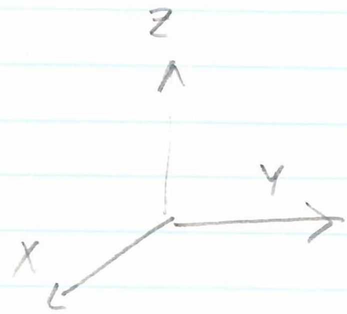
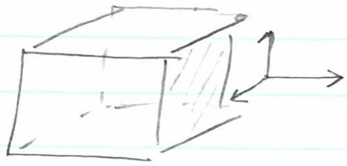
Topics to address

- Stress
  - Stress-balance (static equilibrium)
- Strain
- Stress-strain equilibrium.

Elasticity is a continuum theory. Do not have a microscopic length scale within the theory.

Stress comes from microscopic forces, though. It is the forces applied to an element of material by the surrounding material (via short-ranges interactions – electro, vdW) through the surface of that material volume element





$$\sigma_{YY}\, dx\,dz\, \hat{Y} + \sigma_{ZY}\, dx\,dz\, \hat{Z} + \sigma_{xY}\, dx\,dz\, \hat{X}$$

σ_{ij} : force per area acting in $\hat{i}$ direction on a surface whose normal is in $\hat{j}$ direction

[Units : force/area N/m^2 ← same as pressure
& dimensions Pa

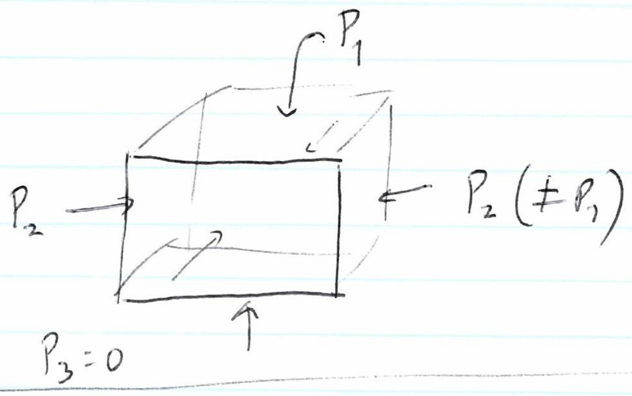
$$dF_i = \sigma_{ij}\, ds_j$$

$$\sigma_{ij} = \begin{Bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{Bmatrix}$$: Cauchy stress tensor (1822)

Hydrostatic stress: $-p\begin{pmatrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix} = -p\,\delta_{ij}$

shear stresses   normal stresses

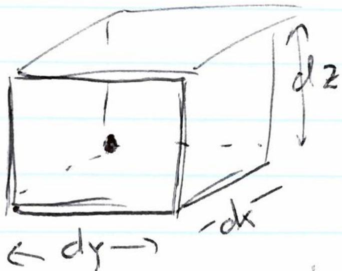
$p = \frac{-1}{3}\,\mathrm{Tr}\,\sigma_{ij}$



What is the stress tensor?

Any shear stresses?

STRESS BALANCE



Cube centred on (x, y, z)

Forces:

① "Body" forces on the volume of the cube. $\vec{F}_i = f_i\, dx_i\, dy_i\, dz_i$

e.g. $\vec{f} = \rho\vec{g}$ for gravity.

② Net stress on the cube:
let's do the Y-component

$$\left[\sigma_{YY}\left(y+\tfrac{dy}{2}, x, z\right) - \sigma_{yy}\left(y-\tfrac{dy}{2}, x, z\right)\right] dx\,dz$$ {from Y-face}

$$+\left[\sigma_{Yz}\left(x, y, z+\tfrac{dz}{2}\right) - \sigma_{yz}\left(x, y, z-\tfrac{dz}{2}\right)\right] dy\,dx$$ {from z-face}

$$+\left[\sigma_{\cancel{YX}}\left(x+\tfrac{dx}{2}, y, z\right) - \sigma_{xz}\left(x-\tfrac{dx}{2}, y, z\right)\right] dy\,dz$$

$$= \left(\partial_y\sigma_{yy} + \partial_z\sigma_{yz} + \partial_x\sigma_{yx}\right) dx\,dy\,dz$$

Force balance: $\boxed{\partial_j\sigma_{ij} + f_i = 0}$

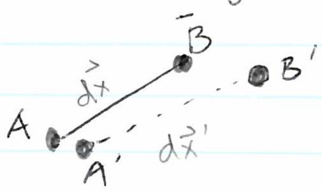
Strain: Let the points in an objects be labelled by coords. $(\vec{x})$ [Initial state: reference state]

After deformation

$$\vec{x} \longrightarrow \vec{x}' = \vec{x} + \underbrace{\vec{u}}_{\text{displacement}}$$

Not all displacements fields yield to strain e.g. translations or rotations

Need change in distance between material pts



$$d\vec{x}' = d\vec{x} + d\vec{u}$$

$$dl^2 = dx_i dx_i \qquad dl'^2 = dx'_i\, dx'_i$$

$$= \left(dx_i + \frac{\partial u_i}{\partial x_j} dx_j\right)^2.$$

$$= dl^2 + \underbrace{2\, dx_i\, dx_j \frac{\partial u_i}{\partial x_j}}_{\text{symmetrize}} + \underbrace{\frac{\partial u_i}{\partial x_k} dx_k \frac{\partial u_i}{\partial u_l} dx_l}_{\text{change labels of summed indices}}$$

$$\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i \to k \quad k \to i \quad j \to l$$

$$dl'^2 = dl^2 + 2\,\varepsilon_{ij}\, dx_i\, dx_j$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{\text{typically small but non linear}}\right) \leftarrow \text{Green strain}$$

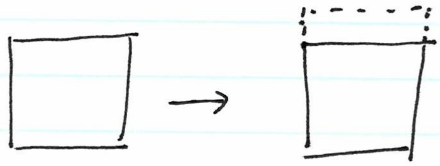
Symmetric tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$$

small displacement: Cauchy strain

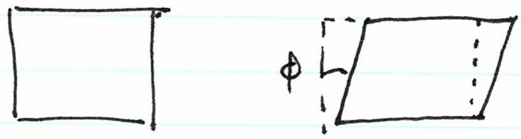
Stress: Use torque balance to show stress tensor is symmetric

H.W.



what is u?

what is ε?



What is u?

What is ε?

As with stress, you can diagonalize strain because it is a symmetric tensor.

$$\begin{pmatrix} \epsilon_1 & & 0 \\ & \epsilon_2 & \\ 0 & & \epsilon_3 \end{pmatrix}$$

These are called principal strains

$$dV' = dx'dy'dz' = dx\,dy\,dz\,(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)$$

$$\simeq dV\,(1+\epsilon_1+\epsilon_2+\epsilon_3)$$

fractional change in vol $= \epsilon_1+\epsilon_2+\epsilon_3 = \mathrm{Tr}(\epsilon_{ij})$

Stress - strain relationship

For small strains, expect linear relationship between stress and strain, along lines of Hooke's law.

Formally, if ϵ_{ij} and σ_{ij} are linearly related tensors, only possible rotationally invariant relation is

$$\sigma_{ij} = \underbrace{2\mu}_{\text{shear}} \epsilon_{ij} + \underbrace{\lambda}_{\text{bulk}} \delta_{ij} \, Tr(\epsilon_{ij})$$

Lamé 1852

λ, μ are called the Lamé constants

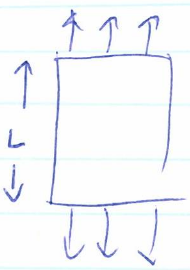
Write out explicitly:

$$\sigma_{xx} = 2\mu \, \epsilon_{xx} + \lambda (\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$$

$$\dots$$

$$\sigma_{xy} = 2\mu \, \epsilon_{xy}$$

Less formally, let's consider components



$$\sigma_{zz} = \frac{f_z}{\text{area}} = \frac{k \cdot u_z}{\text{area}} = \frac{kL}{\text{Area}} \frac{u_z(L)}{L}$$

$$= \underbrace{E}_{\text{Young's modulus}} \epsilon_{zz}$$

Units of Y: same as stress.

Extension in z leads to compression in the other direction:

$$\epsilon_{yy} = \epsilon_{xx} = -\Lambda \, \epsilon_{zz}$$

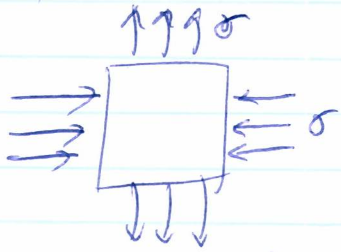
$$\epsilon_{zz} = \frac{1}{E}\sigma_{zz} - \frac{\Lambda}{E}\sigma_{yy} - \frac{\Lambda}{E}\sigma_{xx}$$

$\vdots$

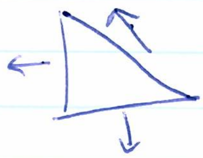
Shear stress

$$\epsilon_{xy} = \frac{2(1+\Lambda)}{E}\sigma_{xy}$$

To show consider a pure shear



and rotate to a 45° plane



(E, Λ) and (λ, μ) equivalent

$$\boxed{E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \qquad \Lambda = \frac{\lambda}{2(\lambda+\mu)}}$$

$\Leftrightarrow$

$$\lambda = \frac{E\Lambda}{(1-2\Lambda)(1+\Lambda)} \qquad \mu = \frac{E}{2(1+\Lambda)}$$

Cauchy - Navier equations

1. Use ϵ_{ij} in terms of u_i

2. $\partial_j \sigma_{ij} + f_i = 0$

3. $\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \, Tr(\epsilon) \delta_{ij}$

Put 3 into 2 and then 1 into that

$$(\mu+\lambda) \nabla (\operatorname{div} \vec{u}) + \mu \nabla^2 \vec{u} + \vec{f} = 0$$

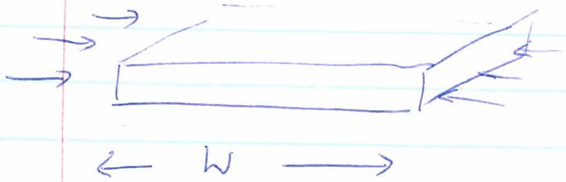
Thin plates:

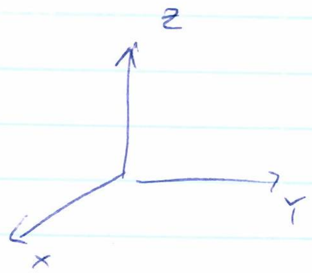
$(W \gg t)$

Not interested in variations across the thickness of plate.

Integrate out z-dependence.

- Stretch or compress in-plane.



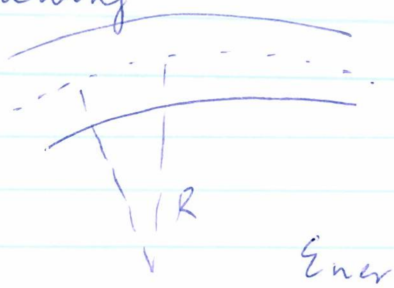


$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} = \frac{f}{E\,t} = \frac{f}{Y}$$

$\boxed{\text{Stretching modulus } Y = Et}$

Stretching energy/area $\quad e_s = \frac{1}{2} Y e_{yy}^2$

- Bending



Deflection from flat shape:

$$\zeta = \zeta(y)$$

Energy can't depend on ζ or ζ' but only on $\zeta'' = \frac{1}{R}$

ζ ← translation, ζ' ← slope

Can't depend on sign $\quad e_B = \frac{1}{2} B (\zeta'')^2$

What is B? Dimensionally units of energy

$$[B] = [E]^a [t]^b \rightarrow B \sim Et^3$$

Recap LECTURE 7

- Stress tensor σ_{ij} force in $\hat{j}$ direction of a area plane w normal in $\hat{i}$ direction

$\{\sigma_{ij}\}$ is symmetric

$$\partial_j \sigma_{ij} + f_i = 0 \qquad (1)$$

↑ add dynamics here easily $\rho \frac{\partial^2 u}{\partial t^2}$

(pressure, principal stresses, units)

- Strain $$\epsilon_{ij} = \frac{1}{2}\underbrace{(\underbrace{\partial_i u_j + \partial_j u_i}_{\text{Cauchy: not rotationally invariant}} + \partial_i u_k \partial_j u_k)}_{\text{Green}} \quad -(2)$$

$\vec{u}$: displacement field

Change in distances between material points

- Hooke's law (E, Λ) or (λ, μ) $-(3)$

[can get Cauchy Navier eqns in a straight forward way]
Put (2) into (3) and then (3) into (1)

[Last time: discussion on linearity.
We'll assume material linearity $-(3)$, while not assuming that geometry is linear]

Going to 2D

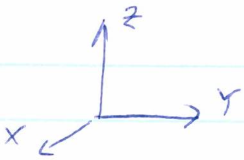
Stretching in-plane: $\sigma_{YY} = Y \epsilon_{YY}$ (Force/line) $\qquad Y = Et$

energy density: $e_s = \frac{1}{2} Y \epsilon_{YY}^2$

Bending:

energy density $e_B = \frac{1}{2} B (\zeta'')^2$ (curvature) $\qquad B \sim Et^3$

Euler buckling of a plate.

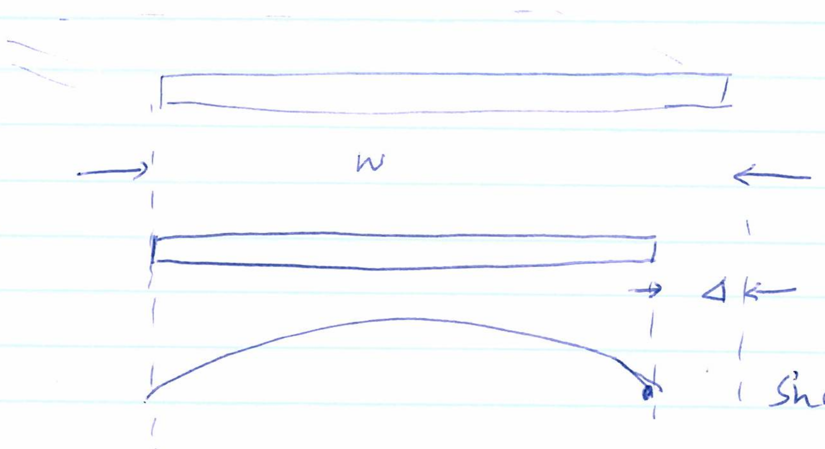


Full energy density: $e = \frac{1}{2} Y \epsilon_{YY}^2 + \frac{1}{2} B K_{YY}^2$

curvature

(1) Compare pure compression against pure bending

Euler elastica



compression $\epsilon_{YY} = \frac{\Delta}{W}$

$e_s = \frac{1}{2} Y \left(\frac{\Delta}{W}\right)^2$

Shape: $\xi(y)$

- Bending: no extension

$$\int_0^{W-\Delta} dy \sqrt{1+\xi'^2} - W = 0$$

Small slope $\xi/W \ll 1$ $\quad \int_0^{W-\Delta} \left(1+\frac{1}{2}\xi'^2\right) dy - W = 0$

$e_b = \frac{1}{2} B \xi''^2$ $\quad$ constraint $\int_0^W \frac{1}{2}\xi'^2 dy = \Delta$

Euler-Lagrange equation: $B\xi'''' - \sigma\xi'' = 0$

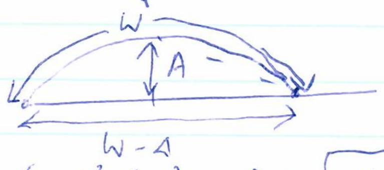
Torque = 0 $\quad \xi(0) = \xi(W) = 0$

$\xi(y) = A \sin\left(\sqrt{\frac{\sigma}{B}}\, y\right)$ $\quad \sigma = \pi^2 B / W^2$

$A = \frac{2}{\pi}\sqrt{\Delta W}$ $\quad e_b = \frac{1}{2} B\, \Delta / W^3$

from integrating the constraint

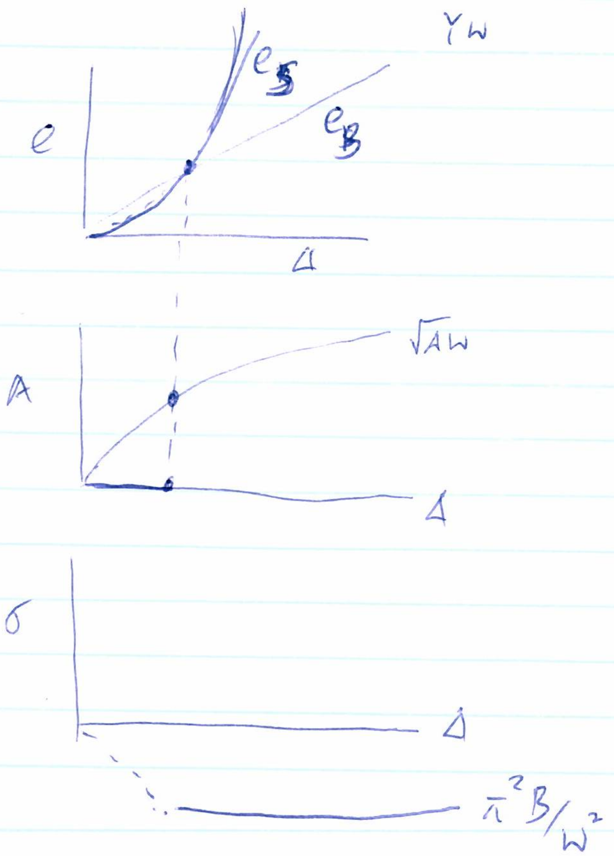
Cheap way to get scale of A



$A^2 + \left(\frac{W-\Delta}{2}\right)^2 = \left(\frac{W}{2}\right)^2 \Rightarrow A \simeq \sqrt{2\Delta W}$

Transition at $\Delta/\Delta_c = 1$ where $\frac{B\Delta_c}{W^3} = Y\frac{\Delta_c^2}{W^2}$

$$\Rightarrow \Delta_c = \frac{B}{YW} \Rightarrow \frac{\Delta_c}{W} = \frac{B}{YW^2} \sim \left(\frac{t}{W}\right)^2 \sim (VK)^{-1}$$



First order jump in A (sub-critical)

Comment: this is unlike a wave eqn. Nonlinearity comes from constraint $A = A(\Delta)$

[Exercise: Try to work this out for another boundary condition like clamped boundaries, where

$$\psi'(0) = 0 = \psi'(W)$$]

[Exercise: Try to check if you can do this for other modes. Do they become unstable at different Δ_c? Is there a barrier between modes?

Approach (2)

Full treatment of Euler buckling:

$\vec{u}(y) = u(y)\hat{y} + \zeta(y)\hat{z}$ DISPLACEMENT

$\epsilon_{yy} = u' + \frac{1}{2}\zeta'^2 + \underbrace{\frac{1}{2}u'^2}_{\text{omit}}$ STRAIN.

$\sigma_{yy} = Y\epsilon_{yy}$ HOOKE'S LAW.

Energy density. $e = \frac{1}{2}Y\epsilon_{yy}^2 + \frac{1}{2}B(\zeta'')^2$

The corresponding Euler-Lagrange equations are

(i) $B\zeta'''' - \sigma_{yy}\zeta'' = 0$ Normal force balance from variation w.r.t. $\zeta(y)$.

(ii) $\sigma_{yy}' = 0 \Rightarrow \sigma_{yy} = \text{constant}$ In-plane force balance from variation wrt $u(y)$

Solve the pair of FvK equations by a series solution. perturbing around planar state

Try $\zeta(y) = A(\Delta)\sin\left(\frac{\pi y}{W}\right) + \text{other} \ldots$

$u(y) = \underbrace{-\frac{\Delta}{W}\cdot y}$ + other term. planar state.

Plug $\zeta(y)$ back into first FvK eqn.

$$B\left(\frac{\pi}{W}\right)^4 A - \sigma_{yy}\left(\frac{\pi}{W}\right)^2 A = 0 \Rightarrow \sigma_{yy} = \frac{B\pi^2}{W^2} \quad (3)$$

But $\sigma_{yy} = Y\epsilon_{yy} = Y\left[-\frac{\Delta}{W} + \frac{1}{2}\frac{A^2\pi^2}{W^2}\sin^2\left(\frac{\pi y}{W}\right)\right]$

$= Y\left[-\frac{\Delta}{W} + \frac{\pi^2 A^2}{W^2 4}\right.$ + oscillating term to be fixed by other terms in the series.

(4)

Put (3) & (4) together, we get

$$A^2 = \frac{2^2}{\pi^2} W (\Delta - \Delta_c) \qquad \text{where } \frac{\Delta_c}{W} = \frac{\pi^2}{W^2} \frac{B}{Y}$$

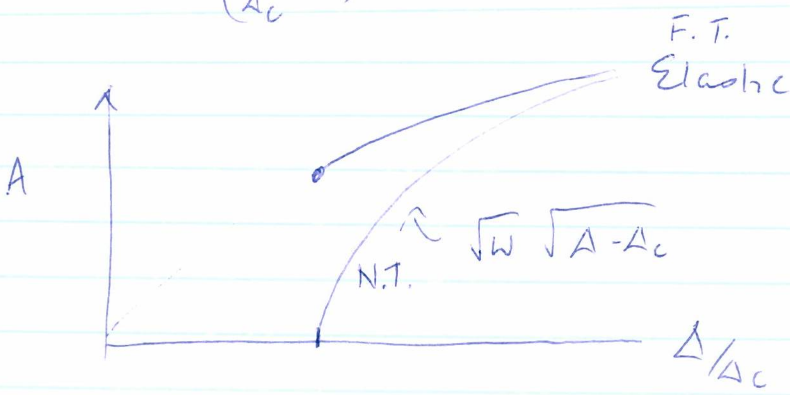
The first term in this series solution already yields the same threshold as our simpler argument ie. $\frac{\Delta_c}{W}$

Further $A \sim \sqrt{W} \sqrt{\Delta - \Delta_c}$ goes to the elastica solution as $\Delta \gg \Delta_c$

We were fortunate in this basic problem to find a solution, but both in this problem and in other problems to come, we can do stability analysis.

For small $\left(\Delta/\Delta_c - 1\right)$: post-buckling

For large $\left(\frac{\Delta}{\Delta_c} - 1\right)$: elastica again.



$$\frac{\Delta_c}{W} = \frac{B}{Y W^2} \sim \left(\frac{t}{W}\right)^2 = (V.K)^{-1}$$ only from sheet geometry

Control parameter $\frac{\Delta}{\Delta_c} = \frac{\Delta}{B/(YW^2)} = \frac{\Delta \cdot Y}{W^2 (B/W^2)} = \left(\frac{\Delta}{W}\right)\left(\frac{Y}{\sigma}\right)$

Not just a material property

1D Wrinkling

[Cerda - Mahadevan]

Different foundations:
- fluid
- solid
- tensioned.

Why do you get a finite wavelength?

Energy density $e = \frac{1}{2} B (\zeta'')^2 + \frac{1}{2} K \zeta^2$

For a fluid foundation $K = \rho g$.

Inextensibility:

$$\frac{\Delta}{W} = \frac{l - \lambda}{l}$$

$$l = \int_0^\lambda \sqrt{1 + \zeta'^2} \, dy \simeq \int_0^\lambda \left(1 + \frac{1}{2}\zeta'^2\right) dy = \lambda + \lambda \frac{q^2 A^2}{4}$$

$$\frac{q^2 A^2}{4} = \frac{\Delta}{W} \Rightarrow qA = 2\sqrt{\frac{\Delta}{W}}$$

Put this in energy density:

$$e = \frac{1}{2}\left[B (q^2 A)^2 + K A^2\right]$$

$$= \frac{1}{2}\left[B q^2 \left(\frac{4\Delta}{W}\right) + K \frac{1}{q^2}\left(\frac{4\Delta}{W}\right)\right]$$

Minimize, to get $q \sim \left(\sqrt{\frac{K}{B}}\right)^{1/4}$

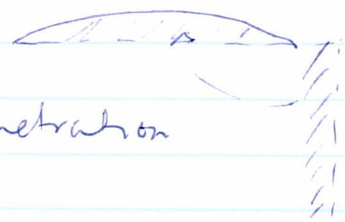
Tensional case: $\frac{T}{2}\left(\partial_x \zeta\right)^2$

Here $K = T/\lambda^2$

- - - - - - - - - - - - -

Soft substrate:

Incompressible substrate.



Typical displacement of u, over a penetration depth l_p $\Rightarrow$ $A\lambda \sim u\, l_p$

Typical strain $\epsilon \sim \frac{u}{l_p} \sim \frac{A\lambda}{l_p^2}$

Energy/area $\sim \underset{\text{substrate}}{E_s}\, \epsilon^2 \left(\frac{l_p \lambda}{\lambda}\right) \sim E_s \left[\frac{(A\lambda)}{l_p^2}\right]^2 l_p$

$\sim E_s \frac{\lambda^2 A^2}{l_p^3}$

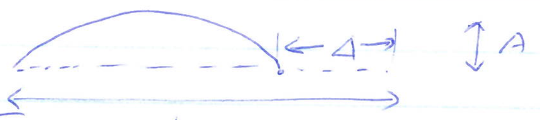
For a deep substrate assume $l_p \sim \lambda \rightarrow K = \frac{E_s}{\lambda}$

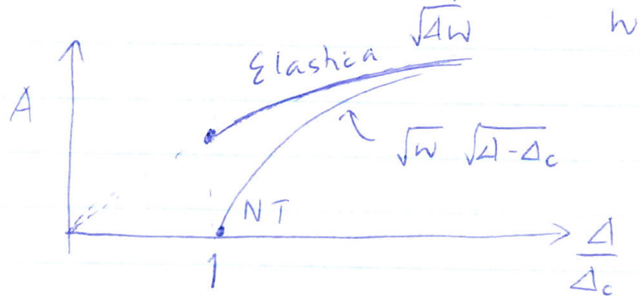
For a shallow substrate of depth H. assume $l_p \sim H \rightarrow K = \frac{E_s \lambda^2}{H^3}$

Recap of lecture 2

- Finished discussion of reduction to 2D $\to (B, Y)$ from (E, Λ, t)

- Euler buckling





(where $\frac{\Delta_c}{W} = \frac{B}{YW^2} \sim \left(\frac{t}{W}\right)^2 \sim \nu k$

i) Elastica

e = bending energy + σ Inextensibility constraint

ii) Full problem

e = bending energy + Stretching energy.

$$\to \partial_y \sigma_{yy} = 0 \quad \text{(in-plane)}$$

$$B \zeta'''' - \sigma_y \zeta'' = 0 \quad \text{(normal)}$$

(nonlinear, coupled)

Solved near-threshold in powers of A or $\left(\Delta/\Delta_c - 1\right)$

(post-buckling / Landau expansion / amplitude expansion)

NT solution happens to agree with ET at $\left(\Delta/\Delta_c\right)^{-1}$ large.

Only parameter here is $\frac{\Delta}{\Delta_c} = \frac{\Delta}{W} \cdot \frac{Y}{\sigma}$

1D Wrinkling

[Cerda-Mahadevan]

Different foundations:
- fluid
- solid
- tensioned.

Why do you get a finite wavelength?

Energy density $e = \frac{1}{2} B(\zeta'')^2 + \frac{1}{2} K \zeta^2$

For a fluid foundation $K = \rho g$.

Inextensibility:

$\zeta = A \sin(qy)$

$$\frac{\Delta}{W} = \frac{l-\lambda}{l}$$

$$l = \int_0^\lambda \sqrt{1+\zeta'^2}\, dy \simeq \int_0^\lambda \left(1+\frac{1}{2}\zeta'^2\right) dy = \lambda + \lambda \frac{q^2A^2}{4}$$

$$\frac{q^2A^2}{4} = \frac{\Delta}{W} \Rightarrow qA = 2\sqrt{\frac{\Delta}{W}}$$

Put this in energy density:

$$e = \frac{1}{2}\left[B\left(q^2 A\right)^2 + KA^2\right]$$

$$= \frac{1}{2}\left[B q^2 \left(\frac{4\Delta}{W}\right) + K \frac{1}{q^2}\left(\frac{4\Delta}{W}\right)\right]$$

Minimize to get

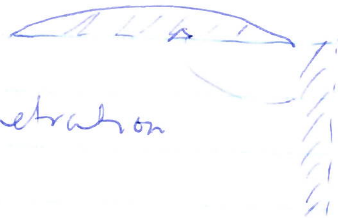
$$q \sim \left(\sqrt{\frac{K}{B}}\right)^{1/4}$$

Tensional case: $\frac{T}{2}\left(\partial_x \zeta\right)^2$

Here $K = T/\lambda^2$

Soft substrate:

Incompressible substrate.



Typical displacement of u, over a penetration depth l_p $\Rightarrow$ $A\lambda \sim u\, l_p$

Typical strain $\epsilon \sim \frac{u}{l_p} \sim \frac{A\lambda}{l_p^2}$

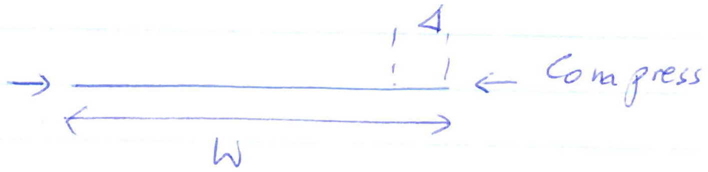
$$\text{Energy/area} \sim \underbrace{E_s}_{\text{substrate}} \epsilon^2 \left(\frac{l_p \lambda}{\lambda}\right) \sim E_s \left[\frac{(A\lambda)}{l_p^2}\right]^2 l_p$$

$$\sim E_s \frac{\lambda^2 A^2}{l_p^3}$$

For a deep substrate assume $l_p \sim \lambda$. $\rightarrow K = \frac{E_s}{\lambda}$

For a shallow substrate assume $l_p \sim H \rightarrow K = \frac{E_s \lambda^2}{H^3}$
of depth H.

1-dimensional folds

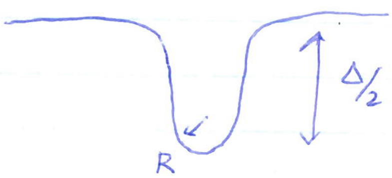


Energy density of wrinkled state $e \simeq \sqrt{BK}\left(\frac{\Delta}{W}\right)$

Folds are local, so we'll write an energy, rather than an energy density. E/length into the board.

Consider a fold of max radius of curvature R

$$E_B \sim B\left(\frac{1}{R}\right)^2 R$$



$$E_K \sim K(\Delta^2)\, R$$

(Doing more carefully, you get $E_K \sim KR\Delta^2 - K\Delta^3$

Minimize

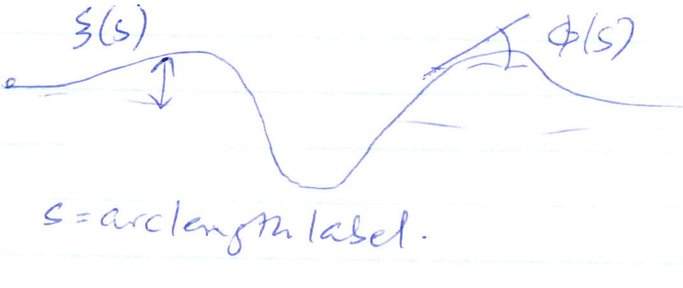
$$E_B + E_K \rightarrow R \sim \left(\frac{B}{K}\right)^{1/2} \frac{1}{\Delta}$$

Has been checked in numerics.

[Expts, not really? Affected by contact energy?]

Exact solution Witten & Diamant 2012

$$\frac{d\xi}{ds} = \dot{\xi} = \sin\phi$$



$$E_B = \frac{B}{2}\int_{-W/2}^{W/2} \dot{\phi}^2\, ds$$

s = arclength label.

$$E_K = \frac{K}{2}\int_{-W/2}^{W/2} ds\, \xi^2 \cos\phi$$

Constraint

$$\Delta = \int ds\,(1-\cos\phi)$$

Set up as an action $k = \int ds \, \mathcal{L}(\xi, \phi, \dot{\xi}, \dot{\phi})$

$$\ddddot{\phi} + \left[\frac{3}{2}\phi^2 + \sigma_{yy}\right]\ddot{\phi} + \sin\phi = 0$$

Lagrange multiplier for inextensible

Borrow kink solutions to a sine-Gordon eqn.

Symm fold $\phi(s) = 4\tan^{-1}\left[\dfrac{\mathcal{K}\sin(ks)}{k\cosh(\mathcal{K}s)}\right]$

$\mathcal{K} = \frac{1}{2}\sqrt{2-\sigma_{yy}}$ $\quad k = \frac{1}{2}\sqrt{2+\sigma_{yy}}$

Antisymm fold $\phi(s) = 4\tan^{-1}\left[\dfrac{\mathcal{K}\cos(ks)}{k\cosh(\mathcal{K}s)}\right]$

① $\mathcal{K} = \Delta/8$ decay length

② $\sigma_{yy} = (2 - \Delta^2/16)$

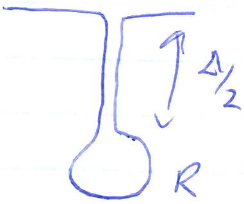
③ Exact solution confirms energy scaling.

Large folds Démery 2014. $U(\Delta) = 2\Delta - \Delta^3/48$

(i) Always less than wrinkles

(ii) has a max

Large symmetric fold $U_{symm} = B\left(\frac{1}{R}\right)^2 R \sim B\frac{1}{R}$



$U_{grav} = \kappa R^2 \Delta$

$\Rightarrow R \sim \Delta^{-1/3}$ different from before self contact

Large antisymm fold

typical size $\sim R$

$$E_B \sim B\left(\frac{1}{R}\right)^2 R \times 2$$

$$E_K \sim K \underbrace{R^2}_{\text{area of bubble}} \underbrace{R}_{\text{height of bubble.}}$$

Minimize and find $R \sim \left(B/K\right)^{1/4}$ independent of Δ

Until self-contact, antisymm & sym fold are degenerate, but this suggests antisymm folds win for large Δ.

Unresolved puzzles:

1. None of these calculations predict a size scale for the fold, yet there appears to be a finite scale in expts. Often get a few folds [illegible] a finite density
2. Why do we even see wrinkles? None of these arguments show a threshold Δ_c for fold onset. System size? No large system limit?
3. Expts show both sym & antisym. folds

What have we missed?

Recap of lecture 3

Wrinkling in 1 Dimension

e = bending energy + substate energy + σ Inextensibility

$$\frac{1}{2} B (\nabla^2 \zeta)^2 + \frac{1}{2} K \zeta^2 \qquad qA \sim \sqrt{\frac{\Delta}{W}}$$

$\zeta = A \sin qx$

$$q = (K/B)^{1/4}$$

For fluid $\quad K = \rho g$

tension $\quad K = T/L^2$

Soft solid $\quad K = E_s/\lambda$ – deep substate

(nasty problem but inspired hand waving in C-M 2003 leads to these guesses) $\quad = \dfrac{E_s \lambda^2}{H^3}$ – shallow substrate

---

Folding in 1 Dim.

Onset:

- Size scale of fold
- Energy of fold.

Deep folds

- Exact solution till self-contact
- breaking of ~~sy~~ degeneracy between sym & antisym folds at large Δ

Puzzles: Why wrinkles? Why finite number of folds?

Setting up a 2D axisymmetric problem

Force balance: (Foppl von Kármán)

$$\partial_j \sigma_{ij} + f_i = 0 \qquad \text{in-plane}$$
$$B \nabla^2 (\mathrm{Tr}\, \mathcal{K}) + \sigma_{ij} \mathcal{K}_{ij} = f_{\text{normal}} \qquad \text{out-of-plane}$$

Hooke's Law

$$\sigma_{rr} = \frac{Y}{1-\Lambda^2} \left(\epsilon_{rr} + \Lambda \epsilon_{\theta\theta} \right)$$
$$\sigma_{\theta\theta} = \frac{Y}{1-\Lambda^2} \left(\epsilon_{\theta\theta} + \Lambda \epsilon_{rr} \right).$$

Strain

$$\vec{u}(r,\theta) = u(r,\theta)\,\hat{r} + u_\theta(r,\theta)\,\hat{\theta} + \xi(r,\theta)\,\hat{z}$$

$$\epsilon_{rr} = \partial_r u_r + \frac{1}{2}\left(\partial_r \xi\right)^2 \qquad \text{[No } \theta\text{-derivatives]}$$
$$\epsilon_{\theta\theta} = \frac{1}{r} u_r$$

FvK

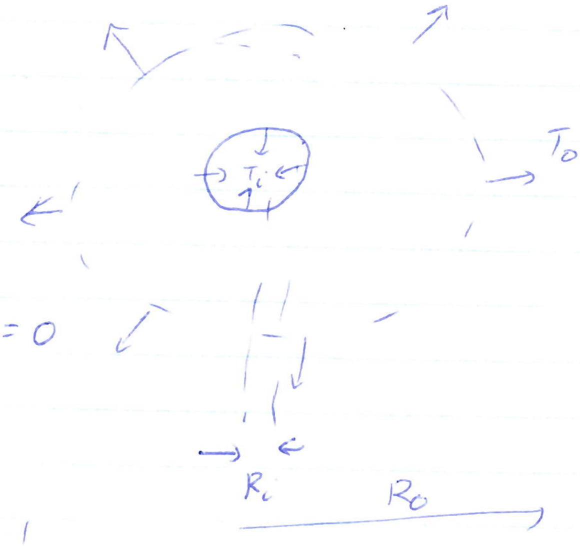
$$\partial_r \sigma_{rr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0 \quad \text{Radial}$$
$$\partial_\theta \sigma_{\theta\theta} = 0 \quad \text{Azimuthal}$$

$$B \Delta^2 \xi - \sigma_{rr} \partial_r^2 \xi - \frac{1}{r^2} \sigma_{\theta\theta} \left(\partial^2 \xi + r \partial_r \xi \right) = f_{\text{nor}}$$

$$(\epsilon_{r\theta} = 0 \text{ and } \sigma_{r\theta} = 0)$$

A specific in-plane problem: Lamé

$T_o = \sigma_{rr}(R_o) \qquad T_i = \sigma_{rr}(R_i)$



Planar problem first i.e.

$\zeta = 0$

FvK $\rightarrow$ $\partial_r \sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$

$\partial_\theta \sigma_{\theta\theta} = 0$

[Solve this for homework!

$$\sigma_{rr} = T_o - (T_o - T_i)\, R_i^2/r^2 = T_o\left[1 - (1-\tau)\frac{R_i^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = T_o + (T_o - T_i)\frac{R_i^2}{r^2} = T_o\left[1 + (1-\tau)\frac{R_i^2}{r^2}\right] \qquad \tau = \frac{T_i}{T_o}$$

For $\tau > 2$ you get azimuthal compression for $R_i < r < \ell$ such that

$$1 + (1-\tau)\, R_i^2/\ell^2 = 0 \Rightarrow \ell^2 = R_i^2(\tau - 2)$$

length of compressed zone $\Rightarrow$ $\ell = R_i\sqrt{\tau - 2}$

Compression leads to buckling above a threshold that depends on B. length of wrinkles $= \ell$ for $B = 0$

Thus you need two parameters to describe 2D wrinkling.

1. Confinement parameter $T = T_i / T_o$
   - this tells when you have compression.

2. Bendability parameter ϵ' where
$$\epsilon = \frac{B}{R_i^2 T_o}$$
   - this tells you when you buckle.

This is to be contrasted with 1D, where you need only one parameter Δ/Δ_c

Large bendability $\epsilon^{-1} \gg 1$, Sheet is very floppy.

Near threshold answer for annulus problem

Get l_{NT} $(\epsilon = 0)$ by solving the planar Lamé problem (use planar FvK eqns.)

Take the normal FvK eqs.

① Plug in $\sigma_{\theta\theta}$ ~~and~~ and σ_{rr} from planar solution.

② Assume wrinkles are $\zeta(r,\theta) = f(r)\cos(m\theta)$ where f is small.

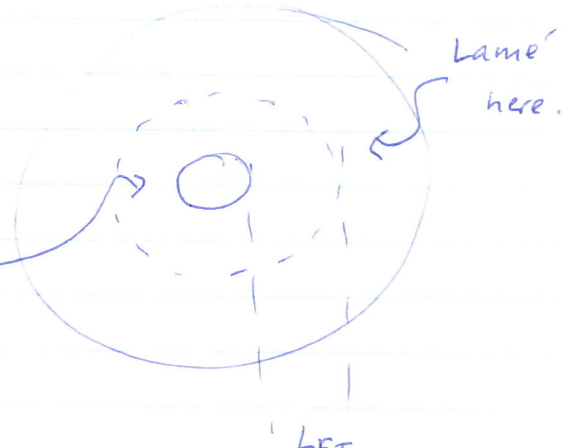
③ Solve, or extract scaling, by ~~bending~~ balancing terms of bending $B\Delta^2\zeta$ and stretching $\sigma\nabla^2\zeta$

Get $T_c = 2 \pm (\epsilon)^{1/4} = 2 + \left(\frac{B}{R^2 T_i}\right)^{1/4}$

$$\frac{l_{NT}}{R_i} - 1 \sim \left(\frac{B}{R^2 T_c}\right)^{1/4} \ldots = l_{NT}^{(0)} + \epsilon^{-1/4}$$

$$m(\epsilon) \sim \left(\frac{B}{R^2 T_c}\right)^{-3/8} \text{ or } \epsilon^{-3/8}$$

Far-from-threshold answer



For $r > l_{FT}$: Lamé solution

For $r < l_{FT}$ $\quad$ $\sigma_{\theta\theta} = 0$ here

use $\sigma_{\theta\theta} = 0$

$\sigma_{rr} = T_i R_i / r$ - from ~~force~~ balance.

Why use $\sigma_{\theta\theta} = 0$ rather than inextensibility?

$\sigma_{yy} = 0$ is equivalent to $\epsilon_{yy} = 0$ in 1D but not in 2D

$\epsilon_{rr} = \frac{\sigma_{rr}}{Y} \Rightarrow \epsilon_{\theta\theta} = -\Lambda \epsilon_{rr} = -\Lambda \frac{\sigma_{rr}}{Y} \neq 0$

Get l_{FT} by minimizing

$$E = \begin{pmatrix} \text{radial stretching} \\ \text{energy inside } r < l_{FT} \end{pmatrix} + \begin{pmatrix} \text{radial + azimuthal} \\ \text{energy outside } r > l_{FT} \end{pmatrix}$$

Minimize with no out-of-plane terms

$l_{FT}/R_i = \tau/\tau_c$ ← completely different from NT

$l_{NT}/R_i = \sqrt{\tau - \tau_c}$

<u>To get number of wrinkles</u>

Take $\zeta(r,\theta) = f(r)\cos(m\theta)$

Minimize smaller energy terms <u>inside</u> wrinkled region.

azimuthal bending	+	azimuthal compression	+	radial stretch due to wrinkles
$B m^4 f/L^4$		$\epsilon_{\theta\theta} \frac{m^2}{L^2} f$		$\sigma_{rr} \frac{f}{L^2}$

get $m \sim \epsilon^{-1/4} = \left(\frac{BT_i}{R_i^2}\right)^{-1/4}$

Some comments on bendability $\epsilon^{-1} = \frac{W^2 T}{B}$

- Unlike vK number, which is purely geometric, bendability includes information on external forcing.

- $$\epsilon^{-1} = \frac{W^2 T}{B} = \frac{T}{Y} \cdot \frac{W^2}{t^2}$$
$$= \underbrace{\text{strain}}_{\text{typically small}} \cdot \underbrace{\text{VK}}_{\text{large for a sheet}}$$

Thus ϵ^{-1} for a plate can be either large or small.

- Small and large ϵ^{-1} represent two different limits of FvK equations

$$B \nabla^4 \xi - \sigma_{ij} \partial_i \partial_j \xi = f_{normal}$$

Bendability $\epsilon^{-1} \sim \dfrac{T \xi / W^2}{B \xi / W^4} \sim \dfrac{T W^2}{B}$

For large ϵ, bending resists external force

small ϵ, in-plane force resists normal force.

Recap of lecture 4

- Axisymmetric 2D wrinkling.

Studied as an example the so-called Lamé problem in two ways. Two parameters in problem – confinement $T = T_i / T_o$

– bendability $\epsilon^{-1} = \frac{T W^2}{B}$

NT

FT

Set $\sigma_{\theta\theta} = 0$ *

Solve for stress in planar state

Find compressive stress in interior

$\rightarrow \ell_{NT}$

Solve for preferred wrinkle number in out-of-plane FvK eqns, using planar radial stresses

Find the wrinkled zone that minimizes in-plane energies

$\rightarrow \ell_{FT}$

Use this stress distribution (not the one you get from planar state and plug into normal FvK eqn.

$\ell_{FT} \sim T/\epsilon$

Completely diff answers: $\ell_{NT} \sim \sqrt{\epsilon \tau_c}$

- Bendability

$$\epsilon^{-1} = \frac{W^2 T}{B} \simeq \underbrace{(vK)^2}_{\text{large}} \underbrace{(\text{strain})}_{\text{small}}$$

large ϵ^{-1} In-plane stresses balances out-plane forces. Almost no NT regime.

- Small ϵ^{-1} Bending balances out-of-plane forces

Different limits of FvK eqns.

Why $\sigma_{\theta\theta} = 0$ instead of inextensibility?

These are equivalent in 1D, but not so in 2D

$$\epsilon_{\theta\theta} = \frac{-\Lambda \sigma_{rr}}{E} + \frac{\sigma_{\theta\theta}}{E} \Rightarrow \epsilon_{\theta\theta} = -\Lambda \epsilon_{rr}$$

which is non-zero

Ridge analysis (Lobkovsky, Witten)

Strain along ridge.

$$\frac{\sqrt{X^2+\delta^2}-X}{X} \sim \frac{1}{2}\frac{\delta^2}{X^2}$$

$$\frac{R}{R+\delta} = \sin(\alpha)$$

$$\frac{\delta}{R} = 1-\sin\alpha = f(\alpha)$$

$$E_{stretch} \sim Y\left(\frac{\delta^2}{X^2}\right)^2 \cdot RX = YR^5/X^3$$

$$E_{bend} \sim B\left(\frac{1}{R^2}\right) \cdot RX = BX/R$$

Minimize: $E_S + E_B \rightarrow \frac{5YR^4}{X^3} - \frac{BX}{R^2} = 0$

$$\Rightarrow R^6 \sim \frac{B}{Y}X^4 \Rightarrow R \sim X^{2/3}t^{1/3}$$

(as $B/Y \sim t^2$)

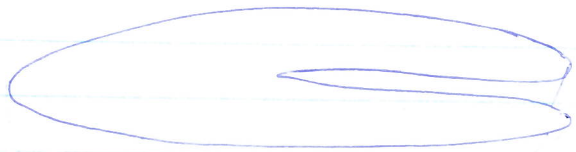
Strain is localized

$$\frac{RX}{X^2} \sim \frac{X^{2/3}t^{1/3}X}{X^2} \sim \left(\frac{X}{t}\right)^{-1/3}$$

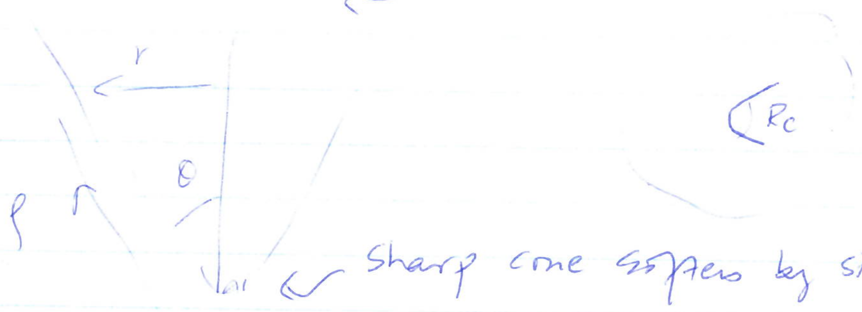
As sheet become thinner this is a vanishingly small fraction

Energetics of a d-cone

developable (follow Cerda 1999)
(At any given point, there's a flat direction)



pure bending.



Sharp cone softens by stretching

Cone : $r = \rho \sin\theta$

$$E_{bending}\text{ (away from vertex)} = \frac{1}{2}\int_{R_c}^{R} B \frac{1}{r^2} \rho \, d\rho \, 2\pi = \frac{1}{2} B \int_{R_c}^{R} B \frac{\rho \, 2\pi \, d\rho}{\rho^2 \sin^2\theta}$$

$$\simeq \frac{1}{2} \frac{2\pi}{\sin^2\theta} B \ln\left(R/R_c\right)$$

- logarithmic in system size
- depends on R_c, size of cone defect.

Core of vertex / crescent singularity

R_c found to depend on system size! $R_c \sim R^{2/3}$

Find energy by making bending & stretching comparable.

$$E_s \equiv Y \gamma^2 R_c^2$$

$$E_B \equiv B \kappa^2 R_c^2$$

$$Y \gamma^2 = B \kappa^2 \qquad \gamma \sim \kappa \sqrt{B/Y}$$

Gauss' Theorema Egregium: (The remarkable theorem).

Gaussian curvature is preserved by smooth isometric transformations

Isometric transformations preserve lengths between material points.

Curvature tensor. $\zeta(x,y)$: $K = \begin{pmatrix} \partial_x^2 \zeta & \partial_x \partial_y \zeta \\ \partial_y \partial_x \zeta & \partial_y^2 \zeta \end{pmatrix}$

Two invariants are Tr and Det.

Along principal axes: $\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$

Bending couples to Trace

$B \, \mathrm{Tr} \, K^2$ in energy

Mean curvature: $\frac{1}{2} \mathrm{Tr} \, K$ plane = 0 Sphere = $\frac{R+R}{2} = R$.

Catenoid = 0 Saddles = ?

depends.

Stretching couples to det K

Gaussian curvature: $K_1 K_2$

plane = 0 cone = 0

Changing Gaussian curvature energetically costly → involves strain.