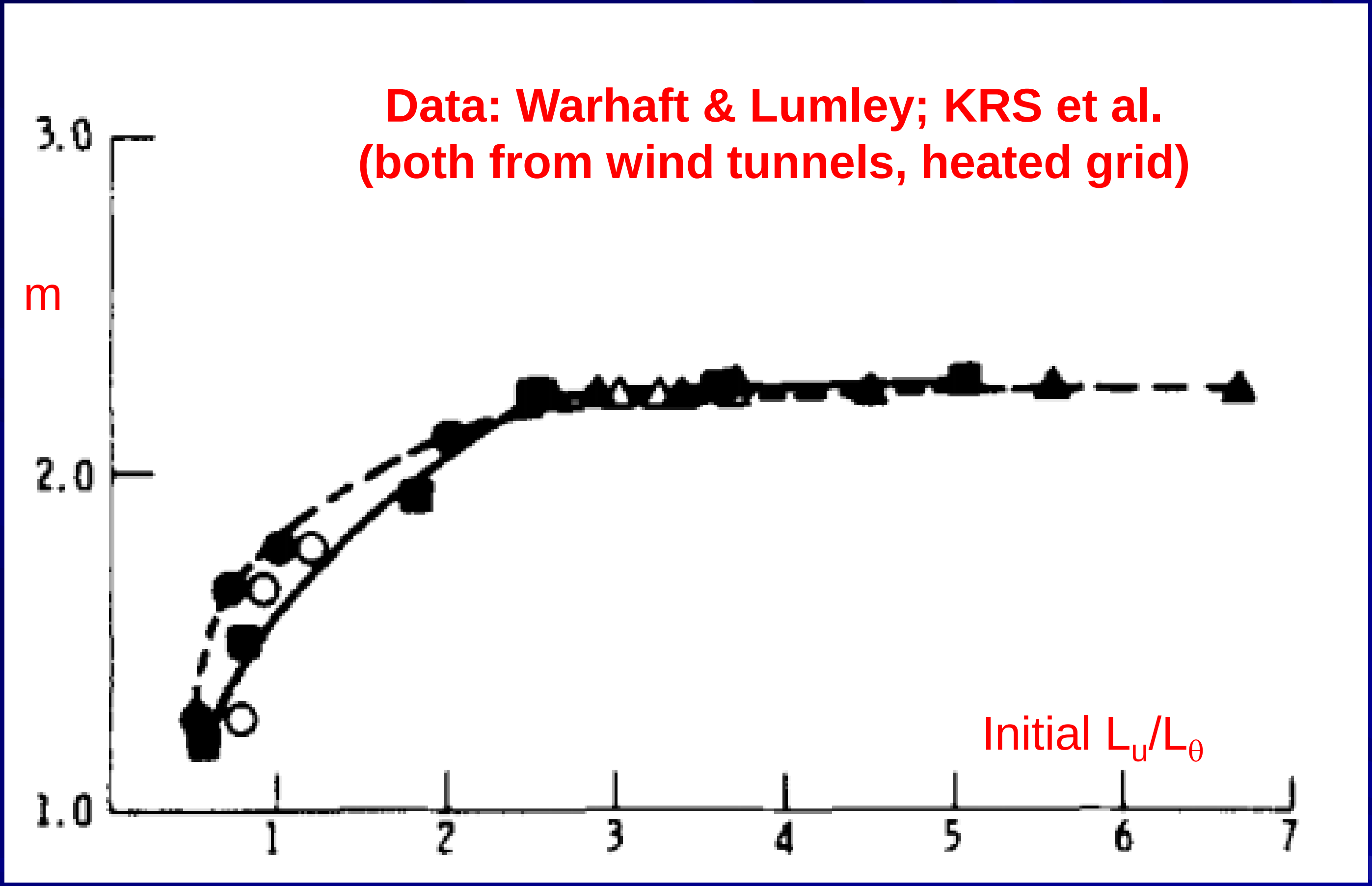
- L_u is set by the mesh size
- L_θ can be set independently and the ratio L_u/L_θ can be varied



$\langle\theta^2\rangle \sim t^{-m}$ (variable m)

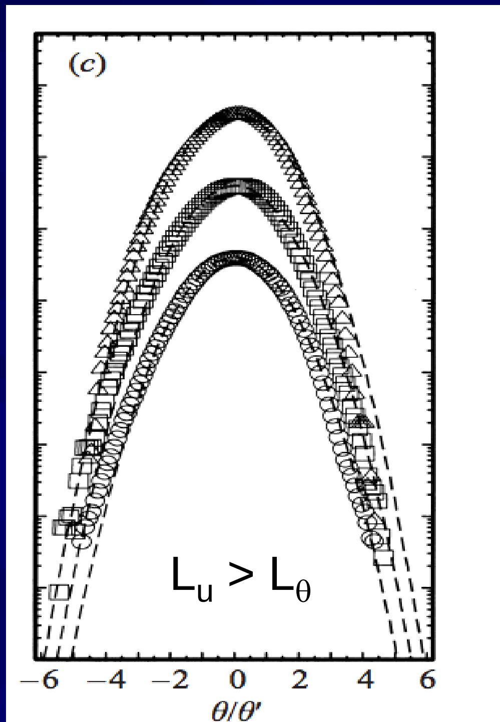
On what does m_0 depend? Conflicting experimental data in the eary days

Non-uniqueness of the exponent is not difficult to understand.



P.A. Durbin, Phys. Fluids 25:1326 (1982)

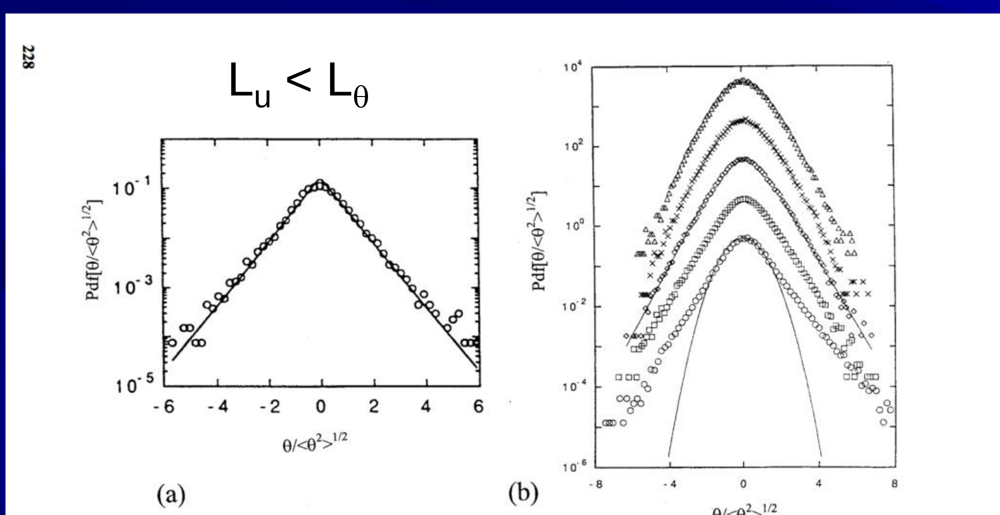
Effect of length-scale ratio: PDF of θ in stationary turbulence



Both PDFs are for stationary velocity and scalar fields, under comparable Reynolds and Schmidt numbers.

Passive scalars in homogeneous flows most often have Gaussian tails, but long tails are observed for column-integrated tracer distributions in horizontally homogeneous atmospheres.

Models of Bourlioux & Majda, *Phys. Fluids* **14**, 881 (2002), closely connected with models studied by Avellaneda & Majda



Probability density function of the passive scalar

Top: Ferchichi & Tavoularis (2002)

Bottom: Warhaft (2000)

Model studies

- Assume some artificial velocity field satisfying div **u** = 0 (see A.J. Majda & P.R. Kramer, *Phys. Rep.* **314**, 239, 1999)

Broad-brush summary of "large-scale, long-time" results

1. For smooth velocity fields (e.g., periodic and deterministic), homogenization is possible. That is,

   $$\langle \mathbf{u}(\mathbf{x};t).\nabla(\theta)\rangle = \nabla(\boldsymbol{\kappa}_T \cdot \nabla(\theta(\mathbf{x};t))$$

   where κ_T is an effective diffusivity (Varadhan, Papanicolaou, Majda, and others)
2. Velocity is a homogeneous random field, but a scale separation exists: $L_u/L_\theta \ll 1$. Homogenization is possible here as well.
3. Velocity is a homogeneous random field but delta correlated in time, $L_u/L_\theta = O(1)$; eddy diffusivity can be computed.
4. For the special case of shearing velocity (with and without transverse drift), the problem can be solved essentially completely: eddy diffusivity, anomalous diffusion, etc., can be calculated without any scale separation. See, e.g., G. Glimm, B. Lundquist, F. Pereira, R. Peierls, *Math. Appl. Comp.* **11**, 187 (1992); M. Avellaneda & A.J. Majda, *Phil. Trans. Roy. Soc. Lond. A* **346**, 205 (1994); G. Ben Arous & H. Owhadi, *Comp. Math. Phys.* **237**, 281 (2002)

My perspective:

While we may not be able to explain everything, we seem to have reached a state at which we can string a plausible story.