Bangalore Statistical Physics School

Mechanics of wrinkles, folds, crumples

Narayanan Menon

Day 1

Objects that are flexible purely for geometric reasons (sheets, filaments and ribbons) make an overwhelming variety of patterns in nature and in the technological world.

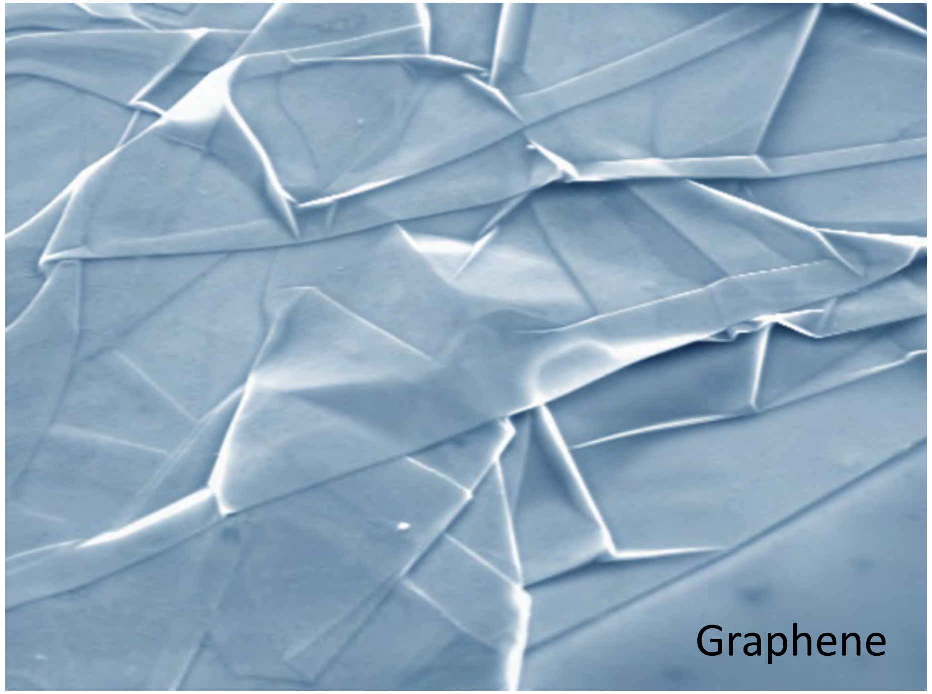
Sea urchin

Sharon, Swinney, Marder

Yva Momatiuk and John Eastcott/PhotoResearchers, Inc.

that a leaf or flower—just like a torn sheet of plastic—can use an enhanced, uniform growth at its margins to generate such complex patterns. Examples of wavy edges in nature include, from left to right, some lichens (shown, *Sticta limbata*), orchids (shown, *Schomborgkia beysiana*), sea slugs (represented by *Glossodoris hikuerensis*) and ornamental cabbage. (Lichen photograph courtesy of Stephen Sharnoff; sea slug photograph courtesy of Jeff Jeffords.)

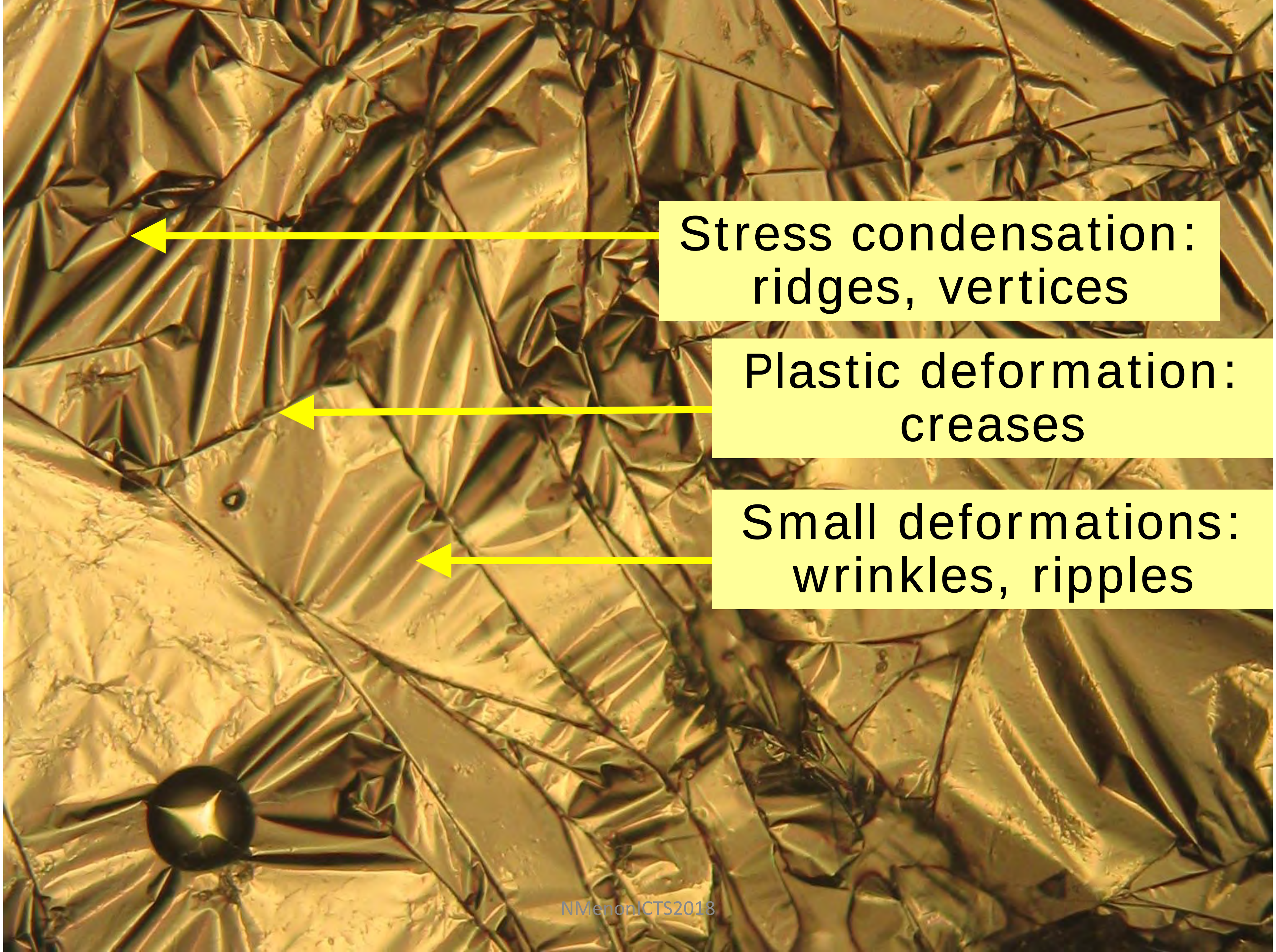

Fabric


Graphene


Earth's skin
Everest
Lhotse
Makalu
5 km

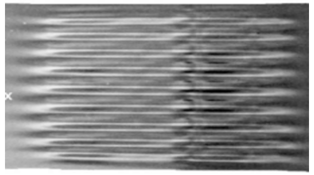
Objects that are flexible purely for geometric reasons (sheets, filaments and ribbons) make an overwhelming variety of patterns in nature and our technological world.

Can we organize this profusion of shape and form by identifying building blocks? Are there elementary excitations of elastic materials that we can study?

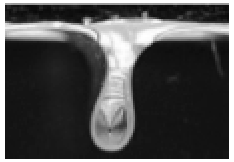

Stress condensation: ridges, vertices
Plastic deformation: creases
Small deformations: wrinkles, ripples

Overall goals of our discussion

Wrinkles

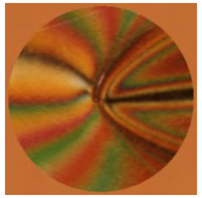


Folds



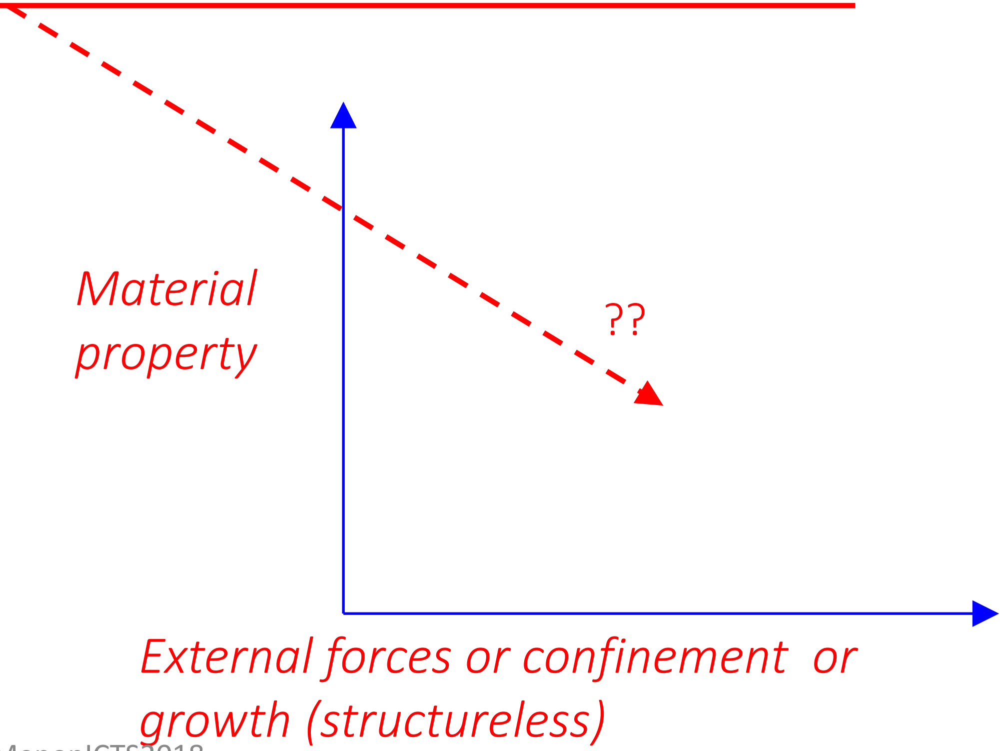
Ridges

d-cones



Creases (plasticity)

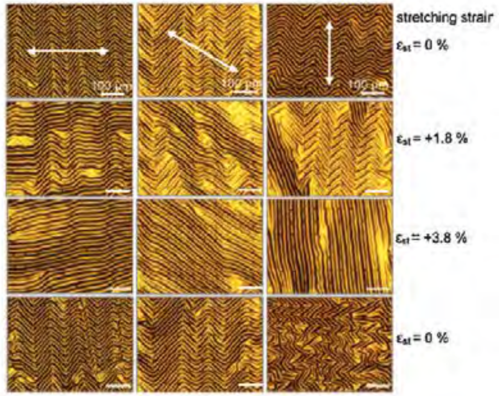
- These structures are generated by elastic instabilities
- What are the energetics and stability of these constructs?
- Where do all these structures belong?
- How to specify these axes?



Instability not as “failure” but technological tool

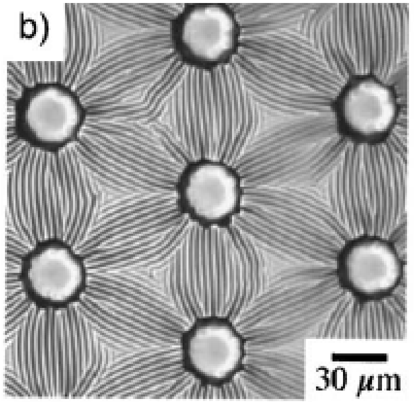
Harnessing instabilities

Stretchable electronics

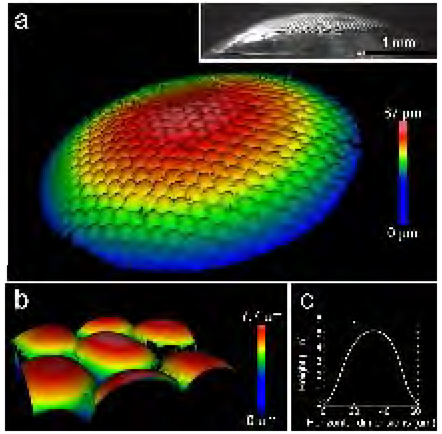


Rogers 2011

Nanoscale elastic patterning



Bowden et al, 1999



Crosby 2010

Mechanical metamaterials

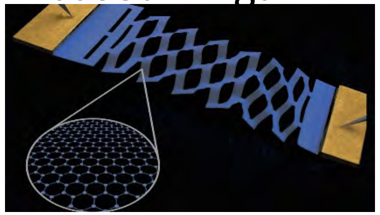
Transformable acoustic waveguides

Bertoldi

Buckliball

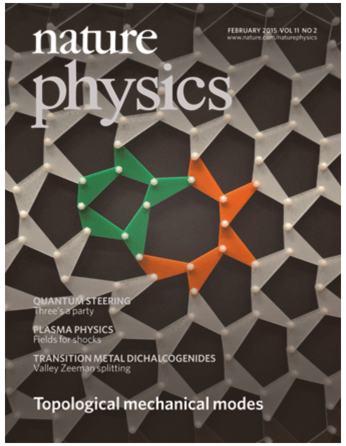
Reis2011

Graphene-based kirigami



McEuen group

Topological material



Chen, Paulose, Vitelli

Patterning (actuatable ones at that), metrology, coatings, surface control

Low-energy modes that allow large deviation

Plan

Overall theme	Pattern formation via elastic instabilities
Intro Elasticity	Stress, large deformation strain, Hooke's Law
2D Elasticity	Moduli for plates, scale separation
1D Euler buckling	Two approaches: near and far from threshold
1D wrinkling	Scaling analysis, generality of "substrate"
1D Folds	mechanical stability, exact solution, system size dependence
2D Wrinkling	Lamé problem as archetype, two limits of FvK, bendability and scale separation; (briefly) other geometries
Crumples	Ridges, d-cones and e-cones
Wrapping	Idea of asymptotic isometry; Folds in 2-D
Thermal effects	Fluctuation induces rigidity; renormalize stiffness of plates, shells

Things I will not do

Mainly mechanics, will not work at thermal scales

Advanced geometry

Working example: sheets; not filaments or ribbons

Focus on statics, not on dynamics (lots of opportunities here)

Why did these break the way they did?

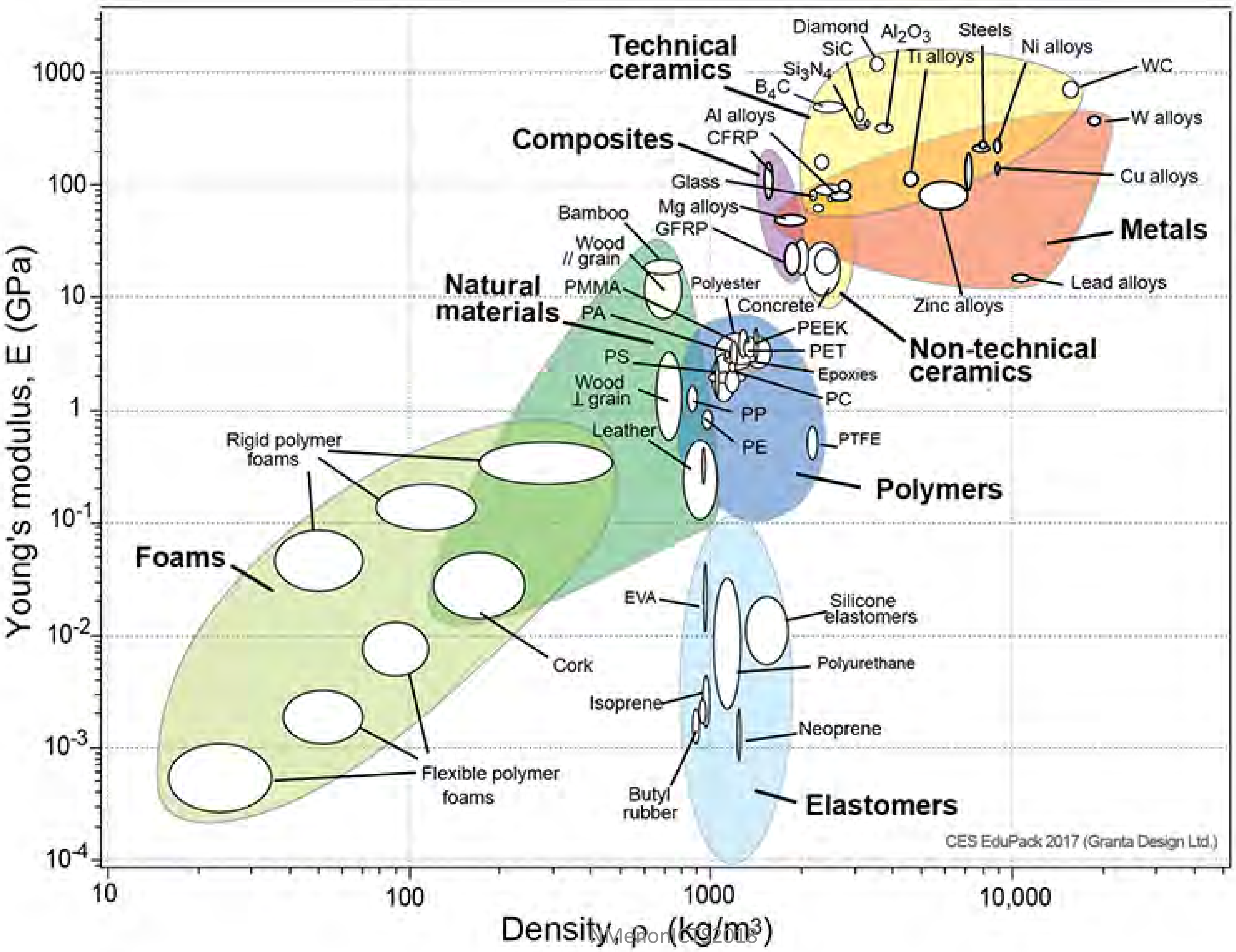
(b)

http://classes.mst.edu/civeng120

Useful (to me) books on elasticity:

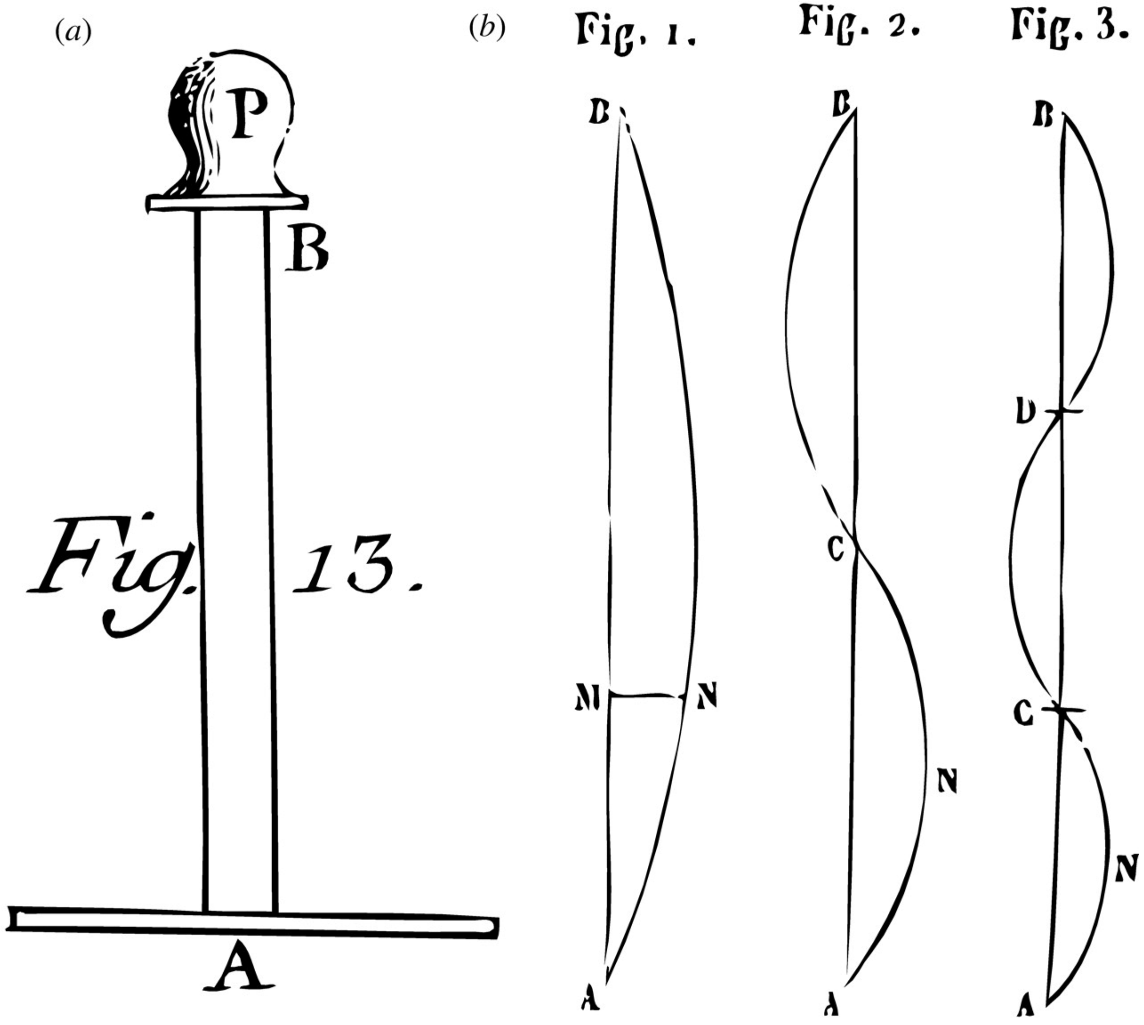
Physics of Continuum Matter by B. Lautrup -- *nice exposition at an introductory level*
Elasticity by Landau and Lifshitz – *no comments needed*
Theory of Elasticity by Timoshenko and Goodier; Plates and Shells by Timoshenko and Woinowsky-Krieger – *both books are detailed pedagogical expositions. Timoshenko is a major figure in engineering mechanics; these are good places to look up solutions for specific geometries*
Elasticity and Geometry by Audoly and Pomeau – *elegant and modern book, with most relevance to thin objects*

Day 2



Young's modulus, E (GPa)
Density, ρ (kg/m³)
Technical ceramics
Composites
Natural materials
Foams
Metals
Non-technical ceramics
Polymers
Elastomers
Diamond
SiC
Si3N4
B4C
Al2O3
Ti alloys
Steels
Ni alloys
WC
W alloys
Al alloys
CFRP
Cu alloys
Glass
Mg alloys
GFRP
Bamboo
Wood // grain
PMMA
Polyester
Concrete
Zinc alloys
Lead alloys
PA
PEEK
PET
PS
Epoxies
PC
Wood ⊥ grain
PP
PE
PTFE
Leather
Rigid polymer foams
Cork
Flexible polymer foams
EVA
Silicone elastomers
Polyurethane
Isoprene
Neoprene
Butyl rubber
CES EduPack 2017 (Granta Design Ltd.)

Euler buckling (a) illustrations from Euler (1744) (b) illustrations from Lagrange 1770.



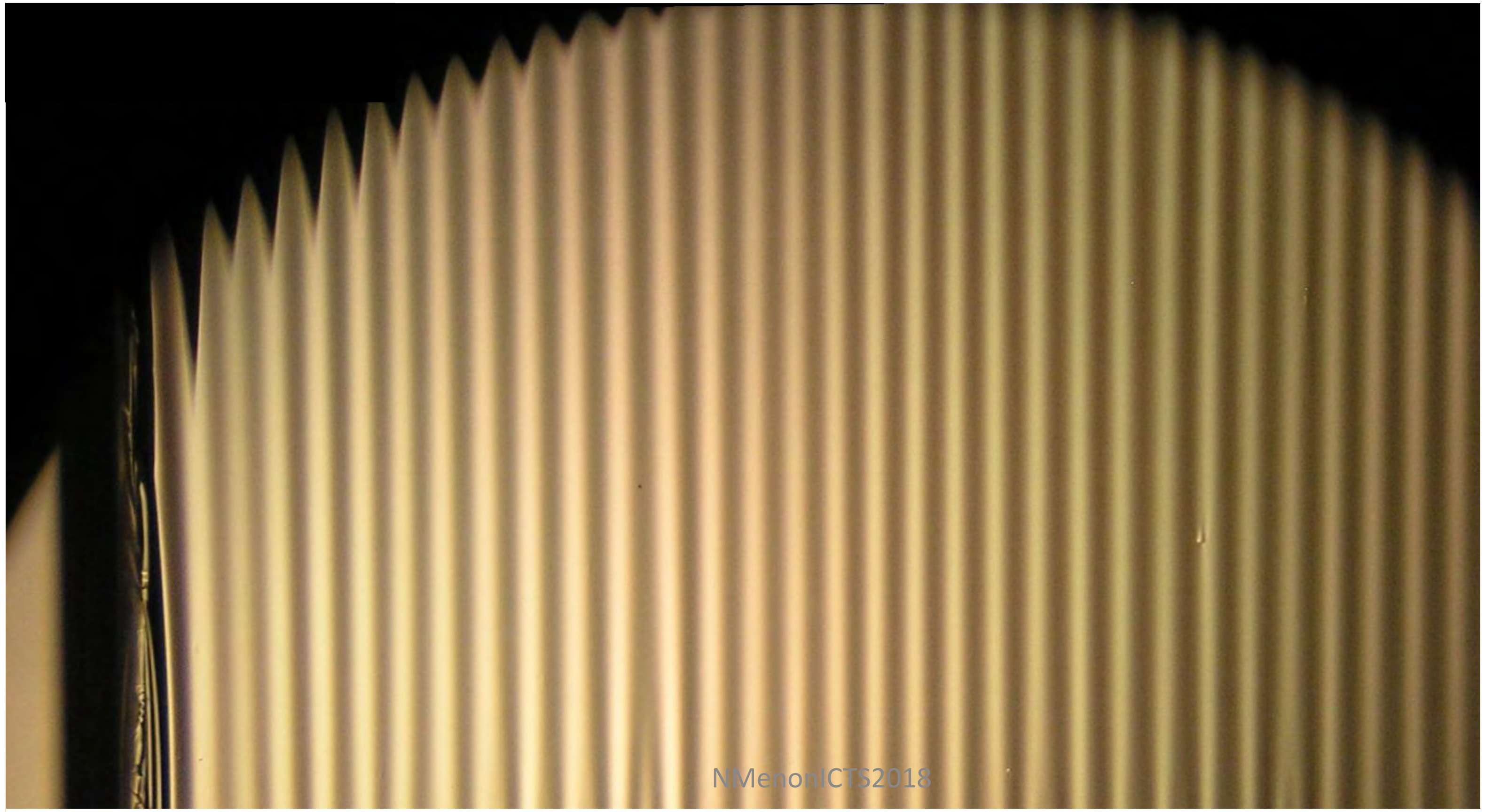
Alain Goriely et al. Proc. R. Soc. A 2008;464:3003-3019

PROCEEDINGS OF THE ROYAL SOCIETY A

Day 3

- 1D wrinkling patterns
- 1D localized solutions – folds
- 2D axisymmetric patterns

Wrinkles in 1D



Huang PRL 2010

Wrinkles in 1D

Cerda and Mahadevan PRL 2003

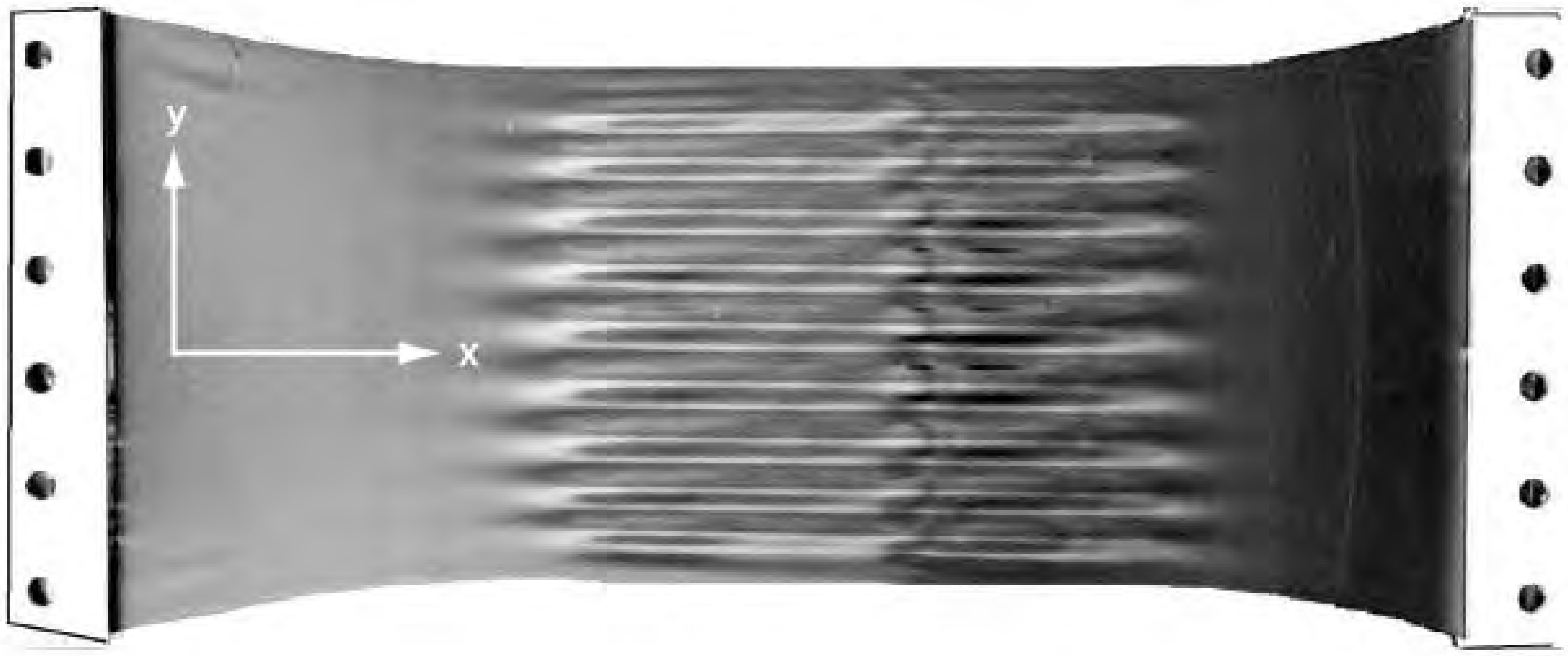
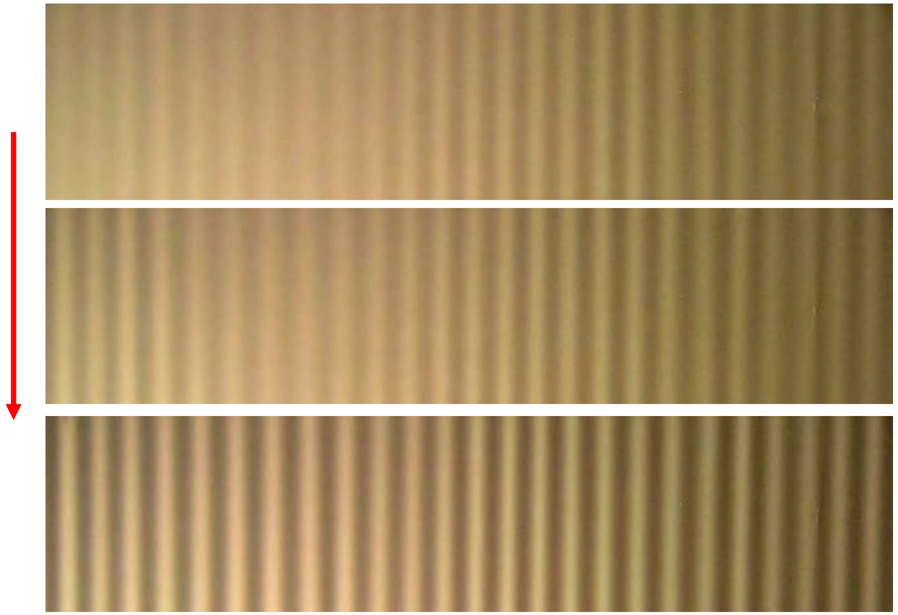
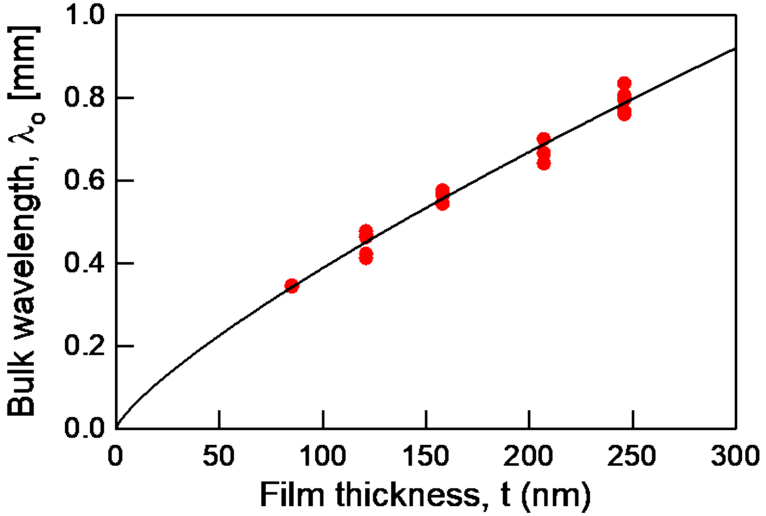


FIG. 1. Wrinkles in a polyethylene sheet of length $L \approx$ 25 cm, width $W \approx 10$ cm, and thickness $t \approx 0.01$ cm under a uniaxial tensile strain $\gamma \approx 0.10$. (Figure courtesy of K. Ravi-Chandar)

Wrinkles in 1D – fluid substrate



t=246 nm, increasing compression



Film thickness

Wavelength independent of amplitude

$$q_o^{-1} = \lambda_o = \left(\frac{B}{\rho g}\right)^{1/4}$$

Tuning wavelength through B

Thickness

Young's modulus

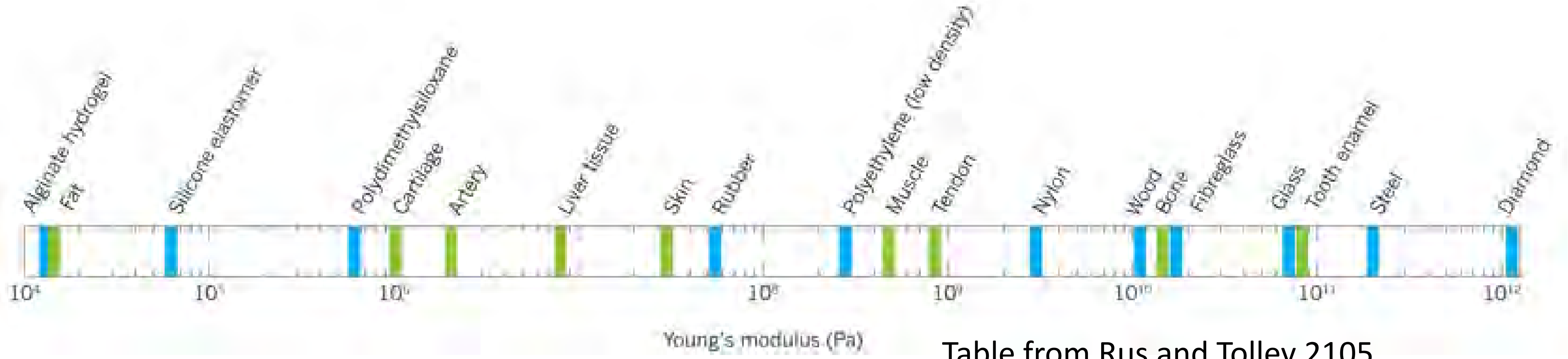


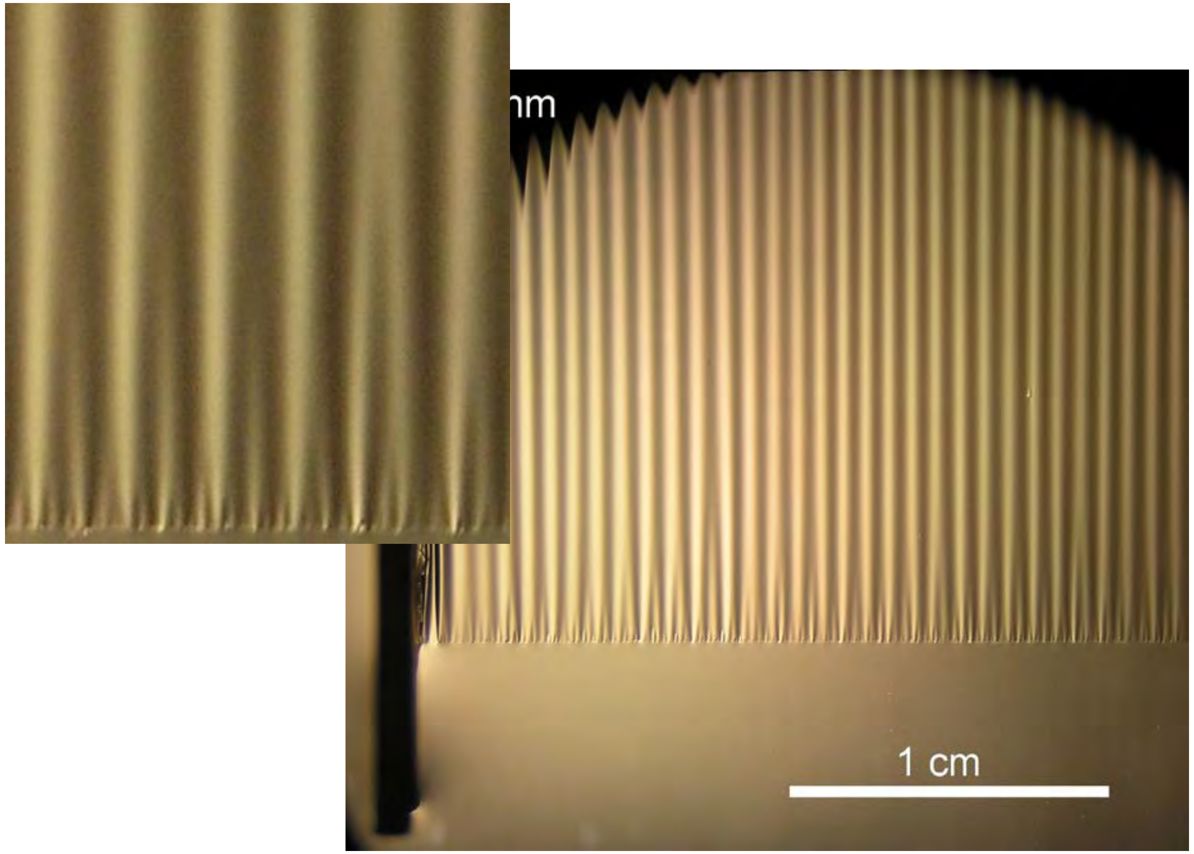
Table from Rus and Tolley 2105

Finger rafting

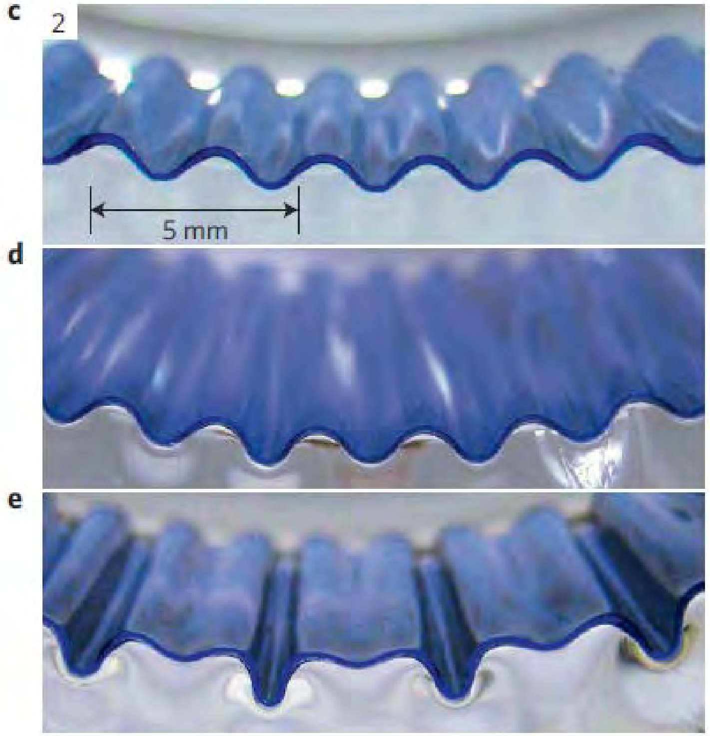
Vella and Wettlaufer, PRL 2004

Finger rafting is the block zippered pattern that forms when thin ice sheets floating on water collide creating "fingers" that push over and under each other alternately. This photo was taken off the Antarctic coast. (Credit: W.F. Weeks)

Wrinkles in 1D – beyond single mode

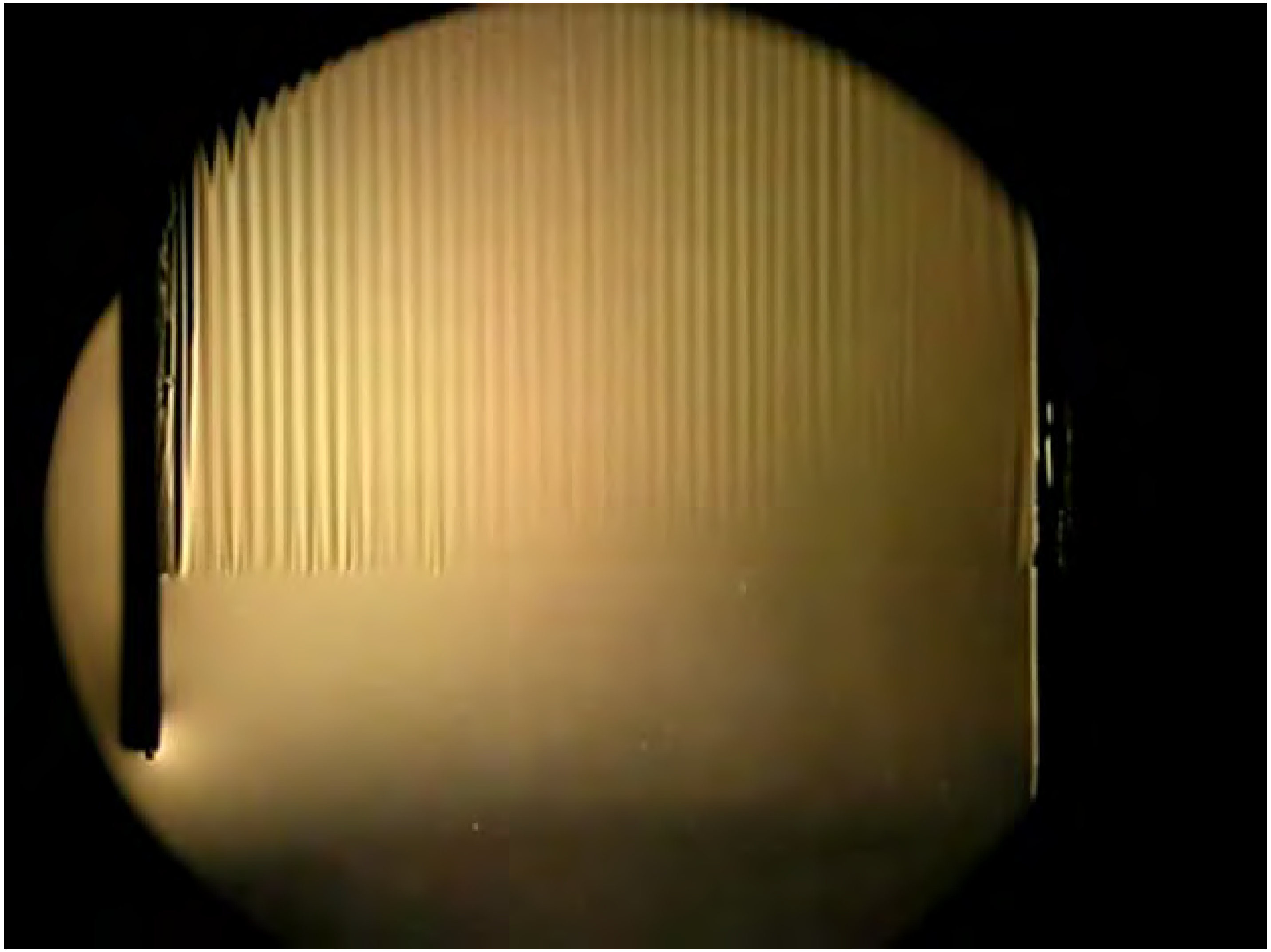


Cascade between two wavelengths, Huang 2010



Period doubling phenomena Brau et al 2010

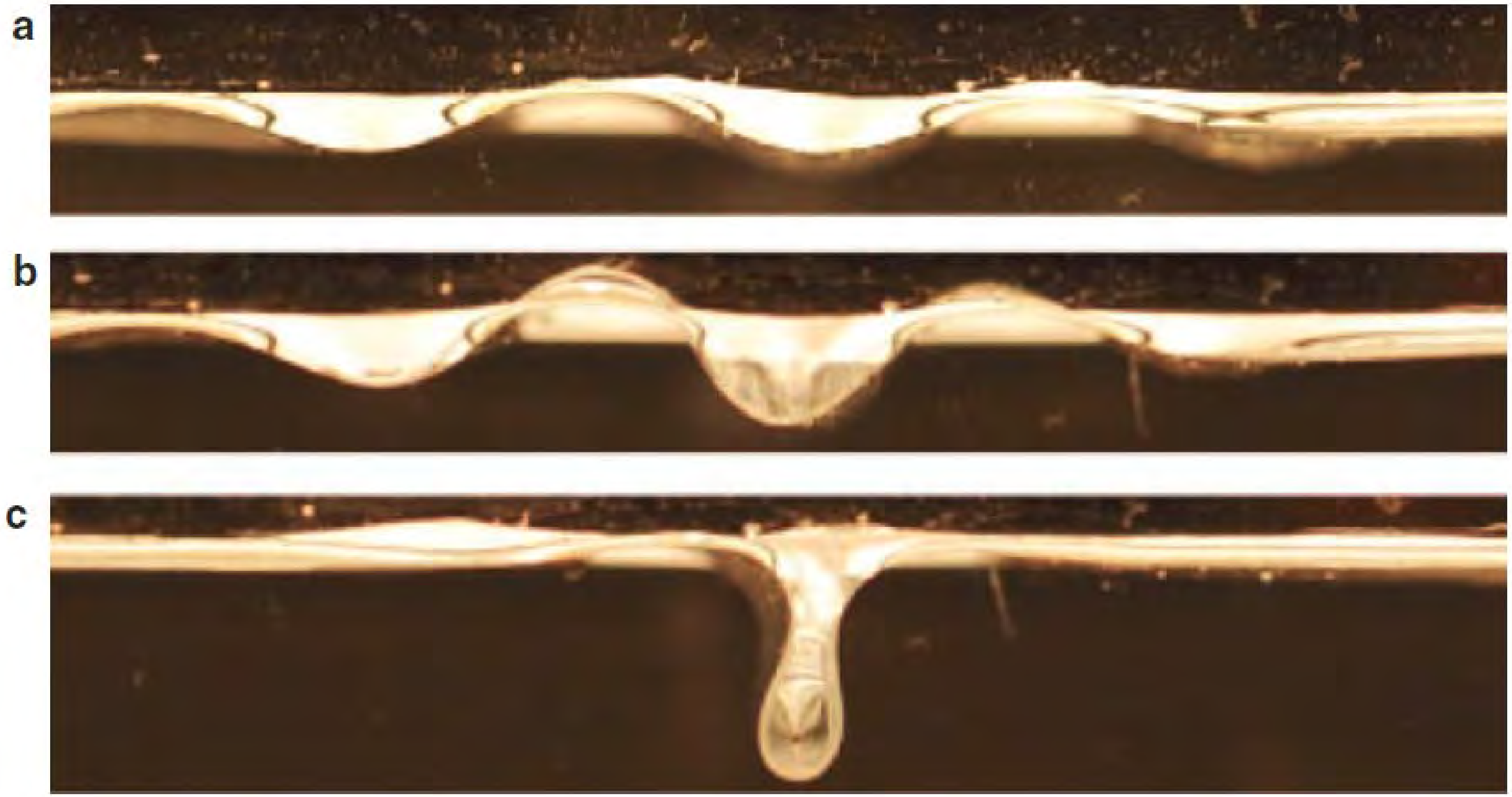
Folding in 1D



Huang thesis 2010

(this is a video)

Folding in 1D



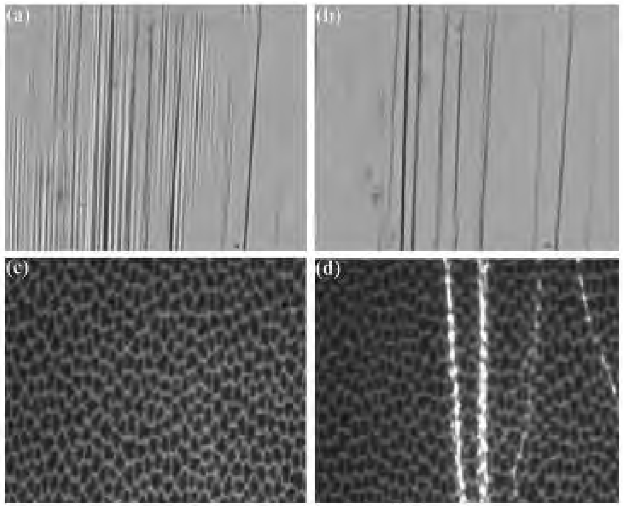


Fig. 2 (a) Bright field microscopy images of a trilayer of colloidal gold nanoparticles on a glycerol/water surface held together by van der Waals forces[20] showing a wrinkled surface when slightly compressed. Here the layer has $h \sim 15$ nm (as determined by AFM) and the wrinkles observed have a wavelength $\lambda \sim 10$ μm. (b) The same layer at further compression has the wrinkles collapse into a pattern of localized folds. (c) and (d) show fluorescence images (750 microns across) of a model lung surfactant system ($h \sim 2$ nm) composed of a 7 : 3, mol : mol mixture of 1,2-dipalmitoyl-sn-glycero-3-phosphocholine (DPPC) and 1-palmitoyl-2-oleoyl-sn-glycero-3-[phospho-rac-(1-glycerol)] (POPG)[25] at an air/water interface. Even at high compression, low-amplitude wrinkles are not observed (c) (likely due to the poor scattering of the mostly hydrocarbon lipids). However, folds (d) (appearing as bright lines running perpendicular to the direction of compression) are easily visualized with fluorescence due to the high density of surface lipids and dye pulled into a given fold. The amount of material pulled into a given fold has been previously carefully measured.[26] We used the size of the folds and our scaling law $\lambda \sim \ell$ to extract out the bending stiffness of the lung surfactant monolayer to be on the order of 10 kT in agreement with previous work.[7,17]

Plastic sheet (left)

Gold nanoparticles, lung surfactant

Pocivavsek Science 2008, Soft Matter 2009

Folding in 1D

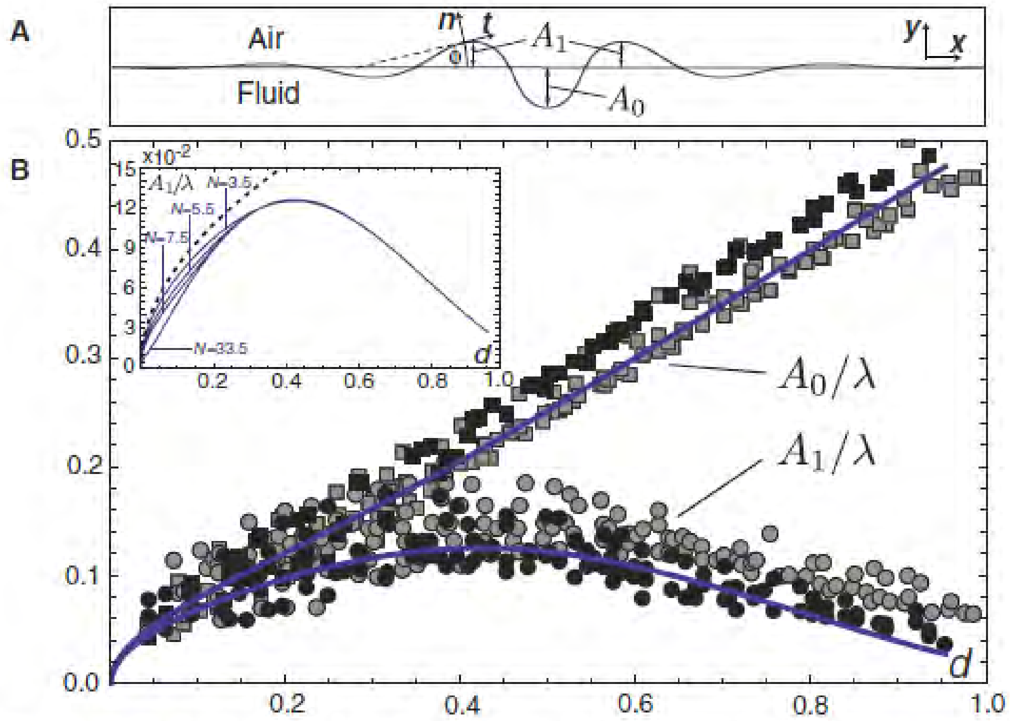


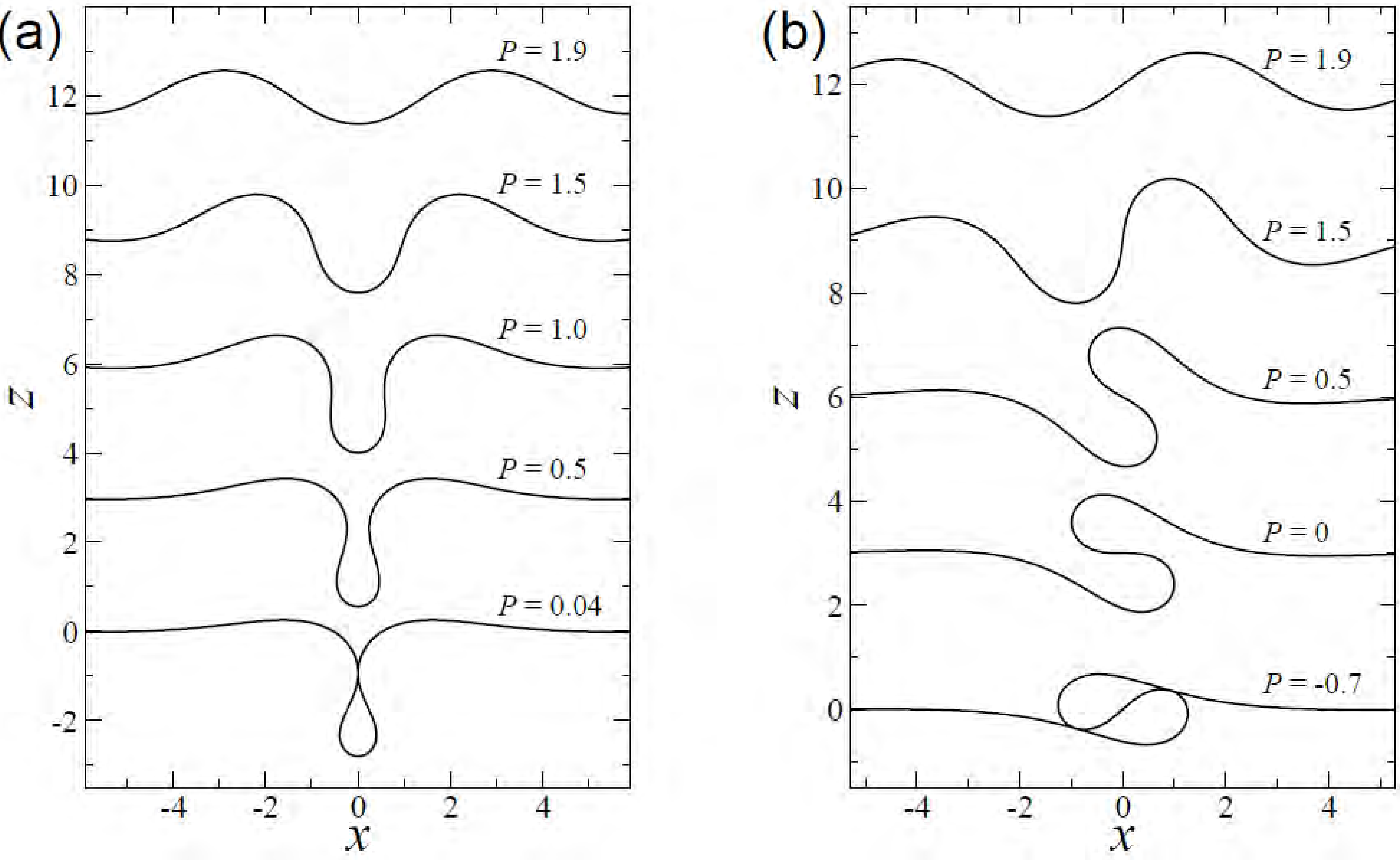
Fig. 2. (**A**) The figure defines A_0 and A_1 and the geometrical parameters describing a confined sheet. The deformation can be described by using a two-dimensional coordinate system. Here t and n are the tangent and normal to the surface, respectively. ϕ gives the position of the tangent with respect to the horizontal direction. (**B**) Experimental results for polyester on water for A_0 (squares) and A_1 (circles). Experimental data were taken for several membrane sizes, including when $N = 3.5$, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, and 8.0. Dark solid lines show numerical results for a sheet with $L = 3.5\lambda$. Both the physical polyester and numerical data are made dimensionless. A_1, A_0, and Δ are scaled to λ. (Inset) A_1 versus horizontal displacement for several numerical systems of different sizes (solid blue lines). The dashed line is the theoretical curve $A = [(\sqrt{2})/\pi]\lambda\sqrt{(d/3.5)}$ (20) that follows the numerical curve for $N = 3.5$ and $d << 1$. In both numerical and physical cases, the data are more scattered for $d < 0.3$ and then collapse onto more compact (perfectly so in numerical case) curves past this point. This behavior is indicative of the size-dependent behavior in the wrinkling ($d < 0.3$) regime and size-independent behavior in the folding ($d > 0.3$) regime.

- Pocivavsek 2008

- Transition to fold at around $A = 0.3\lambda$

Exact solution

Diamant and Witten 2012



The symmetric (left) and antisymmetric (right) solutions are degenerate

Both cost less than the wrinkle solution at all Δ

After self-contact, get penetration and nonphysical solutions

Large folds

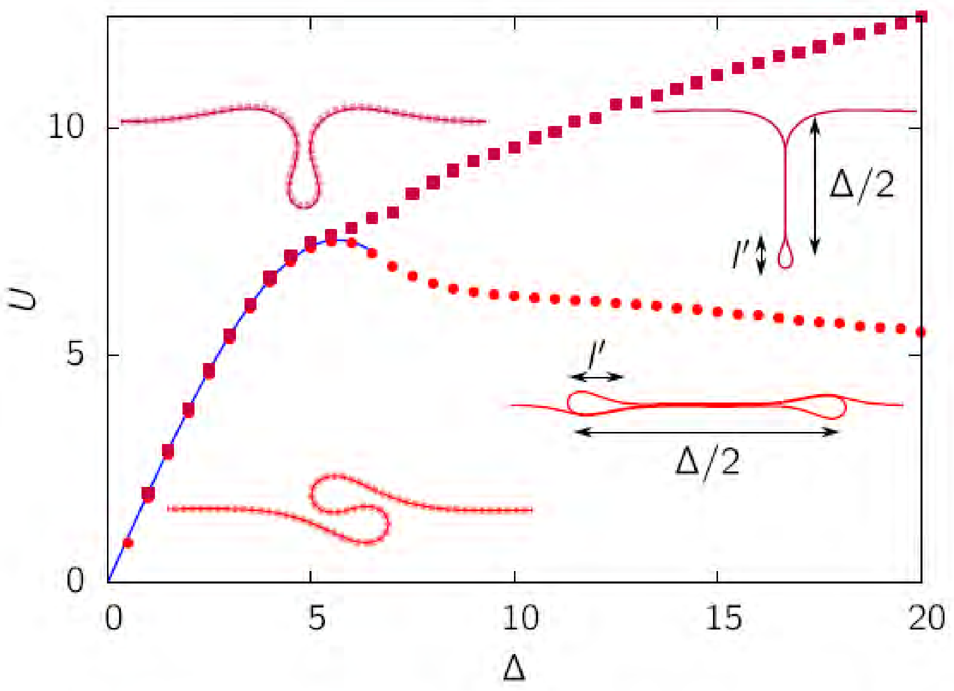


FIG. 2. (Color online) Fold energy as a function of the imposed displacement for the symmetric (squares) and antisymmetric (circle) folds. The solid blue line is the exact solution, Eq. (3), valid before self-contact. Symmetric (top) and antisymmetric (bottom) configurations are shown before self-contact (left, exact solutions from Diamant and Witten [22] are shown as thick dashed lines) and after self-contact (right). After self-contact, the size of the fold $\Delta/2$ absorbs the excess length, while bending is localized in highly curved zones of length l'.

Demery et al 2014

Goes beyond self-contact

Symm and antisymm degenerate till self-contact, but anti-symmetric wins for larger folds

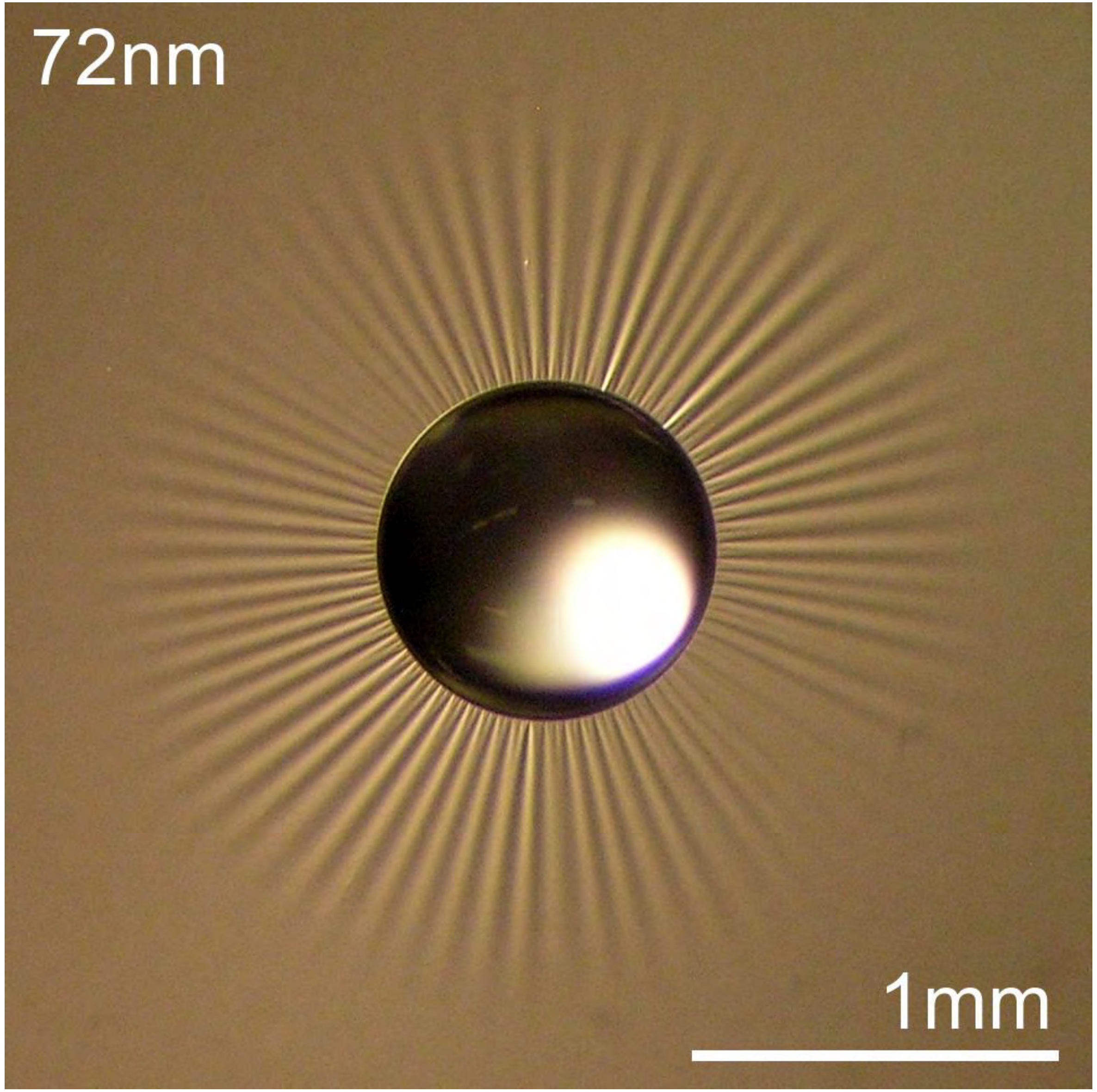
Main source for 1D wrinkling calculation -
Cerda, E., & Mahadevan, L. (2003). Geometry and physics of wrinkling. *Physical review letters*, *90*(7), 074302.

More recent – Paulsen et al "Curvature-induced stiffness and the spatial variation of wavelength in wrinkled sheets." *PNAS* 113, no. 5 (2016): 1144-1149.

1D folds
Pocivavsek, L., Dellsy, R., Kern, A., Johnson, S., Lin, B., Lee, K. Y. C., & Cerda, E. (2008). Stress and fold localization in thin elastic membranes. *Science*, *320*(5878), 912-916.
Diamant, H., & Witten, T. A. (2011). Compression induced folding of a sheet: An integrable system. *Physical review letters*, *107*(16), 164302.
Démery, V., Davidovitch, B., & Santangelo, C. D. (2014). Mechanics of large folds in thin interfacial films. *Physical Review E*, *90*(4), 042401.

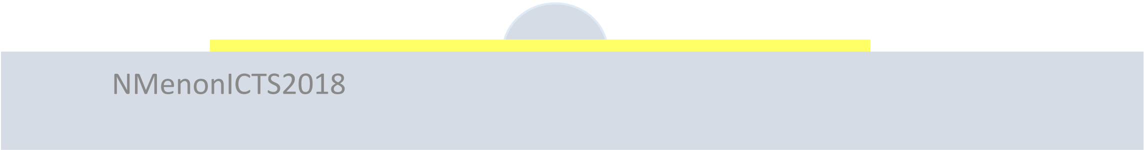
Discussion of Euler buckling regimes follows a pedagogical review in preparation by Benny Davidovitch and myself. Get in touch with me if you want a draft when it is ready

2D wrinkles



Thin sheet of plastic (PS) floating on water with a drop of water in the middle

Huang et al. Science 2007



2D wrinkles

Measure:

Wavenumber, *N*

Length, *L*

Dependence on

- elasticity of sheet

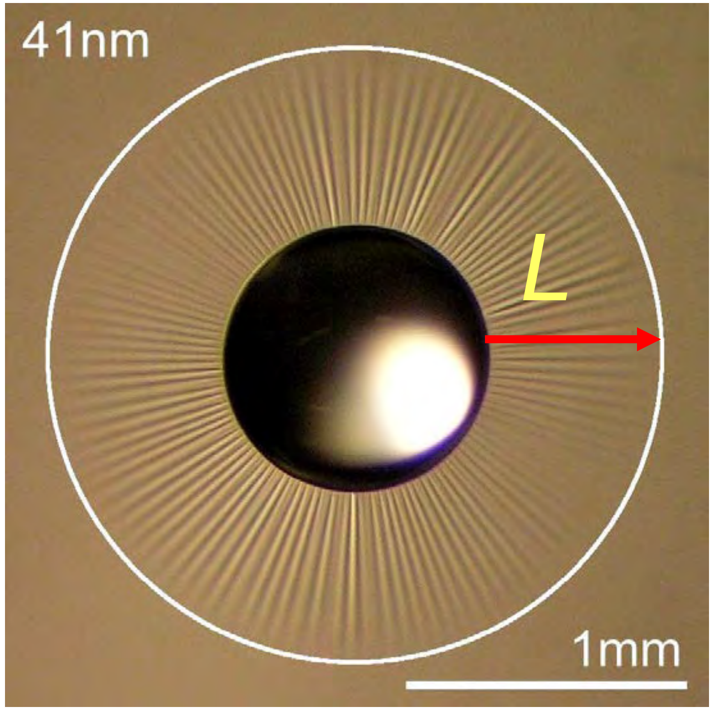
  thickness, *t*,

  Young's Modulus, *E*

- loading

  radius of drop, *a*

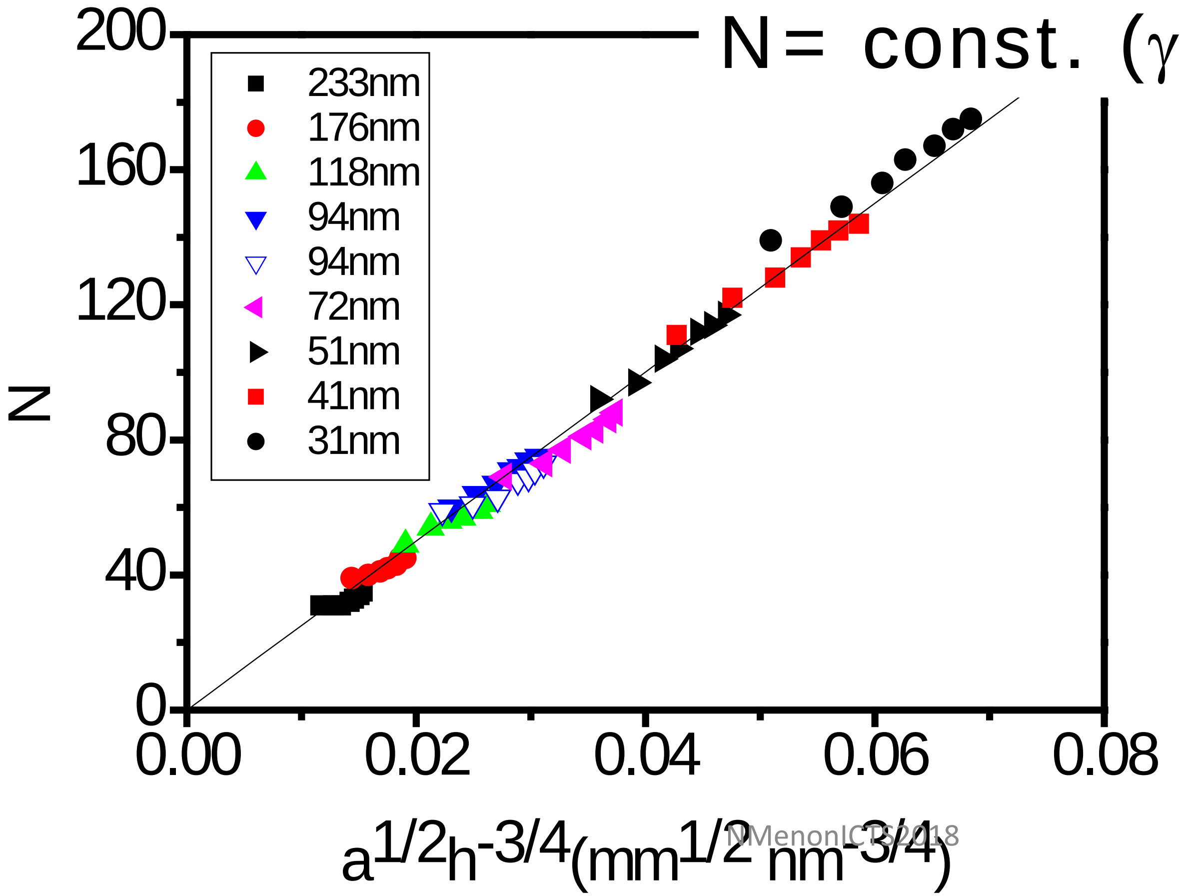
  surface tension, γ



2D wrinkling – wrinkle number

Standard (post-buckling) analysis captures dependence on drop size, film thickness

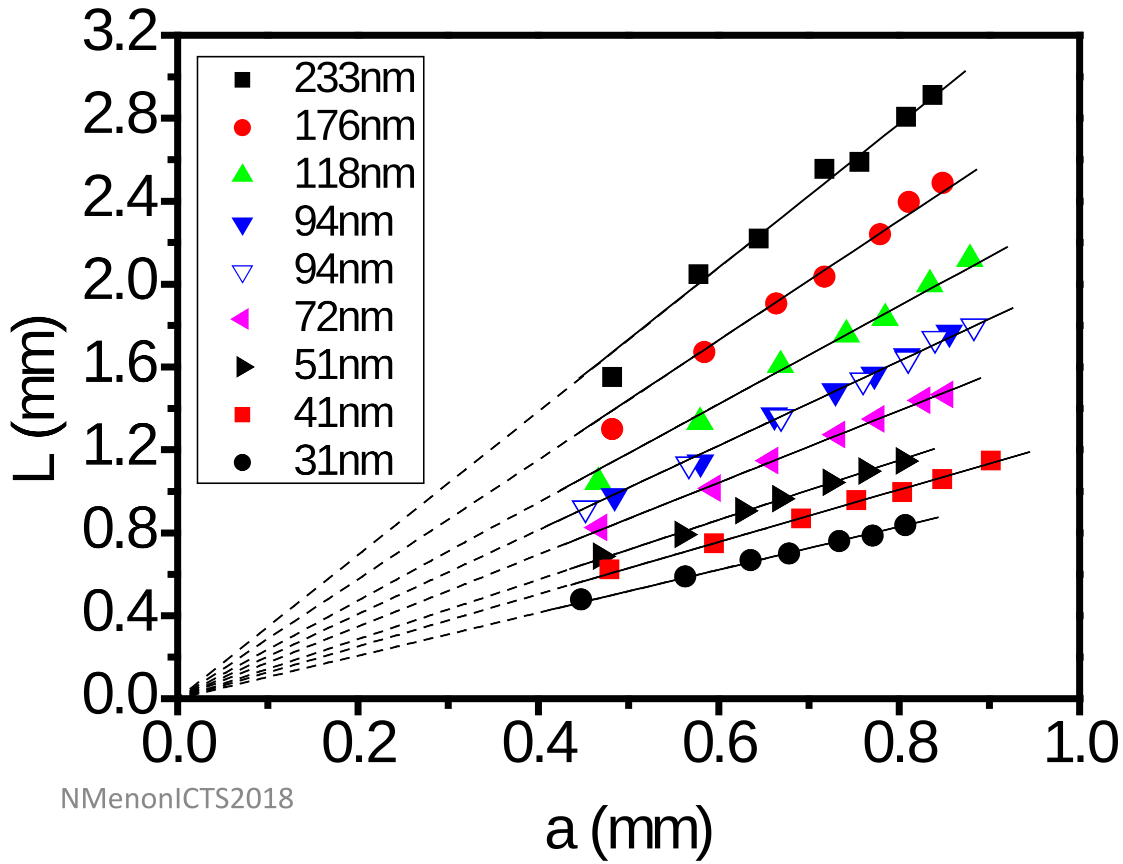
$$N = \text{const.}\ (\gamma/B)^{1/4} a^{1/2}$$



Length of wrinkles

Scaling L ~ a (post-buckling) found in Cerda 2005

L increases with *a*, but thickness dependence, too

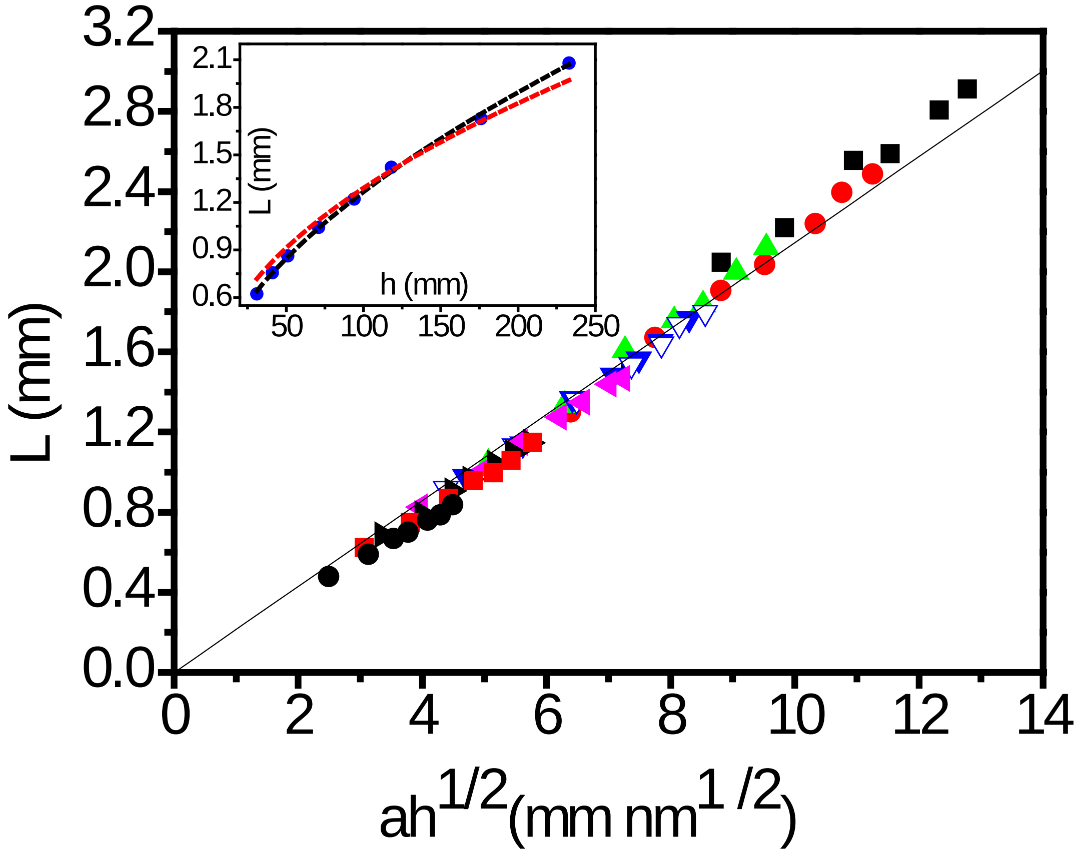


NMenonICTS2018

2D wrinkling - length

Postbuckling scaling does not work *L ~ a* *e.g. Cerda J. Biomech 2005*

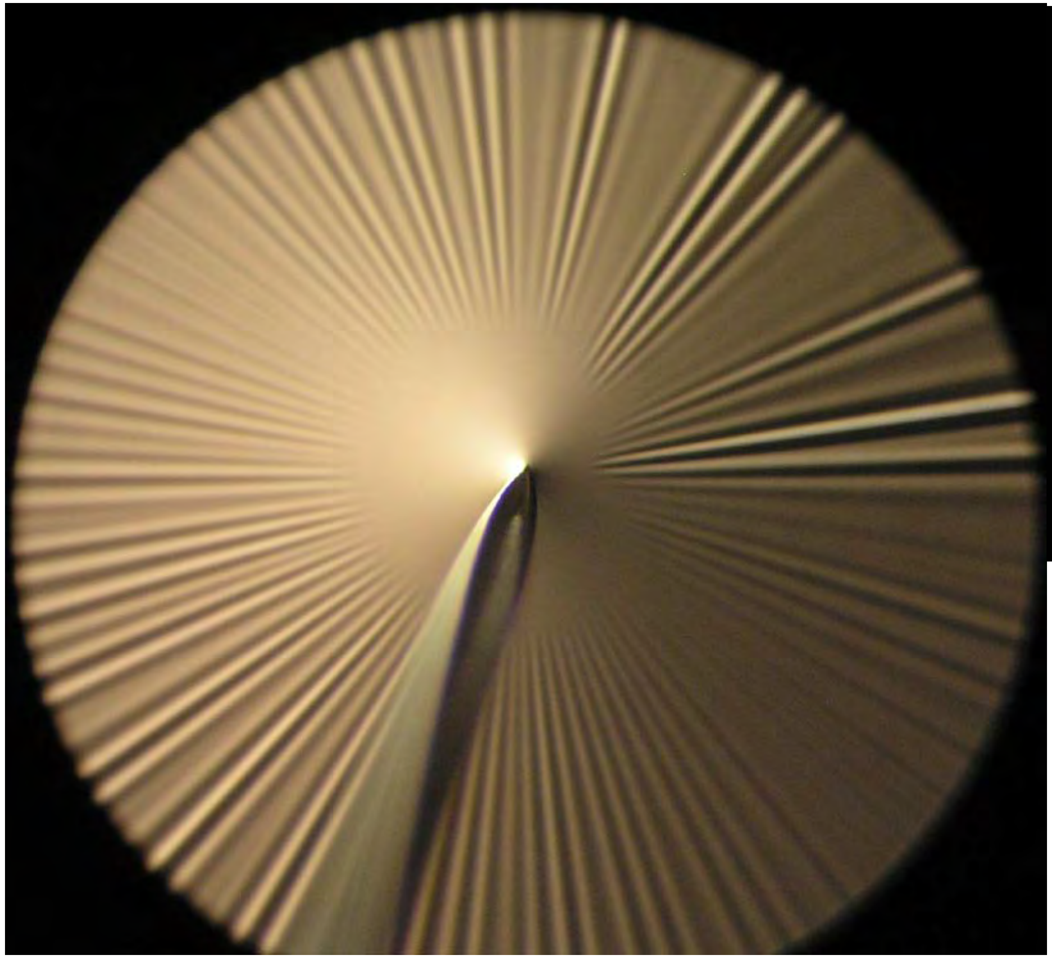
Data approximately collapsed by $L \sim a\, t^{1/2}$



Other variables available to fix dimensions: *E, γ*

Only possible combination: $L = C\, a\, t^{1/2} (E/\gamma)^{-1/2}$

Other axisymmetric geometries



- Sheet on drop – positive Gaussian curvature

King, Schroll, etc PNAS 2012

Postbuckling analysis fails to describe these situations as well

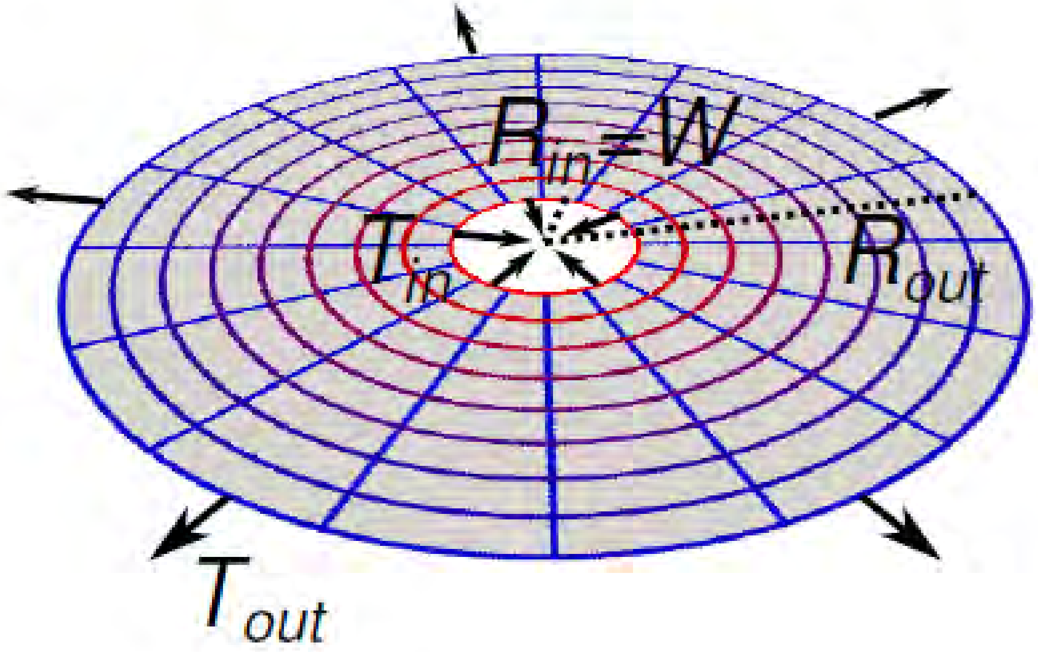
- Poking – negative Gaussian curvature

Vella, Huang, etc 2015

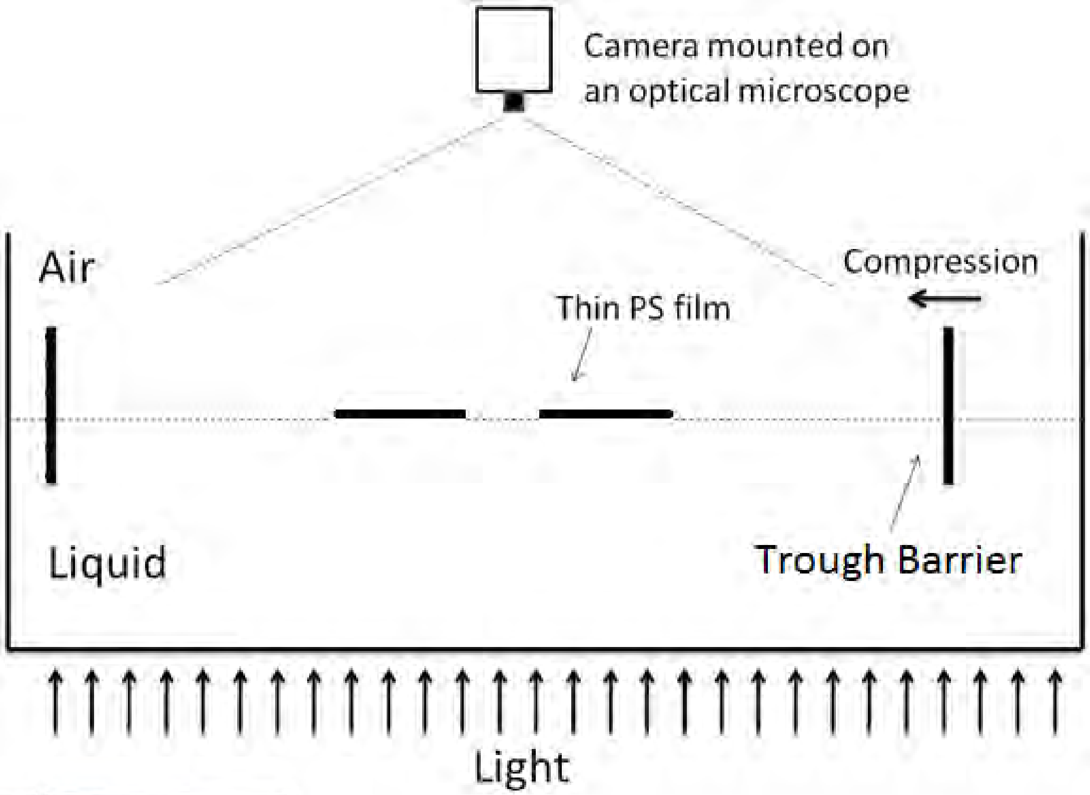
Lecture 4

- Wrinkles in 2D geometries
  - Bendability
  - Near-threshold, far-from-threshold
- Crumples – quick overview
  - Ridges, d-cones

Lamé problem



Davidovitch, et al PNAS 2011



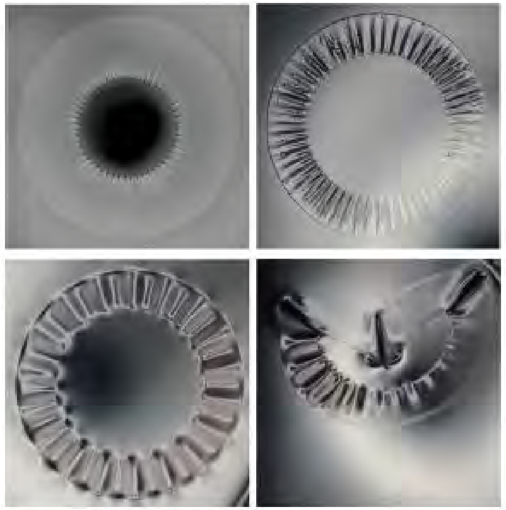
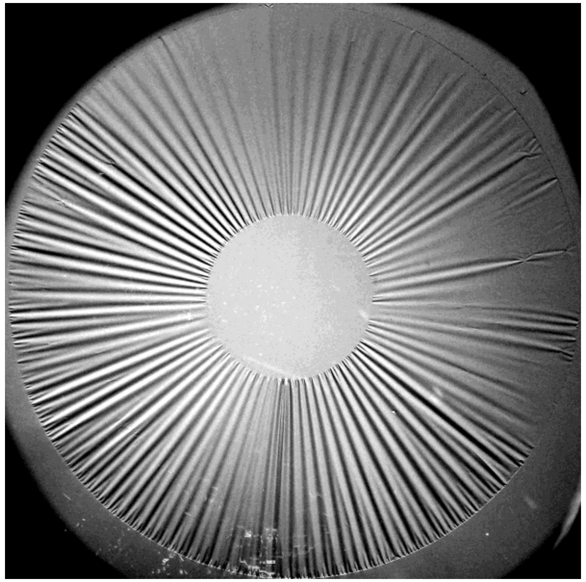


Fig. 2 Typical experiments where wrinkles appear below a critical value of the outer surface tension ($\gamma_i = 71 \pm 1$ mN m^{-1}): (a) "Wide" annulus, $a/b = 0.35$, $t =$ 35 µm, $\gamma_o = 30 \pm 1$ mN m^{-1}, (b) narrow annulus, $a/b = 0.7$, $t = 16$ µm, $\gamma_o = 45 \pm 1$ mN m^{-1}, (c) $a/b = 0.7$, $t = 50$ µm, $\gamma_o = 38.5 \pm 1$ mN m^{-1}, and (d) the same annulus after collapse, $\gamma_o = 32 \pm 1$ mN m^{-1}.

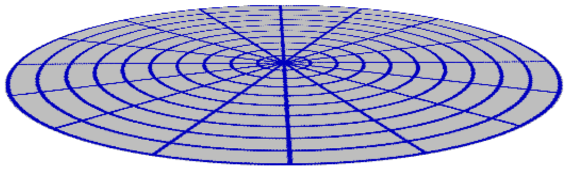
N

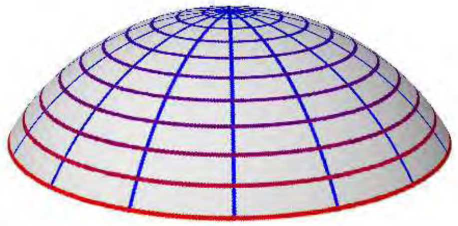
Pineirua Soft Matter

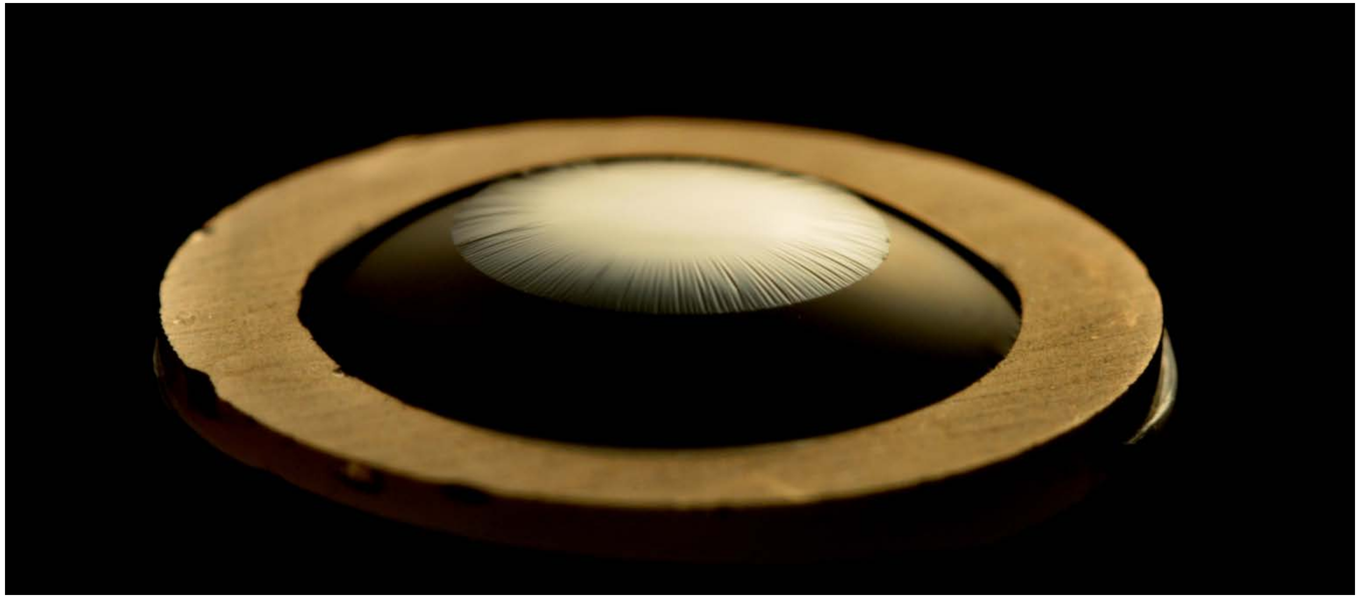
KB Toga thesis



Flat sheet on curved surface



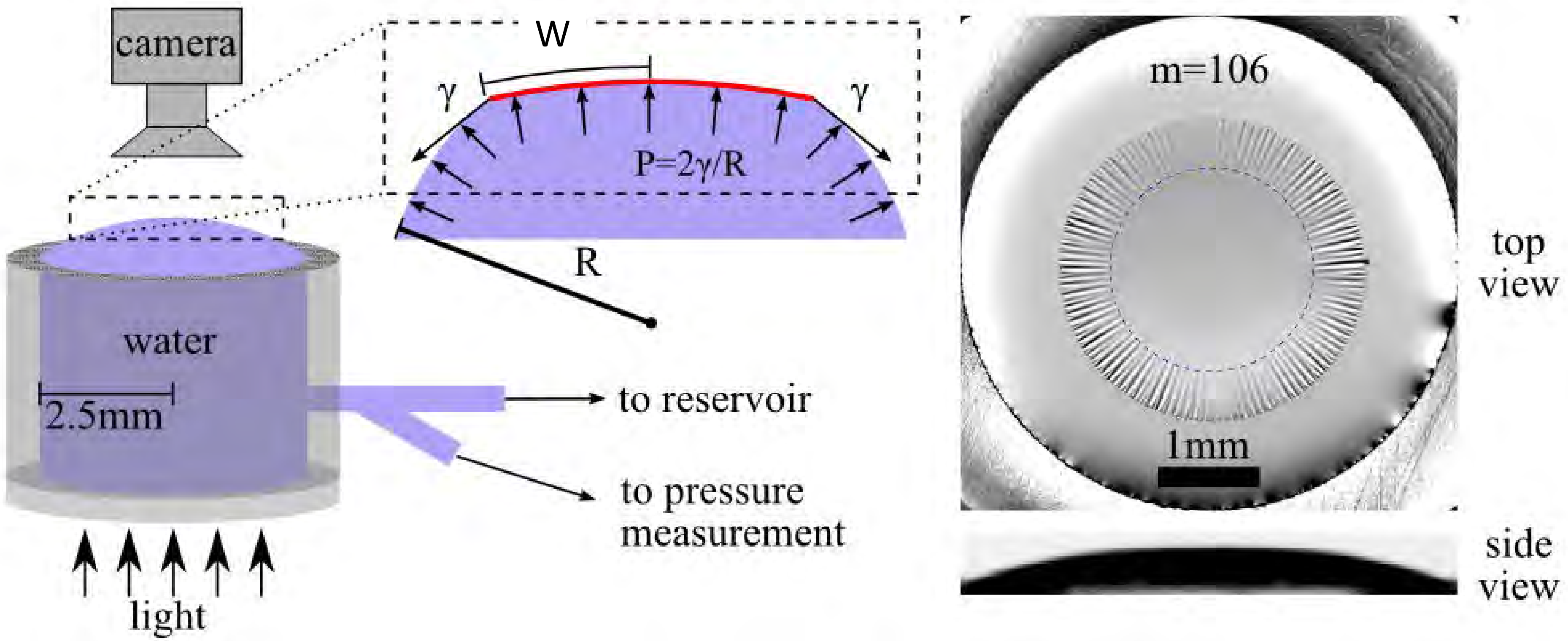




Confinement into smaller perimeters governed by

$\alpha = Y/\gamma\ (W/R)^2$ *Y: stretching modulus*

Experimental Setup



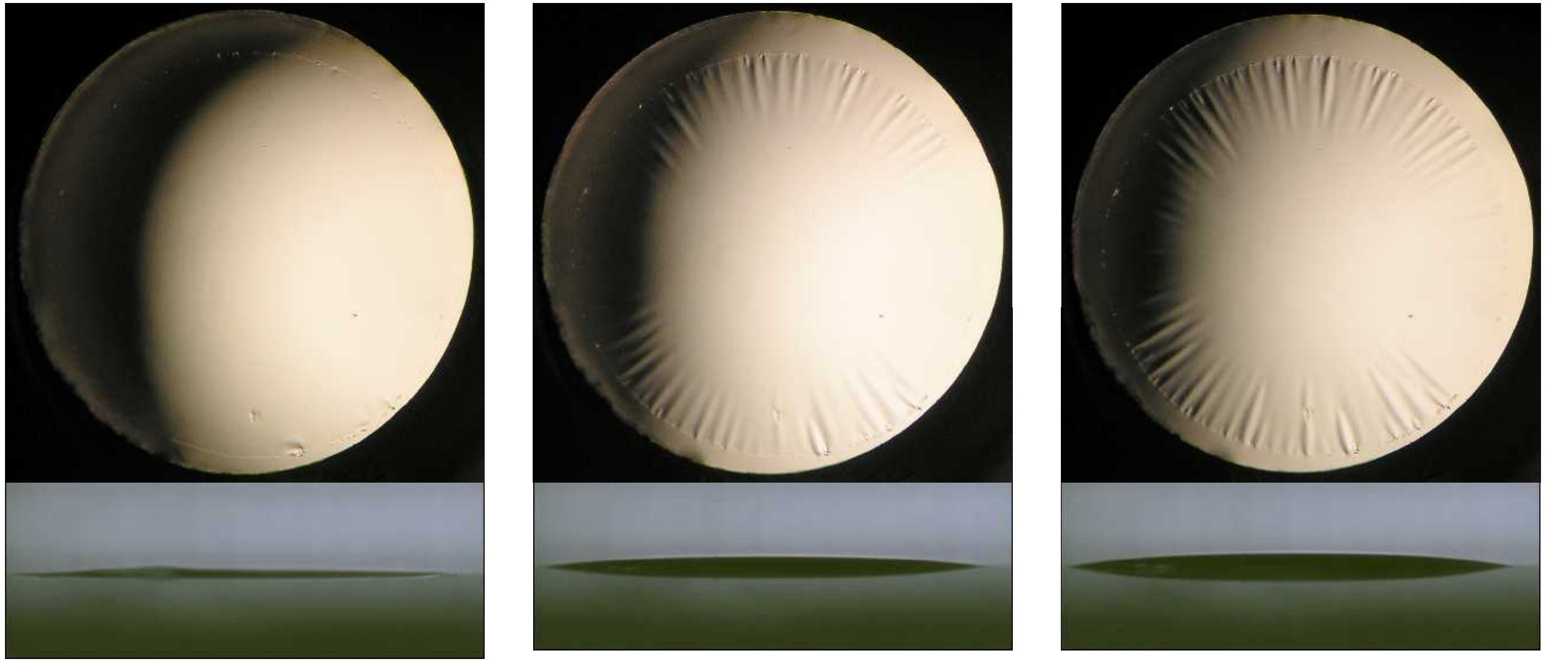
Spin-coated polystyrene film: thickness, t = 40 to 150 nm

Radius of film, W = 0.5 to 1.5 mm

Wrinkles grow inward from edge

Top view

Side view

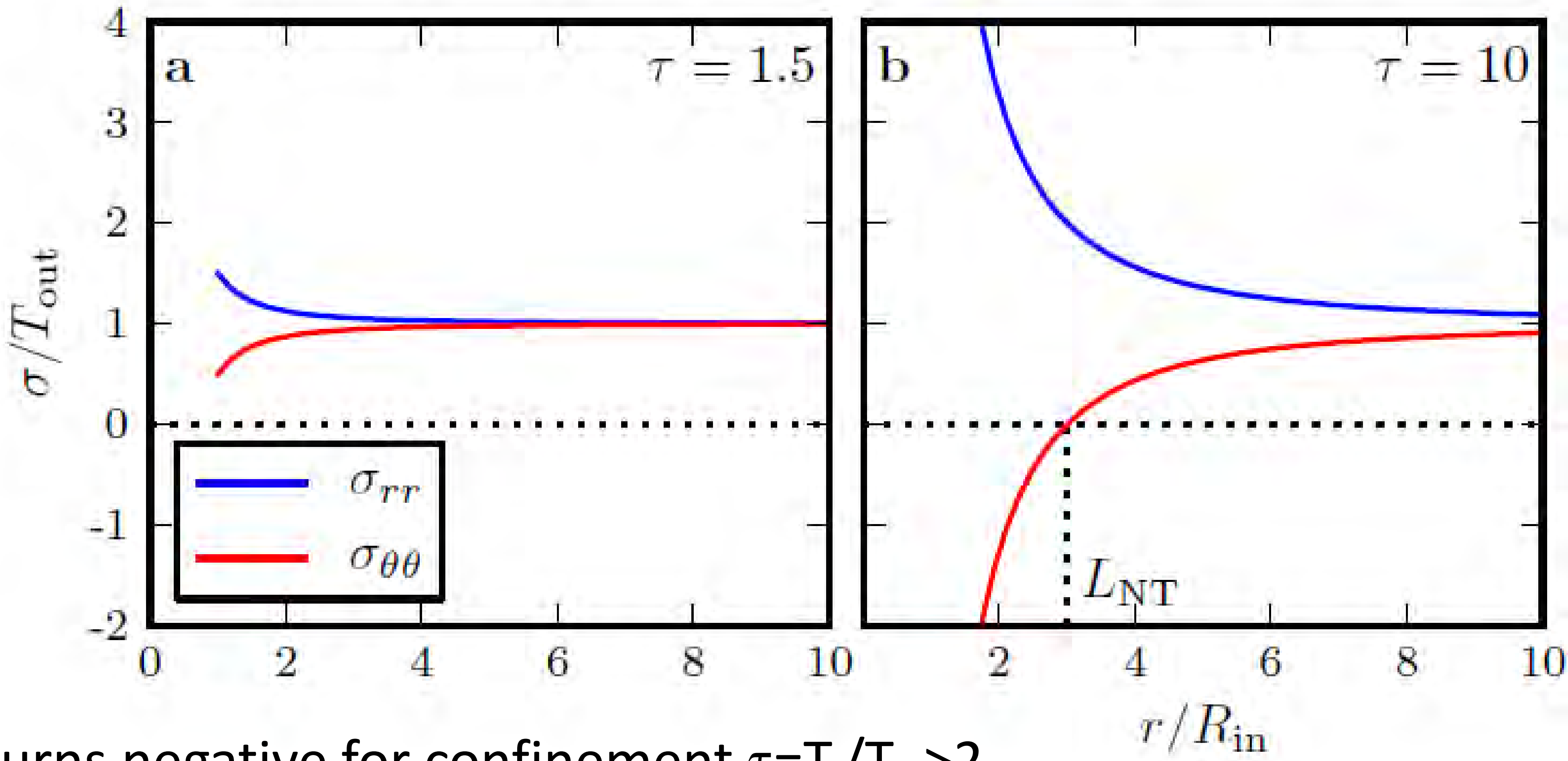


increase drop curvature →
(increase confinement α)

How long are the wrinkles? How many are there?

Lamé solution

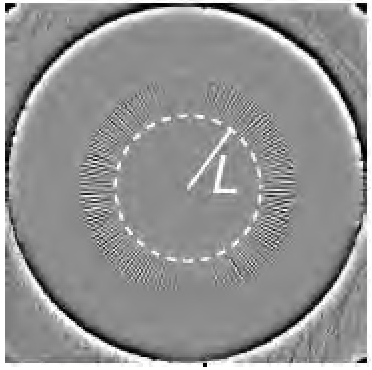
Davidovitch, et al PNAS 2011

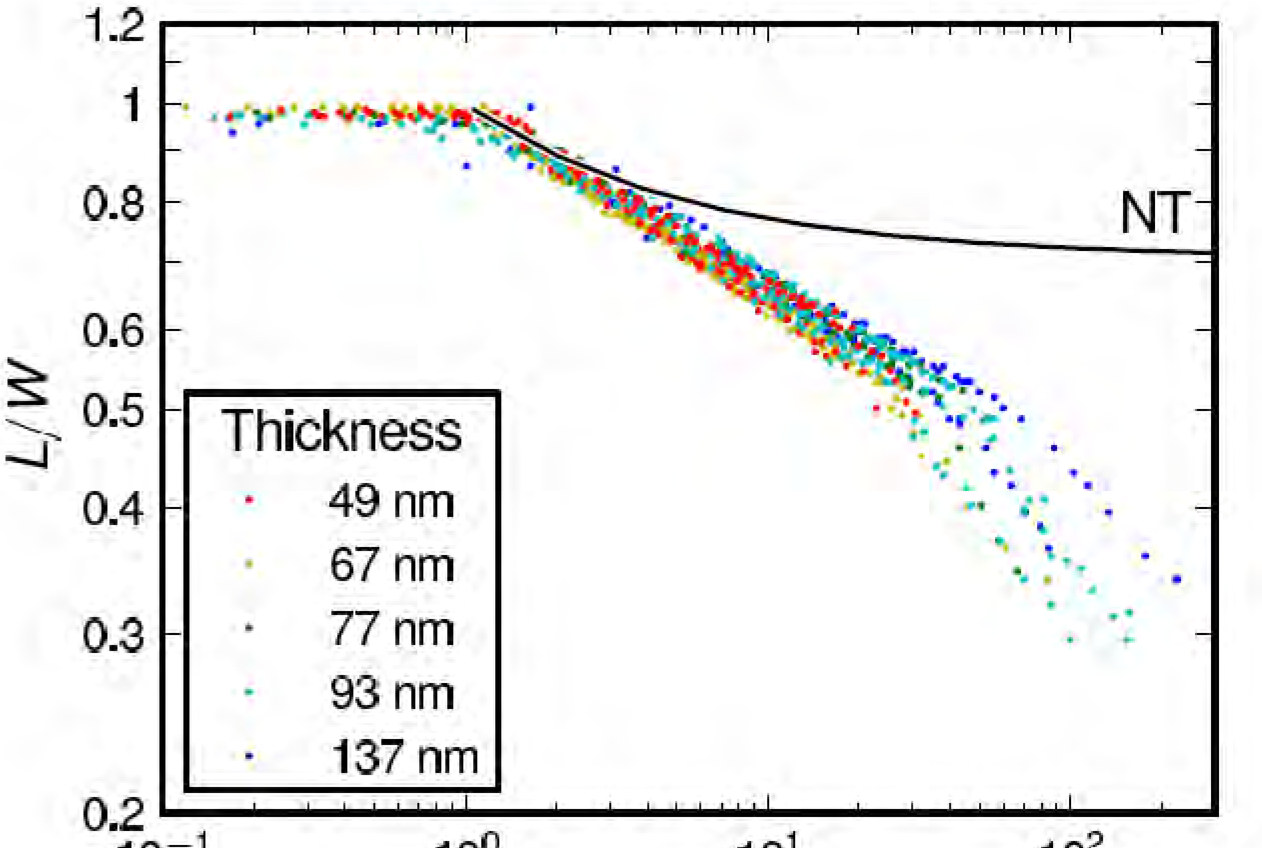


- Azimuthal stress turns negative for confinement $\tau = T_i/T_o > 2$

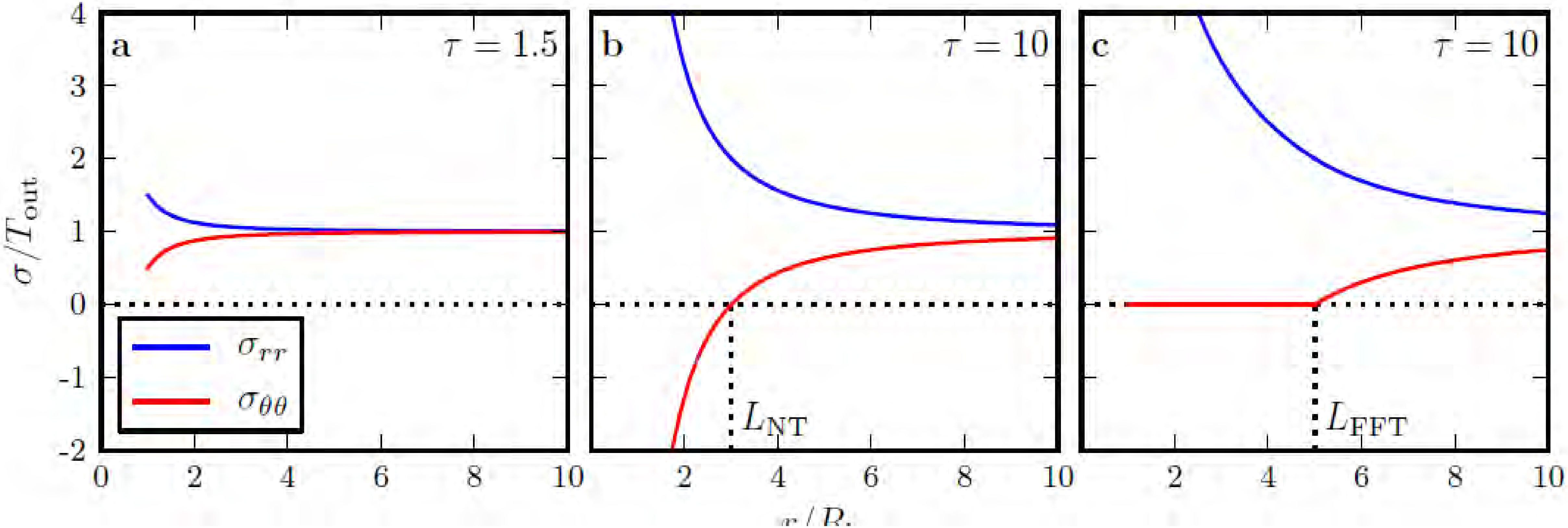
Wrinkle length

Normalized size of unwrinkled zone



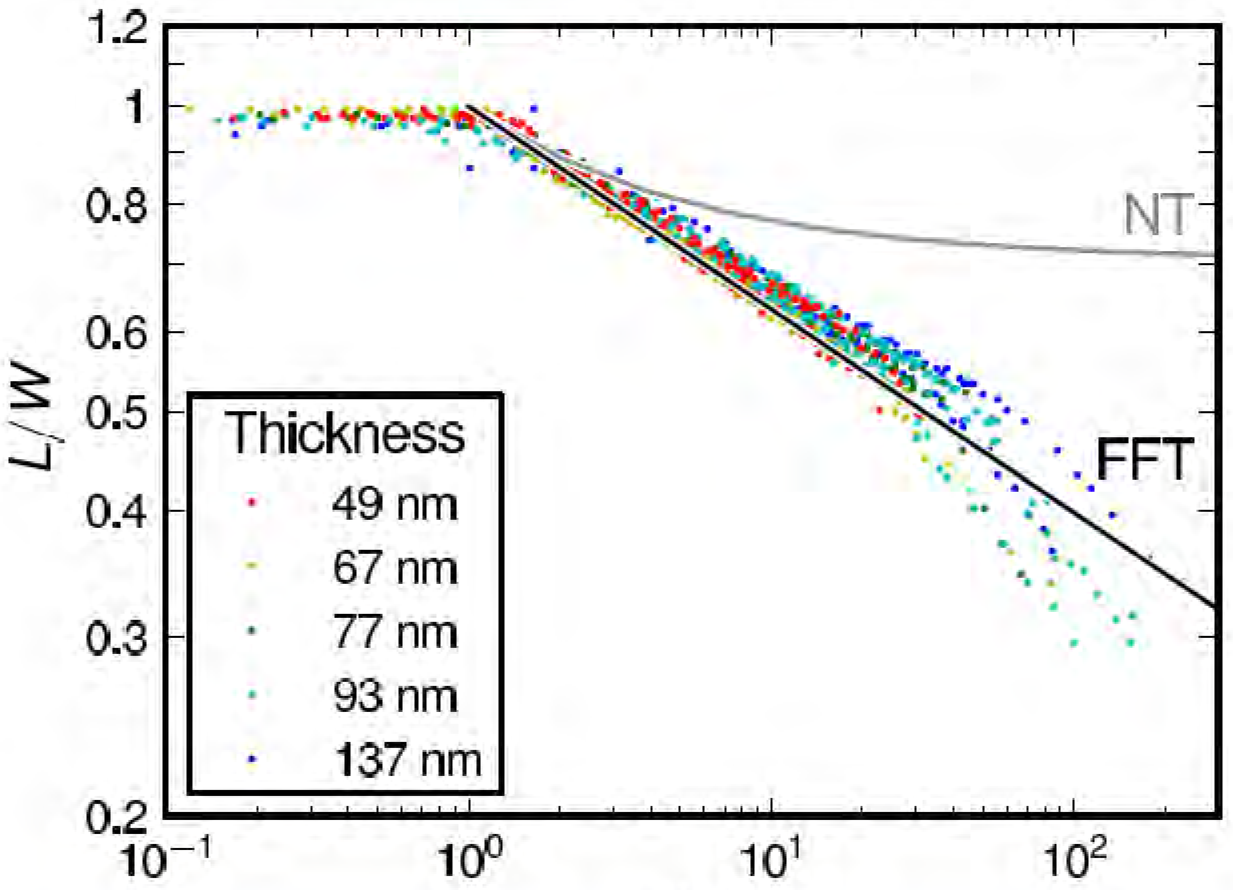


Post-buckling (NT) calculation gets wrinkle length entirely wrong


a
$\tau = 1.5$
b
$\tau = 10$
c
$\tau = 10$
$\sigma / T_{\rm out}$
σ_{rr}
$\sigma_{\theta\theta}$
$L_{\rm NT}$
$L_{\rm FFT}$

Wrinkle length

Normalized size of unwrinkled zone



Confinement α/α^0_{wr}

FT calculation predicts wrinkle length successfully

Lecture 4 references:

2D wrinkling

Huang, J., Juszkiewicz, M., De Jeu, W. H., Cerda, E., Emrick, T., Menon, N., & Russell, T. P. (2007). Capillary wrinkling of floating thin polymer films. *Science*, *317*(5838), 650-653.

Davidovitch, B., Schroll, R. D., Vella, D., Adda-Bedia, M., & Cerda, E. A. (2011). Prototypical model for tensional wrinkling in thin sheets. *Proceedings of the National Academy of Sciences*, *108*(45), 18227-18232.

King, H., Schroll, R. D., Davidovitch, B., & Menon, N. (2012). Elastic sheet on a liquid drop reveals wrinkling and crumpling as distinct symmetry-breaking instabilities. *Proceedings of the National Academy of Sciences*, *109*(25), 9716-9720.

The last two papers --particularly the supplementary info of the 2012 PNAS -- are good resources to follow up my blackboard notes

Paulsen et al "Curvature-induced stiffness and the spatial variation of wavelength in wrinkled sheets." *PNAS* 113, no. 5 (2016): 1144-1149

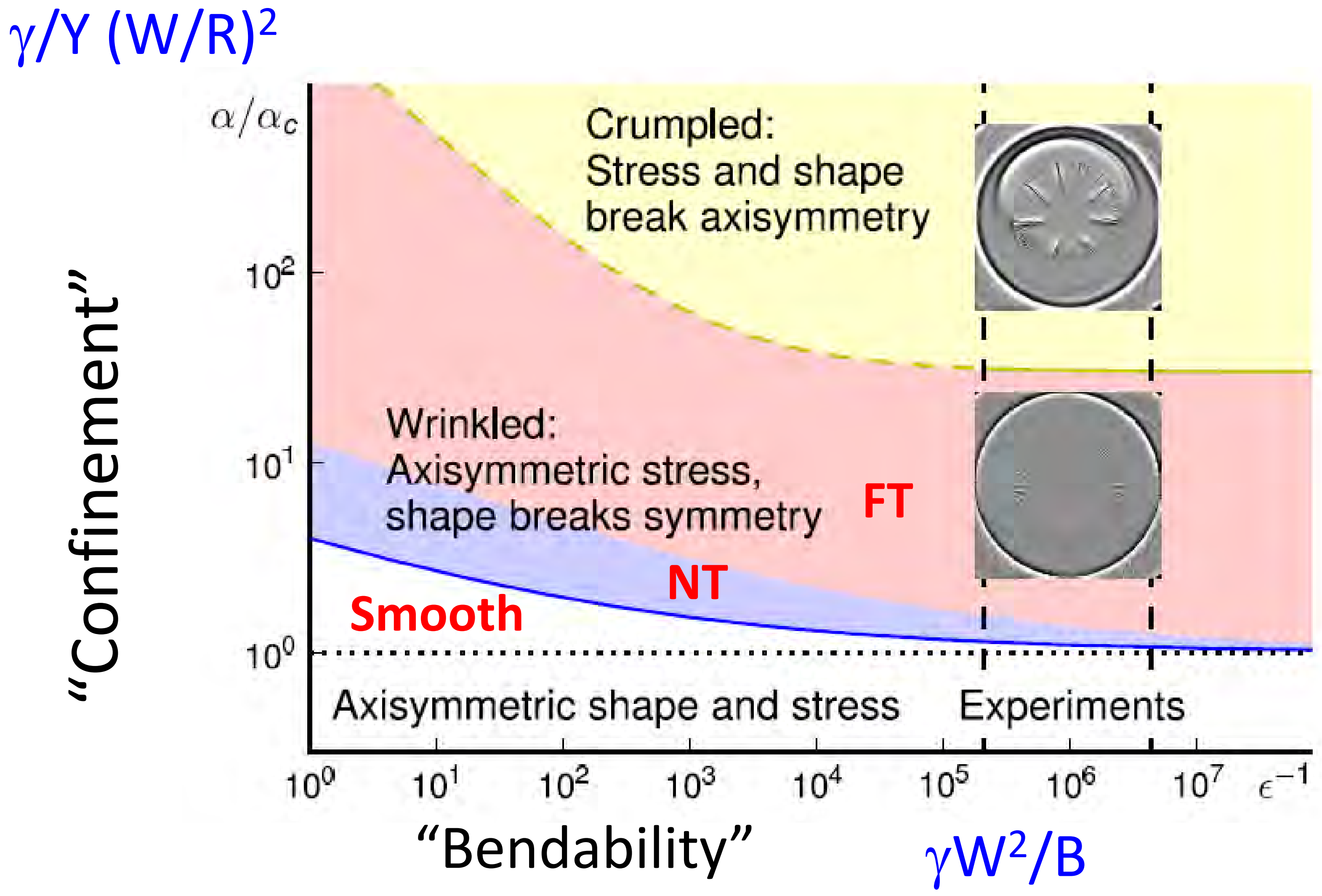
Crumples-

Witten, T. A. (2007). Stress focusing in elastic sheets. *Reviews of Modern Physics*, *79*(2), 643.

Lobkovsky, A., Gentges, S., Li, H., Morse, D., & Witten, T. A. (1995). Scaling properties of stretching ridges in a crumpled elastic sheet. *Science*

Cerda, E., Chaieb, S., Melo, F., & Mahadevan, L. (1999). Conical dislocations in crumpling. *Nature*, *401*(6748), 46-49.

Wrinkling for sheet on drop – “phases”



Almost no regime of near threshold

King et al. PNAS 2012

Analogy from hydrodynamics

Fluids (Navier-Stokes equations)

Stokes

linear (viscous) theory

(expansion in *Re*)

Laminar flow

Euler

non-linear (inertial) theory

inertial flow Re

Elastic sheets (FvK equations)

Near threshold

compression

(amplitude expansion)

low bendability

Far from threshold

compression-free, tension-field theory

(bendability expansion) ε^{-1}

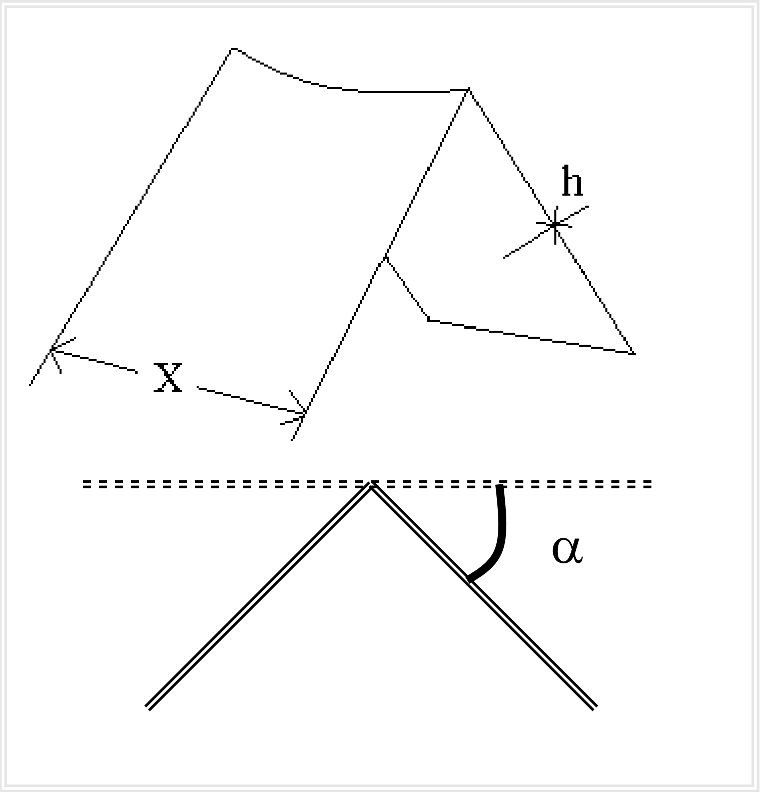
high bendability

Slide from Benny Davidovitch

A little about crumples - ridges

Localizes strain and bending

- Stored energy E ~ $\alpha^{7/3}$ k(X/t)$^{1/3}$
- Mid-ridge radius ~ $\alpha^{-4/3}$ X$^{2/3}$t$^{1/3}$
- Bending energy ~ Stretching energy



Lobkovsky et al. 1997

Ben-Amar, Pomeau

Witten RMP 2009

Lecture 5

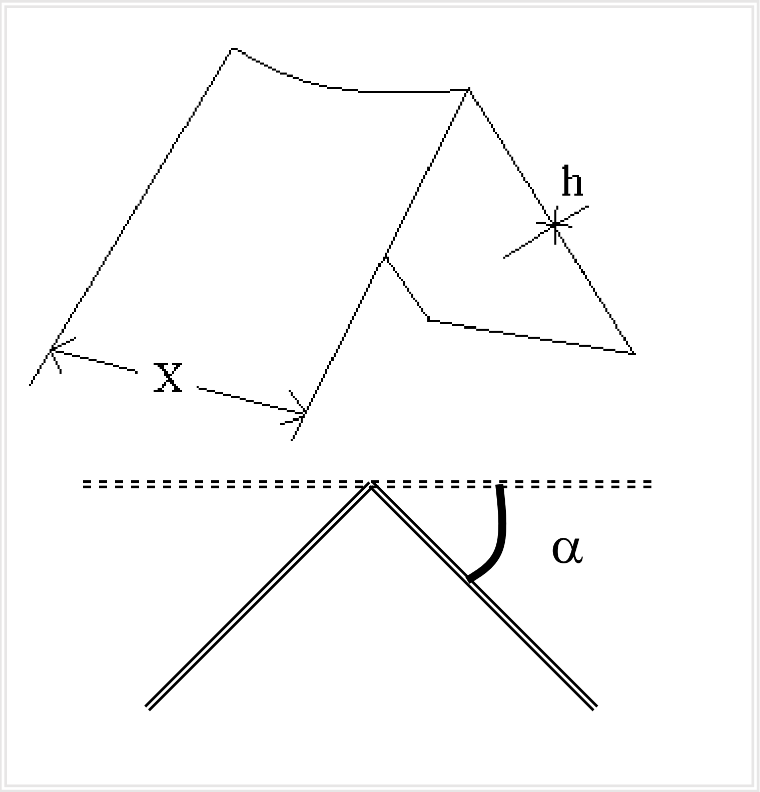
- Crumples – quick overview

- Large deformation – wrappings, asymptotic isometries

A little about crumples - ridges

Localizes strain and bending

- Stored energy E ~ $\alpha^{7/3}$ k$(X/t)^{1/3}$
- Mid-ridge radius ~ $\alpha^{-4/3} X^{2/3} t^{1/3}$
- Bending energy ~ Stretching energy



Lobkovsky et al. 1997

Ben-Amar, Pomeau

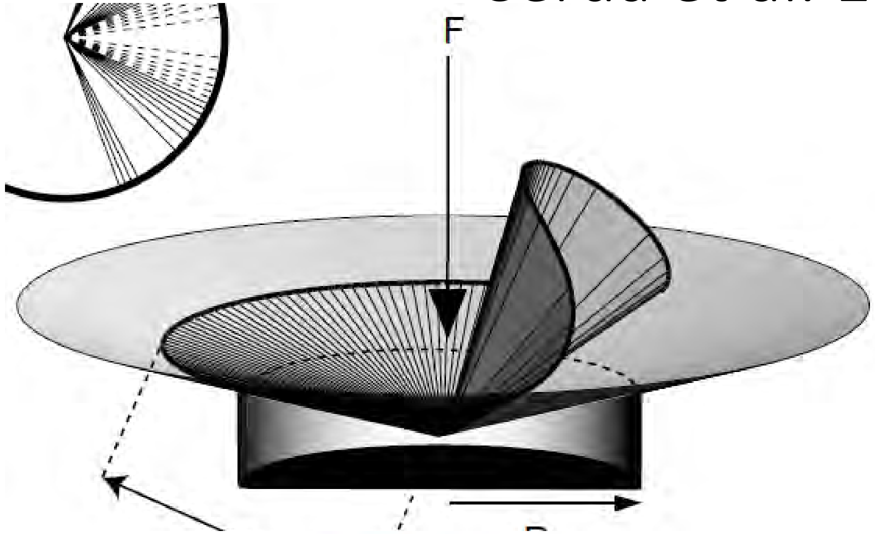
Witten RMP 2009

A little about crumples : d-cones

Cerda et al. 1999

Only bending outside core

Core has comparable stretching and bending



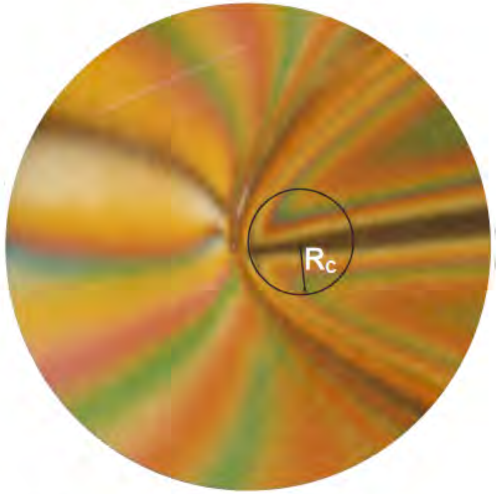


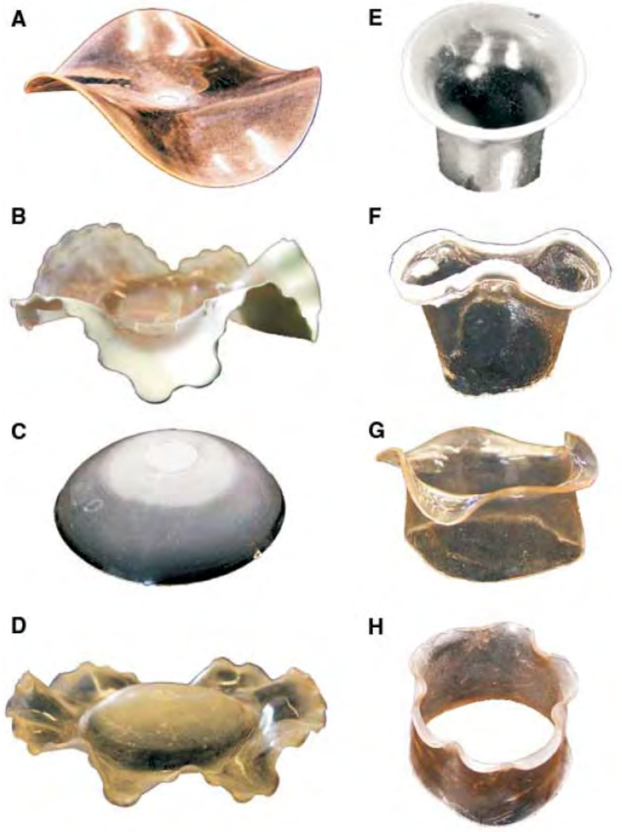
Figure 2 Geometry of a real conical dislocation. Cross-polarizers are used to view the reflected light from a painted sheet deformed into a conical dislocation. Isochromatic lines

A little about crumples : e-cones

- An example with too much material

- Emerges naturally in growth problems

Muller et al 2008



N

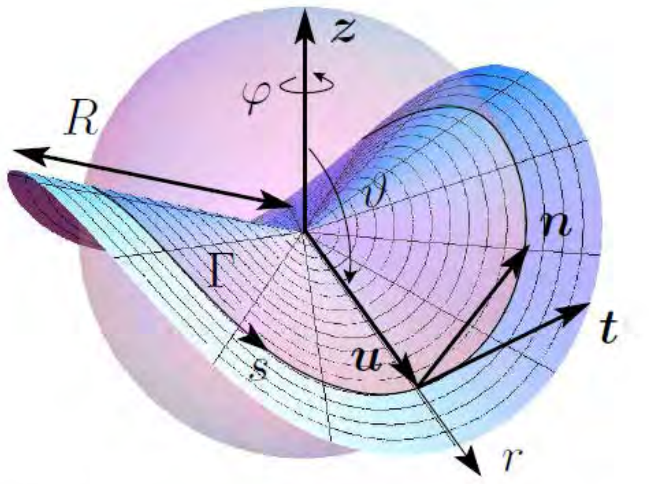


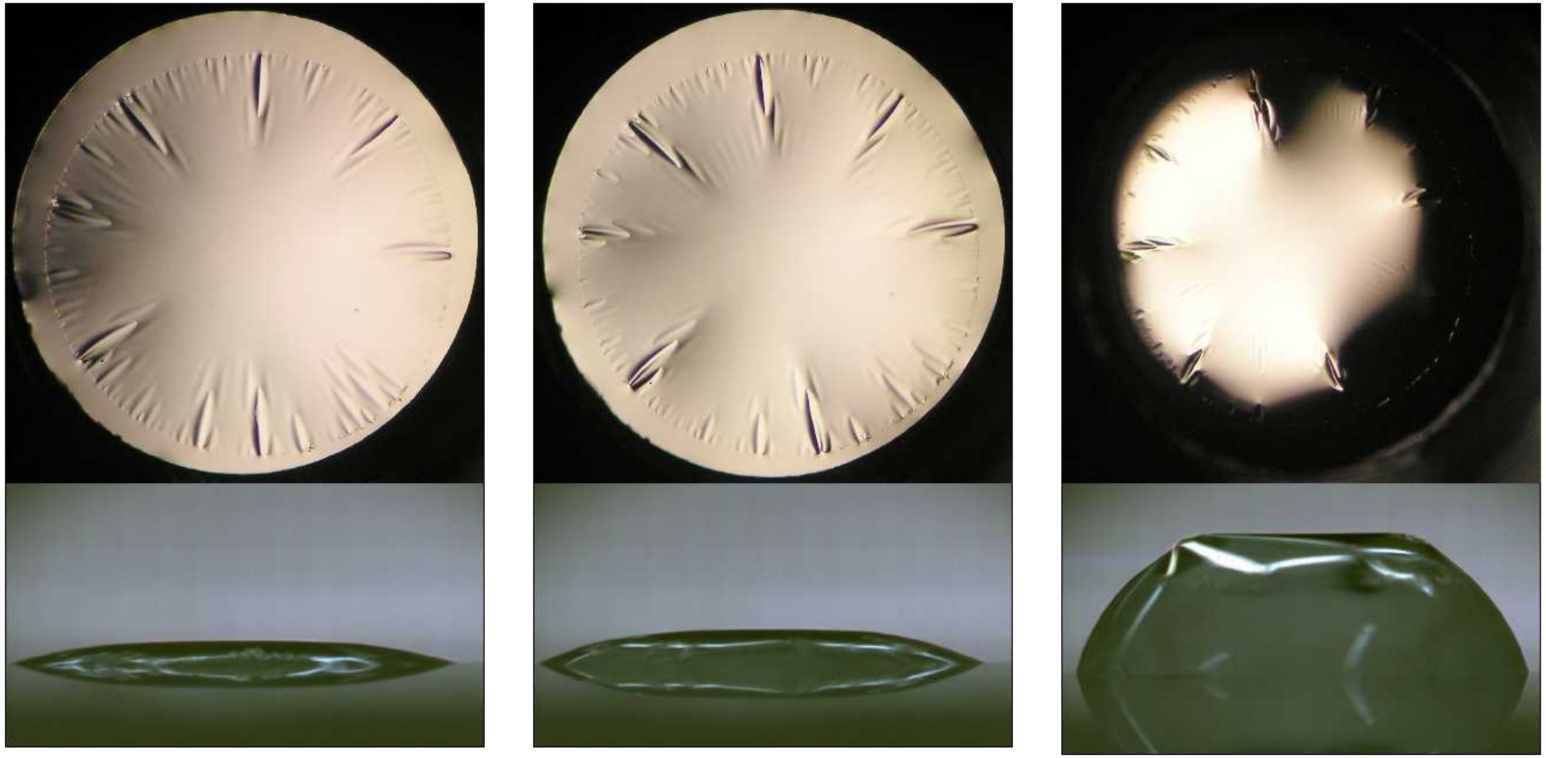
FIG. 1: Geometry of the e-cone with $\varphi_e = \frac{2\pi}{9}$.

- Klein, Efrati, Sharon 2007

Delocalized modes get localized

Top view

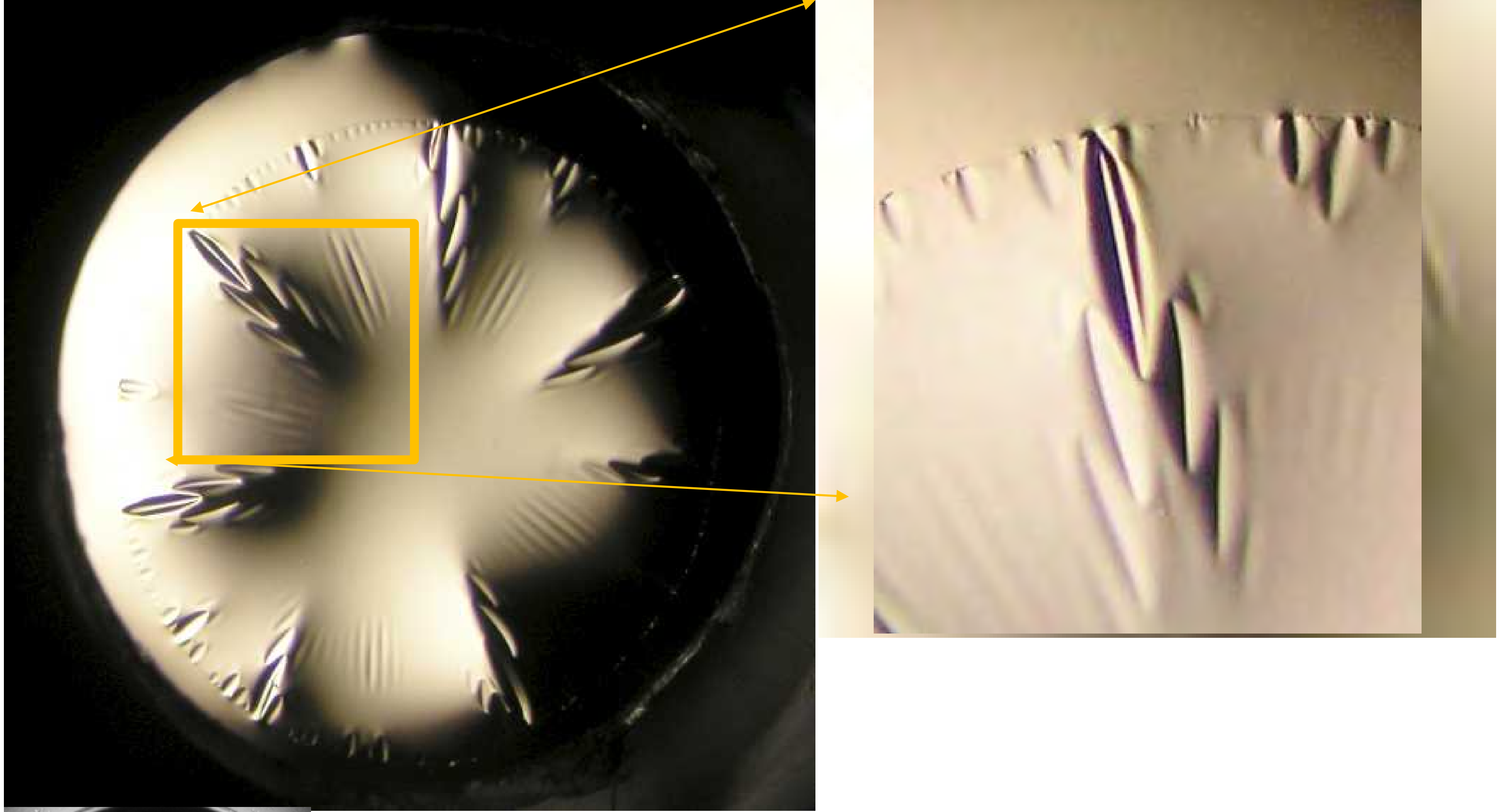
Side view

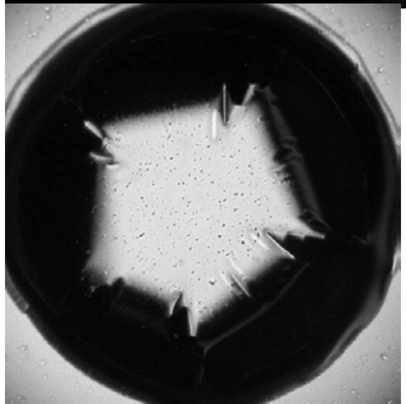


increase drop curvature →

A few wrinkles grow, and sharpen into “crumples”

The others recede





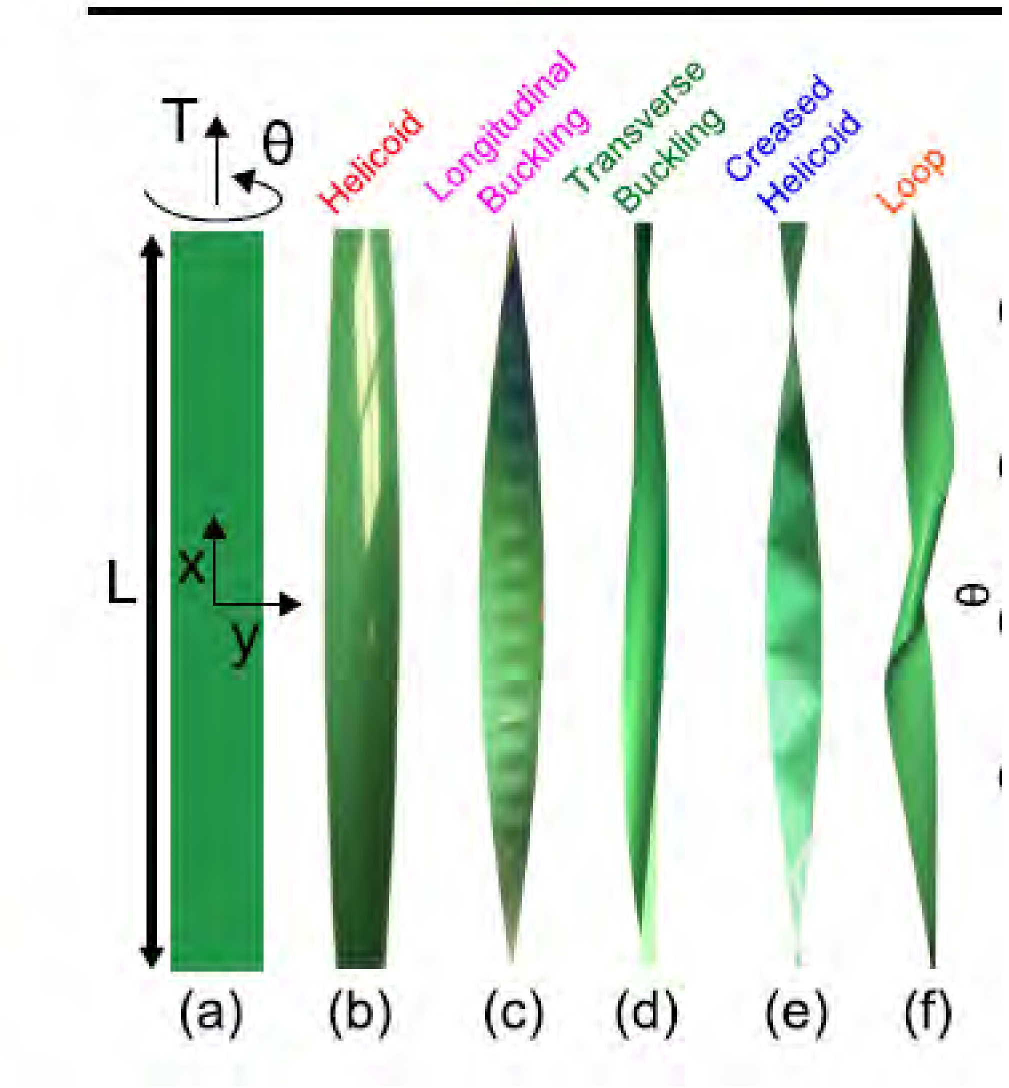
A different way of hiding material

Analogous to scars/disclinations

but fold, not cut, material

emerges from continuum elasticity, no discrete charge

Another wrinkle-to-crumple transition



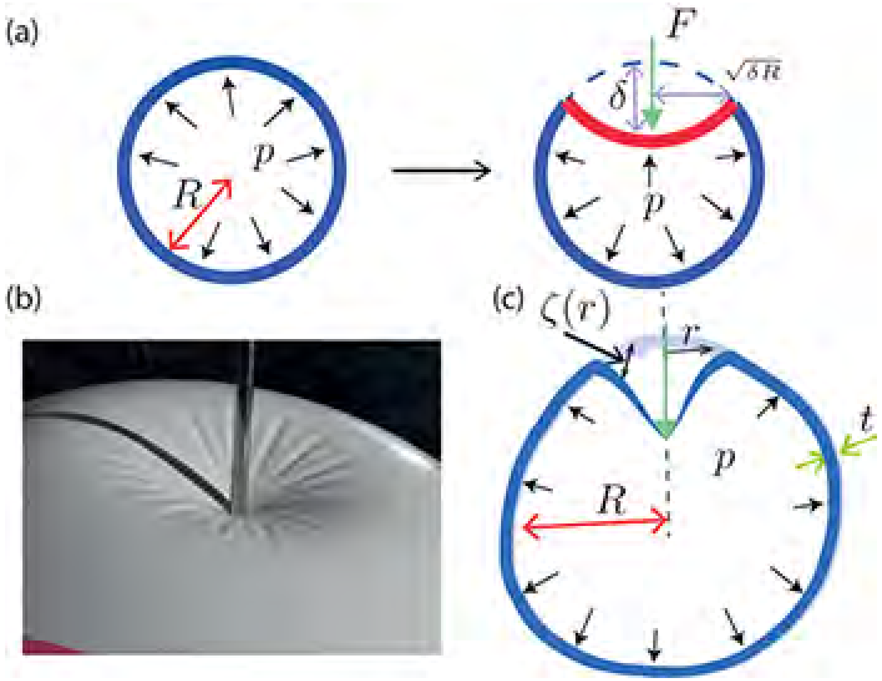
Chopin Kudrolli 2013

Large deformation

What can wrinkles, folds and crumples do for you?

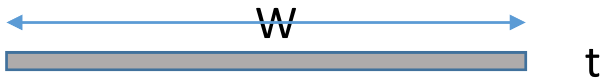
When they are cheap (large bendability), they can achieve nontrivial shapes

Dudte et al Nature Mat 2016

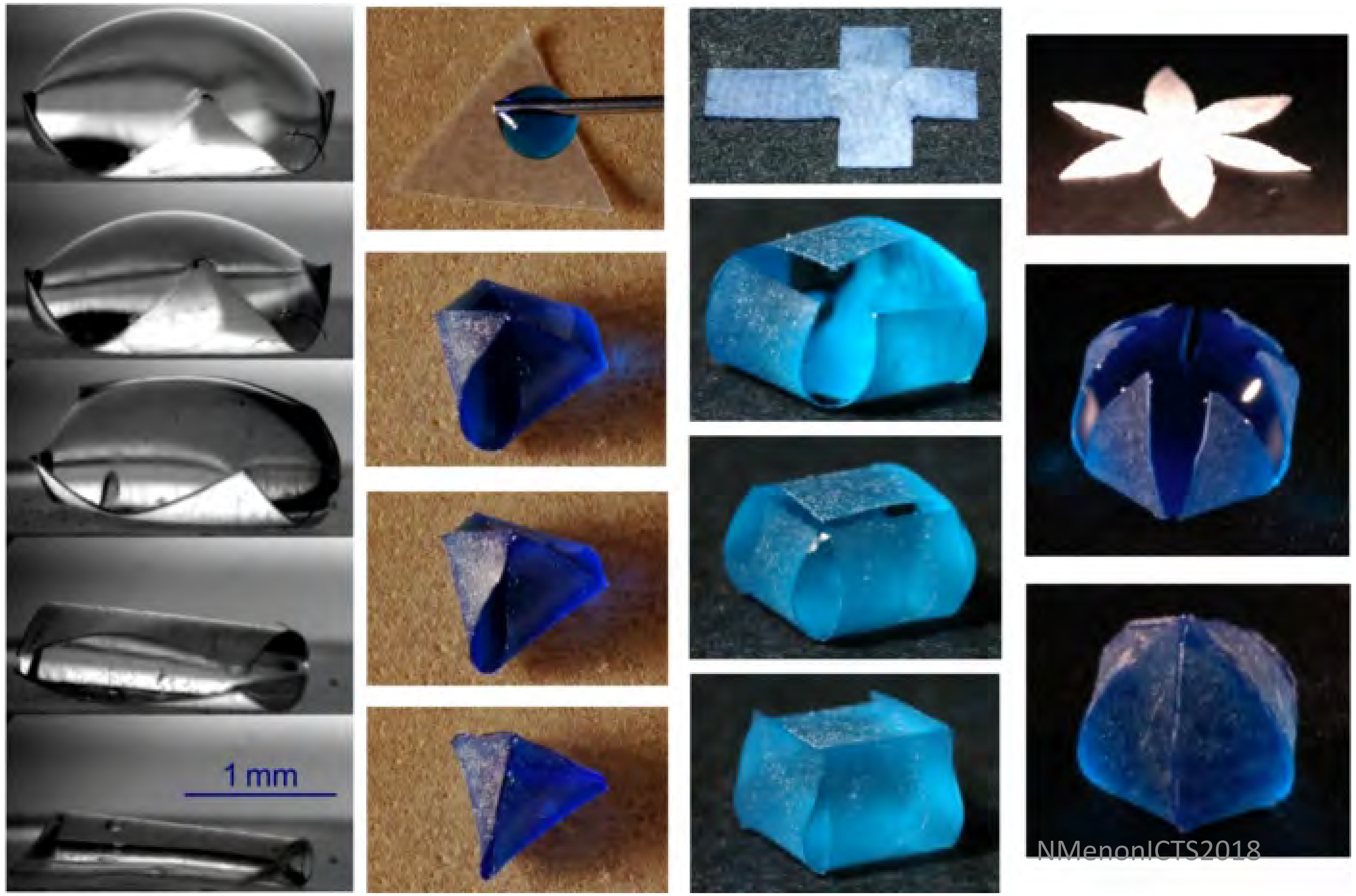


D. Vella *et al* 2015 *EPL* **112** 24007

Wrapping a drop



“Capillary origami” Py, Reverdy, Baroud, Roman, Bico 2006   *t=50μm; W ~ few mm, PDMS*



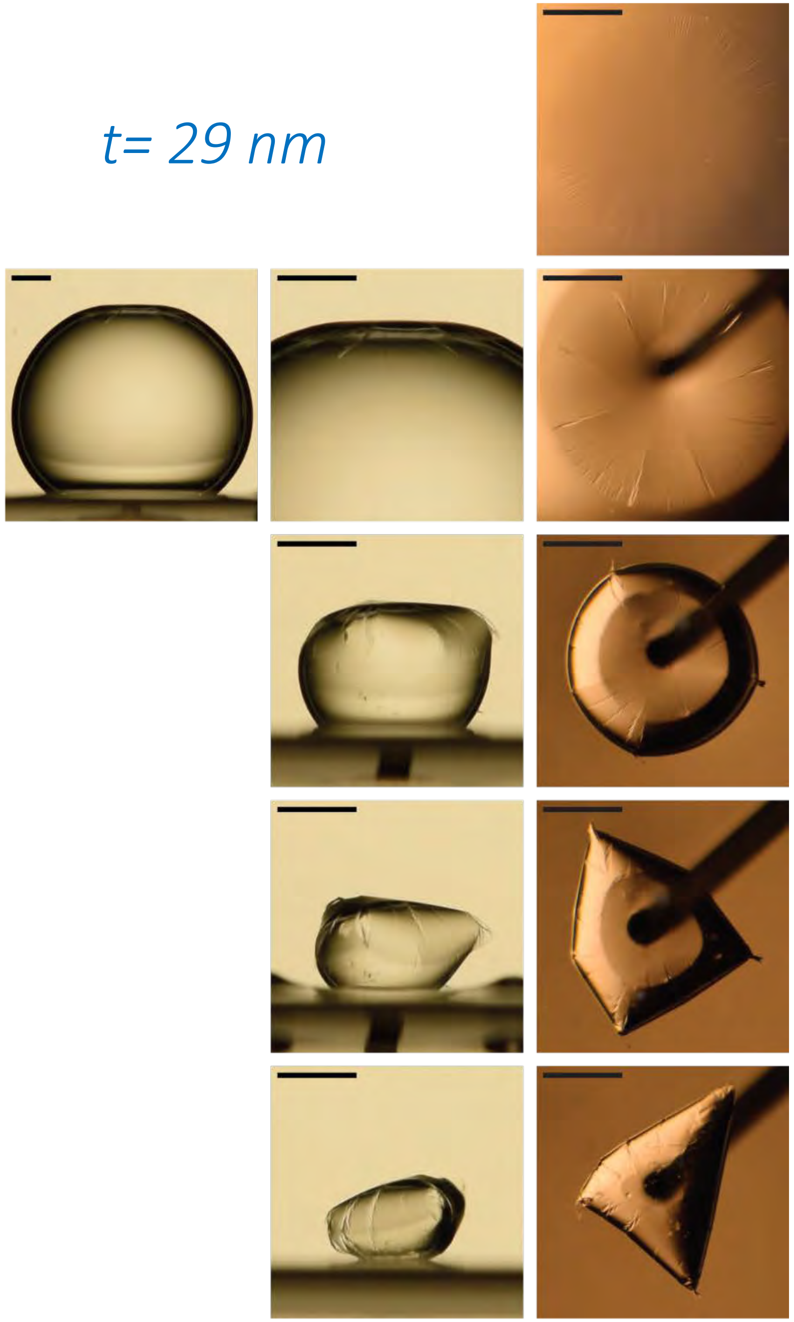
Bending balances torques created by capillary forces

Shapes with flaps cut to allow pure bending (developable shapes)

Now for something thinner….

(but why?)

t= 29 nm



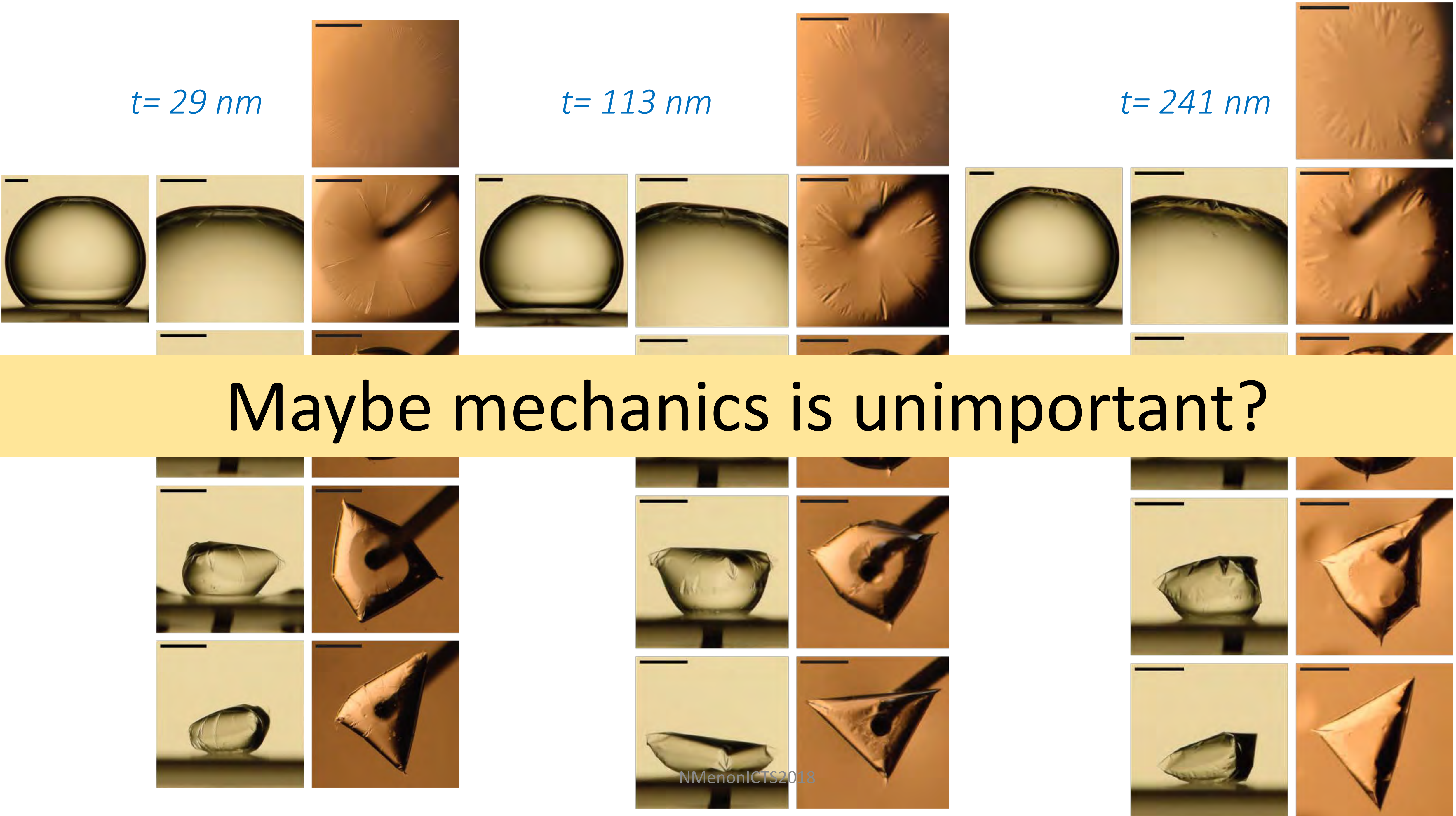
Axially symmetric   wrinkles, crumples

Polygonal shapes   folds, crumples

How to understand this sequence of shapes?

Wrinkles,folds, crumples, all interacting on a curved surface

Maybe mechanics is unimportant?



Wrapping with thin sheets

Describe all shapes with a simple equation:

Energy, $U = \gamma\, A_{free}$

Constraint: free 'compression', but no stretching

Pure geometry, no material parameters!

Wrapping with thin sheets

Describe all shapes with a simple equation:

Energy, $U = \gamma \, A_{free}$

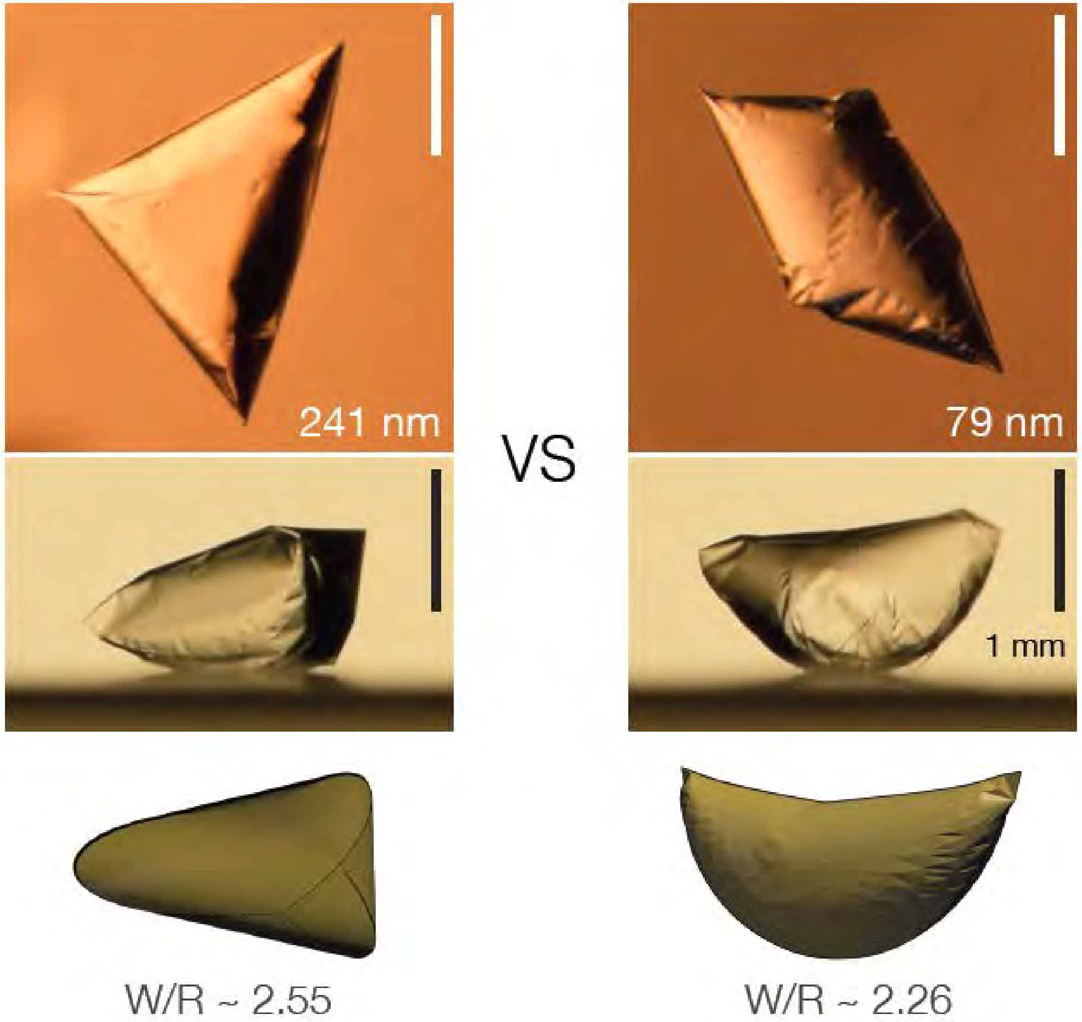
Constraint: free ‘compression’, but no stretching

Works when energy scales are separated (the first inequality is high bendability):

bending << surface << stretching

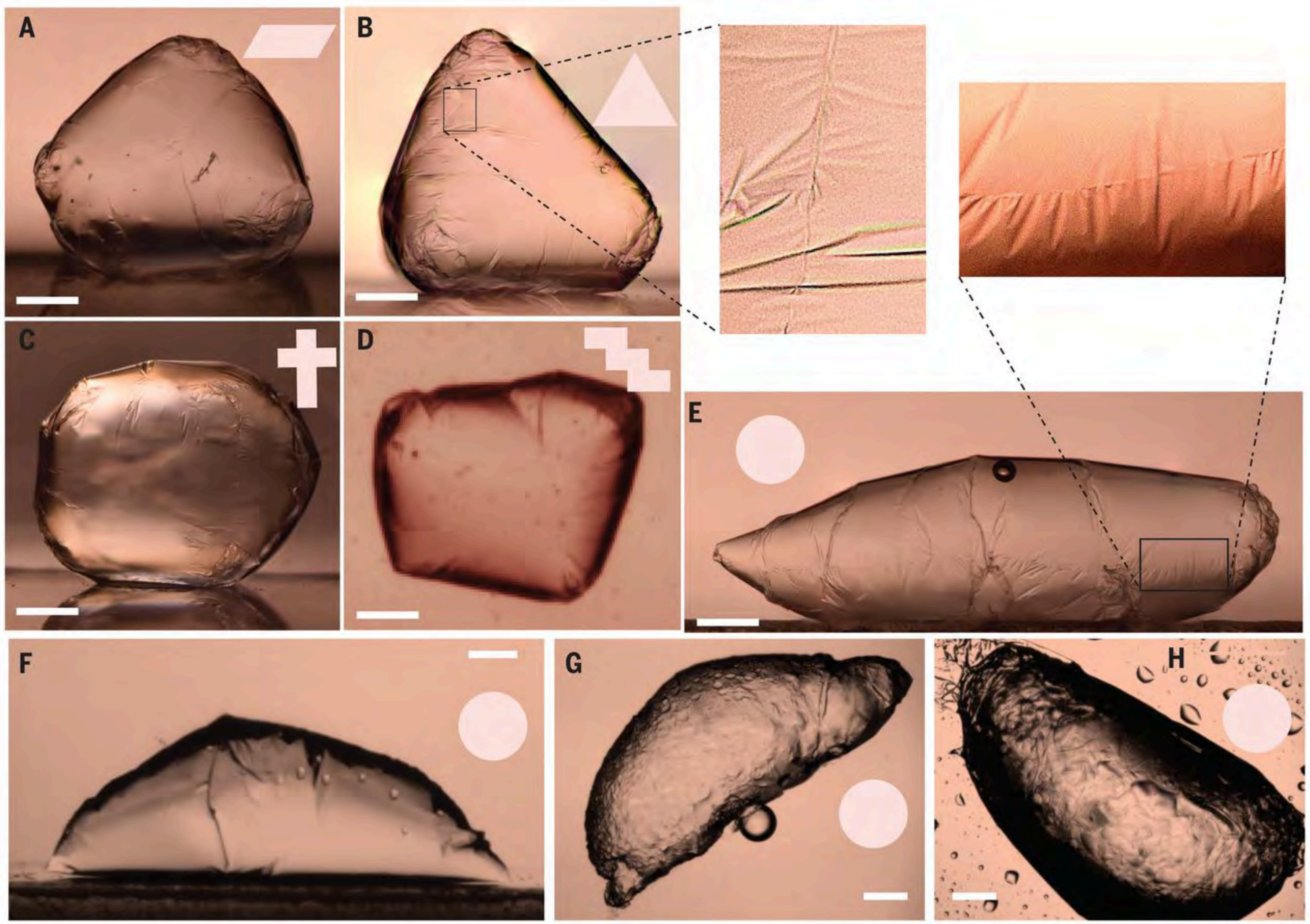
Et^3/W^2 γ **Et**

Predicts non-axisymmetric shapes



Samosa less efficient than gujiya

Other shapes possible as well



Implications

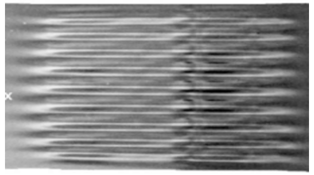
- A ‘thin’ sheet spontaneously achieves the highest wrapping efficiency
  No need for careful design

- Doesn’t rely on material parameters (in high bendability regime)

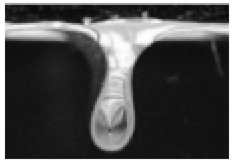
- Shows possibilities of near-isometric deformation if you have high enough bendability (see also Vella 2015)

Overall goals of our discussion

Wrinkles

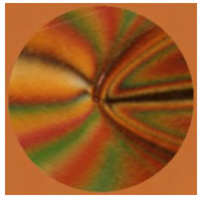


Folds

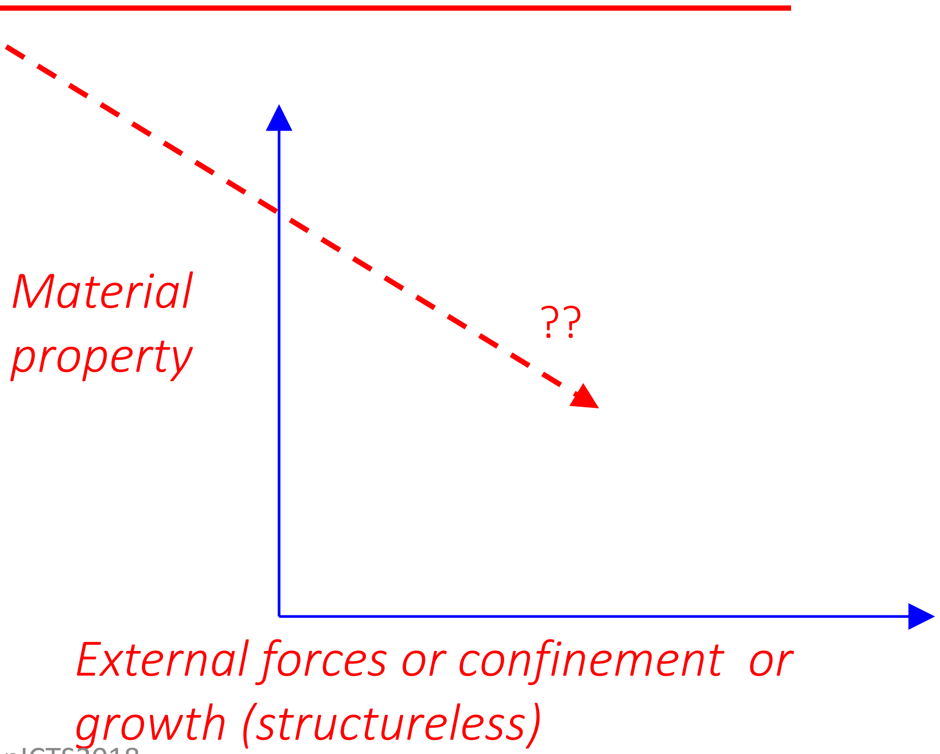


Ridges

d-cones



Creases (plasticity)



- These structures are generated by elastic instabilities
- What are the energetics and stability of these constructs?
- Where do all these structures belong?
- How to specify these axes?

Thanks

- Audience

- Collaborators

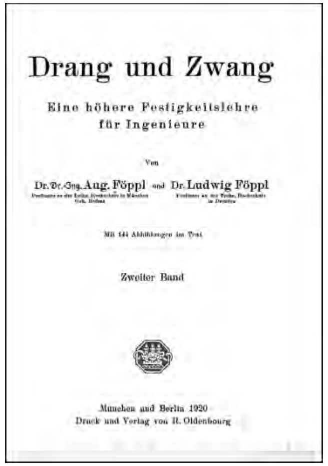
  Expts – Deepak Kumar, Gangaprasath, J. Huang, H. King, KB Toga, JD Paulsen, Tom Russell

  Theory – B. Davidovitch, R Schroll, V. Demery, E. Cerda, D. Vella

- School organizers

  Abhishek, Sanjib, and ICTS staff

2D wrinkles (again)-
Huang, J., Juszkiewicz, M., De Jeu, W. H., Cerda, E., Emrick, T., Menon, N., & Russell, T. P. (2007). Capillary wrinkling of floating thin polymer films. *Science*, *317*(5838), 650-653.
King, H., Schroll, R. D., Davidovitch, B., & Menon, N. (2012). Elastic sheet on a liquid drop reveals wrinkling and crumpling as distinct symmetry-breaking instabilities. *Proceedings of the National Academy of Sciences*, *109*(25), 9716-9720. The SI is useful.
Davidovitch, B., Schroll, R. D., Vella, D., Adda-Bedia, M., & Cerda, E. A. (2011). Prototypical model for tensional wrinkling in thin sheets. *Proceedings of the National Academy of Sciences*, *108*(45), 18227-18232.
Crumples-
Witten, T. A. (2007). Stress focusing in elastic sheets. *Reviews of Modern Physics*, *79*(2), 643.
Lobkovsky, A., Gentges, S., Li, H., Morse, D., & Witten, T. A. (1995). Scaling properties of stretching ridges in a crumpled elastic sheet. *Science*
Cerda, E., Chaieb, S., Melo, F., & Mahadevan, L. (1999). Conical dislocations in crumpling. *Nature*, *401*(6748), 46-49.
Wrapping etc
Py, C., Reverdy, P., Doppler, L., Bico, J., Roman, B., & Baroud, C. N. (2007). Capillary origami: spontaneous wrapping of a droplet with an elastic sheet. *Physical Review Letters*, *98*(15), 156103.
Vella, D., Huang, J., Menon, N., Russell, T. P., & Davidovitch, B. (2015). Indentation of ultrathin elastic films and the emergence of asymptotic isometry. *Physical review letters*, *114*(1), 014301.
JD Paulsen, V. Démery et al. (2015) Optimal wrapping of liquids with ultrathin sheets ***Nature materials 14, 1206 (2015).***



Drang und Zwang

Eine höhere Festigkeitslehre
für Ingenieure

Von

Dr. Dr.-Ing. Aug. Föppl und Dr. Ludwig Föppl

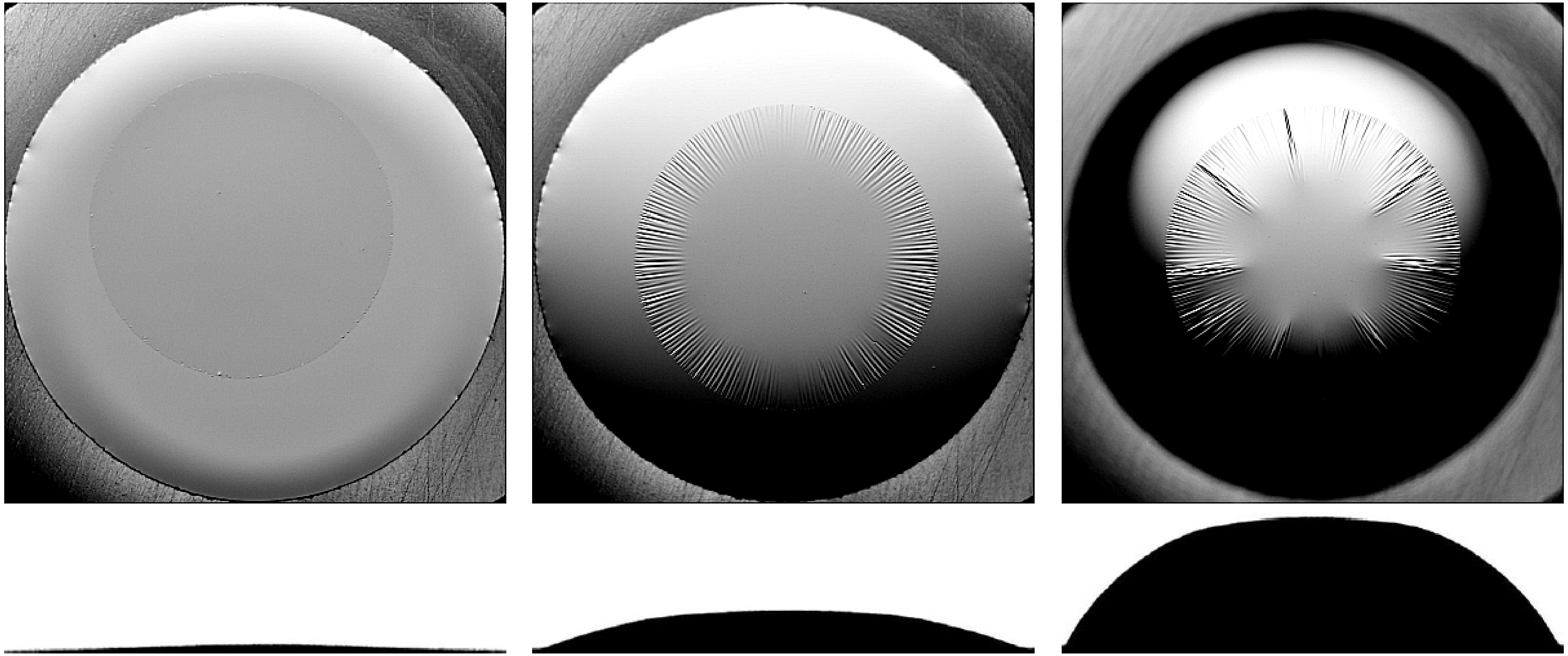
Mit 144 Abbildungen im Text

Zweiter Band

München und Berlin 1920
Druck und Verlag von R. Oldenbourg

Continuous, reversible, wrinkle-to-crumple transition

Increasing curvature



Axisymmetric shape,
Stretched 'cap',
Axisymmetric stress

Broken symmetry shape,
Smooth wrinkles,
Axisymmetric stress

Further symmetry breaking,
Localized features