Canonical transformation

Equilibrium problem revisited :

$$P\left[q(x,0)=\gamma(x)\,;\ q(x,-\infty)=\rho\right] = \int \mathcal{D}[p,q]\, e^{-L\, S[p,q]}$$

$$S[p,q] = \int_{-\infty}^{0} dt \left\{ \int dx\, p\dot{q} - H[p,q] \right\}$$

$$H[p,q] = \frac{\sigma(q)}{2}(\partial_x p)^2 - D(q)(\partial_x q)(\partial_x p)$$

$$= \frac{\sigma(q)}{2}(\partial_x p)\cdot\left\{ \partial_x p - \frac{2D(q)}{\sigma(q)}\cdot \partial_x q \right\}$$

Canonical transformation

$$p(x,t) = -\hat{p}(x,-t)^{*} + \int_{\rho}^{\hat{q}(x,-t)} ds\, \frac{2D(s)}{\sigma(s)}$$

$$q(x,t) = \hat{q}(x,-t)$$

$$\int_{-\infty}^{0} dt \int_{0}^{1} dx\, p\,\dot{q} = \int_{0}^{1} dx \cdot \int_{-\infty}^{0} dt \cdot \frac{d}{dt}\hat{q}(x,-t)\left[\int_{\rho}^{\hat{q}(x,-t)} ds\, \frac{2D(s)}{\sigma(s)}\right] - \int_{-\infty}^{0} dt \int_{0}^{1} dx\, \hat{p}(x,-t)\, \frac{d}{dt}\hat{q}(x,-t)$$

$$= -\int dx \int_{0}^{\infty} dt\, \frac{d}{dt}\hat{q}(x,t)\left[\int_{\rho}^{\hat{q}(x,t)} ds \left[\frac{2D(s)}{\sigma(s)}\right]\right] + \int_{0}^{\infty} dt \int_{0}^{1} dx\, \hat{p}(x,t)\, \frac{d}{dt}\hat{q}(x,t)$$

$$H[P,q] = \frac{\sigma(\hat{q}(x,t))}{2} \cdot \left[\partial_x \hat{P}(x,t) - \frac{2D(\hat{q})}{\sigma(\hat{q})} \cdot \partial_x \hat{q} \right] \partial_x \hat{P}$$

New Action

$$S[P,q] \; \cancel{\hat{S}[\hat{P},\hat{q}]}$$

$$= - \int_0^1 dx \int_0^\infty dt \cdot \dot{\hat{q}} \left[\int_\rho^{\hat{q}} ds \, \frac{2D(s)}{\sigma(s)} \right]$$

$$\underbrace{+ \int_0^\infty dt \int_0^1 dx \left\{ \hat{P} \dot{\hat{q}} - \underbrace{\hat{H}[P,q]}_{\text{Same as } H} \right\}}_{\hat{S}[\hat{P},\hat{q}]}$$

Boundary: $\hat{q}(x,0) = \gamma(x)$

$\hat{q}(x,\infty) = \rho$

$\hat{P} = 0$ at boundary ; $\hat{q} = \rho$ at boundary

First term:

$$- \int_0^1 dx \int_0^\infty dt \, \dot{\hat{q}} \left[f'(\hat{q}(x,t)) - f'(\rho) \right]$$

$$= - \int_0^1 dx \left\{ f'(\rho) - f'(\gamma(x)) - f'(\rho)(\rho - \gamma(x)) \right\}$$

$$= \phi[\gamma(x)]$$

The new Action:

$$\hat{S} = \int_0^\infty dt \int_0^1 dx \left\{ \hat{P} \dot{\hat{q}} - H[\hat{P},\hat{q}] \right\}$$

Downhill $\longrightarrow$ $\hat{S}_{min} = 0$

Explicit solution for SEP using Canonical Transformation.

Introductory example: Equilibrium for general system.

$$P[q(x,0)=r(x),\ q(x,-\infty)=\rho] = \int \mathcal{D}[P,q]\, e^{-L\, S[P,q]}$$

$$S[P,q] = \int_{-\infty}^{0} dt \left\{ \int dx\, P\dot{q} - H[P,q] \right\}$$

$$H[P,q] = \frac{\sigma(q)}{2}(\partial_x P)^2 - D(q)(\partial_x P)(\partial_x q)$$

$$= \frac{\sigma(q)}{2}\cdot \partial_x P \cdot \left\{ \partial_x P - \frac{2D(q)}{\sigma(q)}\cdot \partial_x q \right\}$$

Change of variable:
$$P(x,t) = \hat{P}(x,t) + \int_{\rho}^{q(x,t)} ds\, \frac{2D(s)}{\sigma(s)}$$

$$q(x,t) = q(x,t)$$

The Action

$$S[P,q] = \int_{-\infty}^{0} dt \int_{0}^{1} dx \left[\int_{\rho}^{q(x,t)} ds\, \frac{2D(s)}{\sigma(s)} \right] \dot{q}$$

$$+ \int_{-\infty}^{0} dt \left\{ \int_{0}^{1} dx\, \hat{P}\cdot\dot{q} - \hat{H}[\hat{P},q] \right\}$$

Where
$$\hat{H}[\hat{P},q] = \frac{\sigma(q)}{2}\cdot\left[\partial_x \hat{P} + \frac{2D(q)}{\sigma(q)}\cdot \partial_x q\right]\cdot \partial_x \hat{P}$$

(Different if $\rho=\rho(x)$ and hence fails)

$\Rightarrow$ [Canonical transformation]

$$P(r(x)) = e^{-L\int_{-\infty}^{0} dt \int_0^1 dx \left[\int_{\rho}^{q(x,t)} ds\, \frac{2D(s)}{\sigma(s)}\right]\dot q}$$

$$\int \mathcal{D}[\hat p, q]\, e^{-L\,\hat S[\hat p, q]}$$

• Use $\dfrac{2D(s)}{\sigma(s)} = f''(s)$

$$\Rightarrow \int dx \int_{-\infty}^{0} dt \int_{\rho}^{q(x,t)} ds\, f''(s)\cdot \dot q = \int dx \int_{-\infty}^{0} dt\, \dot q\, [f'(q) - f'(\rho)]$$

$$= \int dx \left[f(r) - f(\rho) - f'(\rho)(r-\rho) \right]$$

$$= \phi[r(x)]$$

• show $\boxed{\min \hat S[\hat p, q] = 0}$

$$\hat H[\hat p, q] = \frac{\sigma(q)}{2}(\partial_x \hat p)^2 + D(q)\cdot \partial_x q \cdot (\partial_x \hat p)$$

$$\Rightarrow \dot q = \frac{\delta \hat H}{\delta \hat p} = -\partial_x \left\{ \sigma(q)(\partial_x \hat p) + D(q)\,\partial_x q \right\}$$

$$\Rightarrow \boxed{\dot q + \partial_x (D(q)\,\partial_x q) = -\partial_x(\sigma(q)\,\partial_x \hat p)}$$

$$\dot{\hat p} = -\frac{\delta \hat H}{\delta q} = -\left\{ \frac{\sigma'(q)}{2}(\partial_x \hat p)^2 + D'(q)\cdot \partial_x q \cdot (\partial_x \hat p) - \partial_x (D(q)\,\partial_x \hat p) \right\}$$

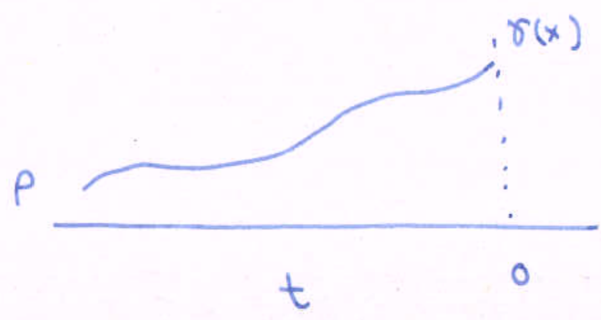
$$\Rightarrow \boxed{\dot{\hat p} - D(q)\,\partial_{xx}\hat p = -\frac{\sigma'(q)}{2}(\partial_x \hat p)^2}$$

<u>Boundary conditions:</u>

$$q(x,-\infty) = \rho \qquad q(x,0) = \gamma(x) \quad \Big| \quad q(0,t)=\rho \qquad q(1,t)=\rho \quad \Big| \quad \hat{p}(0,t) = 0 = \hat{p}(1,t)$$

<u>Solution:</u> "<u>Downhill works</u>"

$$\hat{p}(x,t) = 0$$
$$\dot{q}(x,t) = -\partial_x\left(D(q)\,\partial_x q\right)$$



$$q_R(x,t) = q(x,-t)$$

$$\Rightarrow \hat{S}_{min} = 0 \qquad \dot{q}_R = \partial_x\left(D(q_R)\,\partial_x q_R\right)$$

Symmetric simple exclusion process

* We take similar strategy*

* Find a canonical transformation such that Downhill Solution is the least action path*

$\boxed{\text{Formulation}}$: $D = 1$; $\sigma = 2q(1-q)$

$$H[p,q] = \frac{\sigma(q)}{2}\cdot\partial_x p\cdot\left[\partial_x p - \frac{\partial_x q}{q(1-q)}\right] = \frac{\sigma(q)}{2}\cdot\partial_x p\cdot\partial_x\left[p - \ln\frac{q}{1-q}\right]$$

$$\cancel{= \frac{\sigma(q)}{2}\cdot\partial_x p\cdot\partial_x\left[p - \frac{\delta V}{\delta q}\right] \text{ where } V(q) = \int dx\,\{q\ln q + (1-q)\ln(1-q)\}}$$

(leave as exercise)

First atempt: Try as before

$$p = \hat{p} + \int_{\rho(x)}^{q} ds \frac{1}{s(1-s)} = \hat{p} + \ln\frac{q}{1-q} - \ln\frac{\rho}{1-\rho}$$

$\rho(x) \leftarrow (*)$

⊙ $$\int_{-\infty}^{0} dt \int p\, \dot{q}\, dx = \int dx \int_{-\infty}^{0} dt \left[\ln\frac{q}{1-q} - \ln\frac{\rho}{1-\rho}\right]\cdot \dot{q} + \int dx \int_{-\infty}^{0} dt\, \hat{p}\, \dot{q}$$

$$= \int dx \cdot \left[\gamma \ln\frac{\gamma}{\rho} + (1-\gamma)\ln\frac{1-\gamma}{1-\rho}\right] + \int_{-\infty}^{0} dt \int dx\, \hat{p}\, \dot{q}$$

⊙ $H[p,q] \longrightarrow \hat{H}[\hat{p},q]$

$$= \frac{\sigma(q)}{2}\cdot\left[\partial_x\left(\hat{p} - \ln\frac{\rho}{1-\rho}\right)\right]^2 + \partial_x q \cdot \left[\partial_x\left(\hat{p} - \ln\frac{\rho}{1-\rho}\right)\right]$$

⊙ least Action paths

$$\dot{q} + \partial_{xx} q = -\partial_x\left[\sigma(q)\cdot\partial_x\left(\hat{p} - \ln\frac{\rho}{1-\rho}\right)\right]$$

$$\dot{\hat{p}} - \partial_{xx}\left(\hat{p} - \ln\frac{\rho}{1-\rho}\right) = -\frac{\sigma'(q)}{2}\left[\partial_x\left(\hat{p} - \ln\frac{\rho}{1-\rho}\right)\right]^2$$

⊙ Boundary condition

$q(x,-\infty) = \rho(x)$	$q(0,t) = \rho_a$	$\hat{p}(0,t) = 0$
$q(x,0) = \gamma(x)$	$q(1,t) = \rho_b$	$\hat{p}(1,t) = 0$

⊙ Down-hill solution: $\hat{p} = \ln\frac{\rho}{1-\rho}$ and $\dot{q} + \partial_{xx} q = 0$

$\Rightarrow$ $\boxed{\text{But contradicts } \hat{p}(0,t) = 0 = \hat{p}(1,t)}$

(comment: $\hat{p} = 0$ is not a solution as eqn's are not satisfied)

Second attempt: Transformation

$$p = \ln \frac{1-F}{F} + \ln \frac{q}{1-q}$$

$$q = R \cdot F(1-F)$$

• Step 1: $$\boxed{\int_{-\infty}^{0} dt\, p\dot{q} = \Big[q \ln q + (1-q)\ln(1-q) + q \ln \frac{1-F}{F} \Big]_{t=-\infty}^{t=0} + \int_{-\infty}^{0} dt\, R \dot{F}}$$

[Proof: $$\text{L.H.S} = \int dt \cdot \ln \frac{q}{1-q} \cdot \dot{q} + \int dt \cdot \ln \frac{1-F}{F} \cdot \dot{q}$$

$$= \int dq \ln \frac{q}{1-q} + q \cdot \ln \frac{1-F}{F} \Big|_{t=-\infty}^{t=0} - \int_{-\infty}^{0} dt \cdot q \cdot \frac{d}{dt} \ln \frac{1-F}{F}$$

$$= \Big[q \ln q + (1-q)\ln(1-q) + q \ln \frac{1-F}{F} \Big]_{t=-\infty}^{t=0} + \int_{-\infty}^{0} dt\, \underbrace{\frac{q \cdot \partial_t F}{F(1-F)}}_{R \dot{F}} \Big]$$

• Step 2: $$\boxed{H[p,q] = -\partial_x R \cdot \partial_x F - R \cdot (\partial_x F)^2 \Big[R - \frac{2}{1-F} \Big]} = \hat{H}[R,F]$$

[Proof: $$H = q(1-q)\, \partial_x p \cdot \partial_x \Big[p - \ln \frac{q}{1-q} \Big]$$

$$= q(1-q) \Big[-\frac{\partial_x F}{F(1-F)} + \frac{\partial_x q}{q(1-q)} \Big] \Big(-\frac{\partial_x F}{F(1-F)} \Big)$$

$$= \frac{q(1-q)}{F^2(1-F)^2} (\partial_x F)^2 - \frac{(\partial_x q)(\partial_x F)}{F(1-F)}$$

$q = RF(1-F)$

$$= \frac{1-q}{F(1-F)} R (\partial_x F)^2 - \frac{(\partial_x q)(\partial_x F)}{F(1-F)}$$

we $\dfrac{\partial_x q}{F(1-F)} = \partial_x R + \dfrac{1-2F}{F(1-F)} \cdot R \cdot \partial_x F$

$$\rightarrow \quad R(\partial_x F)^2 \cdot \frac{1-q}{F(1-F)} - R(\partial_x F)^2 \cdot \frac{1-2F}{F(1-F)} - \partial_x R \cdot \partial_x F$$

$$= R(\partial_x F)^2 \cdot \frac{1}{F(1-F)} \left(\cancel{1} - q - \cancel{1} + 2F \right) - \partial_x R \cdot \partial_x F$$

$$= -R(\partial_x F)^2 \cdot \left[R - \frac{2}{1-F} \right] - \partial_x R \cdot \partial_x F$$

Proved

Step3 : $R = \hat{F} + \dfrac{1}{1-F}$

- $$\int_{-\infty}^{0} dt\, R \dot{F} = \int_{-\infty}^{0} dt\, \frac{\dot{F}}{1-F} + \int_{-\infty}^{0} dt\, \hat{F} \cdot \dot{F} = -\ln(1-F)\Big|_{t=-\infty}^{t=0} + \int_{-\infty}^{0} dt\, \hat{F} \cdot \dot{F}$$

- $$\hat{H}[R,F] = -\left(\hat{F} + \frac{1}{1-F}\right)\left(\hat{F} - \frac{1}{1-F}\right)(\partial_x F)^2 - \partial_x \hat{F} \cdot \partial_x F - \frac{\partial_x F}{(1-F)^2} \cdot \partial_x F$$

$$= -\hat{F}^2 (\partial_x F)^2 - (\partial_x \hat{F})(\partial_x F)$$

$$= K[\hat{F}, F]$$

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Finaly

$$S[P,q] = \int dx \cdot \left[q \ln \frac{q}{F} + (1-q) \ln \frac{1-q}{1-F} \right]_{t=-\infty}^{t=0}$$

$$+ \int_{-\infty}^{0} dt \cdot \left\{ \int dx \cdot \hat{F} \cdot \dot{F} - K[\hat{F}, F] \right\} \quad \Bigg| \; K = -\hat{F}^2 (\partial_x F)^2 - (\partial_x \hat{F}) \cdot (\partial_x F$$

where $F = \dfrac{q}{q + (1-q) e^{p}}$ and $\hat{F} = (1-q)(e^{p} - 1) - q(e^{-p} - 1)$

[Proof of the last transformation relation

- $p = \ln \dfrac{1-F}{F} + \ln \dfrac{q}{1-q} \Rightarrow F = \dfrac{q}{q + (1-q) e^{p}}$

- $\hat{F} = R - \dfrac{1}{1-F} = \dfrac{q}{F(1-F)} - \dfrac{1}{1-F} = \dfrac{1}{1-F} \left[\dfrac{q}{F} - 1 \right]$

$$= \frac{1}{1-F} \left[q + (1-q) e^{p} - 1 \right] = (1-q)(e^{p} - 1) \frac{q + e^{p}(1-q)}{(1-q) e^{p}}$$

$$= (1-q)(e^{p} - 1) - q(e^{-p} - 1)$$

where we used $1 - F = \dfrac{(1-q) e^{p}}{q + (1-q) e^{p}}$]

Boundary conditions: $F(0,t) = P_a$, $F(1,t) = P_b$ | $\hat{F}(0,t) = 0$, $\hat{F}(1,t) = 0$

Condition on time: $q(x, -\infty) = \rho(x)$, $q(x, 0) = \gamma(x)$ } Explicit transformation is difficult

~~Ansatz solution:~~ ~~$\hat{F}(x,t) =$~~

Least Action paths

$$\dot{F} = \frac{\delta K}{\delta \hat{F}} = \partial_{xx} F - 2\hat{F}(\partial_x F)^2$$

$$\dot{\hat{F}} = -\frac{\delta K}{\delta F} = -\partial_{xx}\hat{F} - 2\,\partial_x(\hat{F}^2 \cdot \partial_x F)$$

Claim: Solution $\boxed{\hat{F}_{cl} = \frac{\partial_{xx} F_{cl}}{(\partial_x F_{cl})^2} \quad \text{and} \quad \dot{F}_{cl} = -\partial_{xx} F_{cl}}$ ← * Important to denote classical *

Check:
$$\dot{F}_{cl} = \partial_{xx} F_{cl} - 2\frac{\partial_{xx} F}{(\partial_x F)^2} \cdot (\partial_x F)^2 = -\partial_{xx} F$$

$$\dot{\hat{F}} = -\partial_x\left[\partial_x \hat{F} + 2\,\partial_x F \cdot \hat{F}^2\right]$$

$$\text{L.H.S} = \frac{d}{dt}\left(\frac{\partial_{xx} F}{(\partial_x F)^2}\right) = -2 \cdot \frac{\partial_x \partial_t F}{(\partial_x F)^3} \cdot \partial_{xx} F + \frac{\partial_{xx} \dot{F}}{(\partial_x F)^2}$$

$$= + \frac{2\,\partial_x^3 F}{(\partial_x F)^3} \cdot \partial_{xx} F - \frac{\partial_x^4 F}{(\partial_x F)^2}$$

$$\text{R.H.S.} = -\partial_x\left[-2\frac{\partial_{xx} \cancel{F}}{(\cancel{\partial_x} F)^3} \cdot \partial_{xx} F + \frac{\partial_x^3 F}{(\partial_x F)^2} + 2 \cdot \partial_x F \cdot \frac{(\cancel{\partial_x F})^2}{(\cancel{\partial_x} F)^4}\right]$$

$$= -\frac{d}{dx}\left(\frac{\partial_x^3 F}{(\partial_x F)^2}\right) = 2 \cdot \frac{\partial_{xx} F}{(\partial_x F)^3} \cdot \partial_x^3 F - \frac{\partial_x^4 F}{(\partial_x F)^2}$$

Proved

⊙ Boundary condition :

$$\hat{F}_{cl}(0,t) = 0 = \hat{F}_{cl}(1,t)$$

$$\Rightarrow \boxed{\partial_{xx} F_{cl}(x,t) = 0} \text{ at } x = 0,1$$

⊙ Condition on time : First show

$$\boxed{F_{cl} + F_{cl}(1-F_{cl}) \frac{\partial_{xx} F_{cl}}{(\partial_x F_{cl})^2} = q_{cl}(x,t)} \xleftarrow[\text{from}]{\text{comes}} \hat{F}_{cl} = \frac{\partial_{xx} F_{cl}}{(\partial_x F_{cl})^2}$$

[Proof : $\hat{F}_{cl} = \frac{\partial_{xx} F_{cl}}{(\partial_x F)^2}$

$$\hookrightarrow = (1-q)(e^{p}-1) + q(e^{-p}-1)$$

$$\hookrightarrow \text{from } F = \frac{q}{q+(1-q)e^{p}}$$]

This implies

$$F_{cl} + F_{cl}(1-F_{cl}) \cdot \frac{\partial_{xx} F_{cl}}{(\partial_x F_{cl})^2} = \rho(x) \text{ at } t = -\infty$$

$$F_{cl} + F_{cl}(1-F_{cl}) \frac{\partial_{xx} F_{cl}}{(\partial_x F_{cl})^2} = \gamma(x) \text{ at } t = 0$$

~~Lets choose~~ : $\cancel{F_{cl}(x,-\infty) = \rho(x)}$

Works if $F(x,-\infty) = \rho(x)$

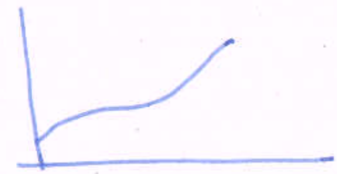
Corresponding ~~minimal~~ $K[\ ] = 0$ [integration by parts]

» minimal action $\hat{S}_{min} = \int_{-\infty}^{0} dt \int dx \frac{\partial_{xx} F}{(\partial_x F)^2} \cdot \dot{F}$

$$= \int dx\, \ln \partial_x F \Big|_{-\infty}^{0}$$

Hamilton - Jacobi approach

$$P_T\left[\rho(x), p(x)\right] \sim e^{-L\cdot\Phi_T[\rho(x),p(x)]}$$

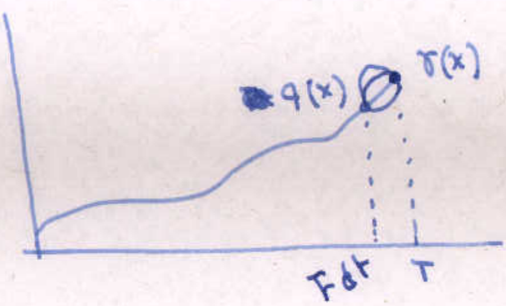


One gets

$$\boxed{\frac{d}{dT}\Phi_T(\rho,p) = -H\left[\underbrace{\frac{\delta\Phi_T}{\delta\rho}}_{p},\rho\right]}$$

Proof 1: generalize the approach given for Langevin equation.

Proof 2:



$$P_T[\rho(x)] = \int \mathcal{D}[q]\, P_{T-dt}[q(x)]\, W_{dt}\left(\rho(x), q(x)\right)$$

$$\left[\ \partial_t\rho = -\partial_x J\right.$$

$$\longrightarrow \rho(x) - q(x) = -dt\,\partial_x J$$

$$\Rightarrow W_{dt}(\rho,q) \simeq \int \mathcal{D}[J]\cdot e^{-L\,dt\int dx\frac{(J+D(\rho)\partial_x\rho)^2}{2\sigma(\rho)}}\cdot\delta\left(\rho - q + dt\,\partial_x J\right)$$

$$\Rightarrow P_T[\rho(x)] = \int\mathcal{D}[J]\, P_{T-dt}\left(\rho + dt\,\partial_x J\right) e^{-L\,dt\int dx\frac{(J+D(\rho)\partial_x\rho)^2}{2\sigma(\rho)}}$$

Substitute $P_T[\rho] \sim e^{-L\,\Phi_T[\rho]}$

$$\Rightarrow e^{-L\Phi[\rho]} \equiv \int\mathcal{D}[J]\cdot e^{-L\,\Phi_{T-dt}\left(\rho+dt\,\partial_x J\right) - L\,dt\int dx\frac{(J+D(\rho)\partial_x\rho)^2}{2\sigma(\rho)}}$$

Final result

$$\phi[\gamma(x)] = \left. \int_0^1 dx \left\{ \gamma(x) \ln \frac{\gamma(x)}{F(x)} + (1-\gamma(x)) \ln \frac{1-\gamma}{1-F} + \ln \partial_x F \right] \right|_{t=-\infty}^{t=0}$$

$$= \int_0^1 dx \left[\gamma(x) \ln \frac{\gamma(x)}{F(x)} + (1-\gamma(x)) \ln \frac{1-\gamma}{1-F} + \ln \frac{\partial_x F}{\rho_b - \rho_a} \right]$$

where $F(x) + F(x)(1-F(x)) \frac{\partial_{xx} F}{(\partial_x F)^2} = \gamma(x)$

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* Remark : $\boxed{\frac{d}{dt} F_a(x,t) = - \frac{d^2}{dx^2} F_a(x,t)}$

with $F_a(x,-\infty) = \rho(x)$

$F_a(x,0) =$ solution of $\left\{ F_a + F_a(1-F_a) \frac{\partial_{xx} F_a}{\partial_x F_a} = \gamma \right\}$

Solution exist because we know F_{TR} exist.

* Remark 2 : The optimal trajectory of density is

$$\boxed{q_a(x,t) = F_a(x,t) + F_a(x,t)\left(1 - F_a(x,t)\right) \frac{\partial_{xx} F_a(x,t)}{\partial_x F_a(x,t)}}$$

Taylors expand

$$= \int \mathcal{D}[J]\, e^{-L\phi_{T-dt}[\gamma] - L\,dt \cdot \int dx \cdot \frac{\delta\phi_{T-dt}}{\delta\gamma} \cdot \partial_x J - L\,dt \int dx \frac{(J + D\partial_x\gamma)^2}{2\sigma(\gamma)}}$$

$$\Rightarrow e^{-L\phi_T[\gamma]} = e^{-L\phi_{T-dt}[\gamma]} \int \mathcal{D}[J] \cdot e^{-L\,dt \int dx \left[\frac{\delta\phi}{\delta\gamma} \cdot \partial_x J + \frac{(J + D\partial_x\gamma)^2}{2\sigma(\gamma)}\right]}$$

[First take $L \to \infty$ then $dt \to 0$]

$$\Rightarrow \boxed{\frac{d\phi_T}{dT} = + \min_J \int_0^1 dx \left[\frac{\delta\phi}{\delta\gamma} \cdot \partial_x J + \frac{(J + D\partial_x\gamma)^2}{2\sigma(\gamma)}\right]}$$

$$= \min_J \int_0^1 dx \left[-\partial_x\left(\frac{\delta\phi}{\delta\gamma}\right) \cdot J + \frac{(J + D\partial_x\gamma)^2}{2\sigma}\right] + J \cdot \frac{\delta\phi}{\delta\gamma}\Big|_{x=0}^{x=1}$$

$$= \int dx \left\{\frac{[D(\gamma)\partial_x\gamma]^2}{2\sigma(\gamma)} - \frac{\left[D(\gamma)\partial_x\gamma - \sigma(\gamma)\partial_x\frac{\delta\phi}{\delta\gamma}\right]^2}{2\sigma(\gamma)}\right\} + J \cdot \frac{\delta\phi}{\delta\gamma}\Big|_{x=0}^{x=1}$$

$$\boxed{= - \int_0^1 dx \left\{\frac{\sigma(\gamma)}{2}\left(\partial_x \frac{\delta\phi}{\delta\gamma}\right)^2 - D(\gamma) \cdot \partial_x\gamma \cdot \partial_x \frac{\delta\phi}{\delta\gamma}\right\}} + J \cdot \frac{\delta\phi}{\delta\gamma}\Big|_{x=0}^{x=1}$$

$$= - H\left[\frac{\delta\phi}{\delta\gamma}, \gamma\right] + J(x) \cdot \frac{\delta\phi}{\delta\gamma(x)}\Big|_{x=0}^{x=1}$$

[Why $J\frac{\delta\phi}{\delta\gamma}=0$? self-consistent solution?]

⊛ choose a boundary condition $\frac{\delta\phi}{\delta\gamma}=0$?

STATIONARY STATE

Example 1: Equilibrium:

indep of time.

$$\phi_T[\gamma(x),\rho] = \int dx \cdot \left\{ f(\gamma) - f(\rho) - f'(\rho)(\gamma-\rho) \right\}$$

$$\Rightarrow \frac{\delta\phi}{\delta\gamma} = f'(\gamma) - f'(\rho)$$

$$\Rightarrow \partial_x \frac{\delta\phi}{\delta\gamma} = f''(\gamma)\cdot\partial_x\gamma = \frac{2D(\gamma)}{\sigma(\gamma)}\cdot\partial_x\gamma$$

In stationary state $\frac{\partial\phi}{\partial\tau}=0$

$$\Rightarrow H\left[\frac{\delta\phi}{\delta\gamma},\gamma\right] = \int dx \underbrace{\left\{ \frac{\sigma(\gamma)}{2}\cdot\partial_x\frac{\delta\phi}{\delta\gamma} - D(\gamma)\cdot\partial_x\gamma \right\}}_{0}\cdot\partial_x\frac{\delta\phi}{\delta\gamma}$$

$$= 0$$

Example 2: Non-interacting particles [Excercise]

Example 3: SEP: $\phi[\gamma] = \int dx \cdot \left\{ \gamma \ln\frac{\gamma}{F} + (1-\gamma)\ln\frac{1-\gamma}{1-F} + \ln\frac{\partial_x F}{\rho_b-\rho_a} \right\}$

$$\gamma = F + F(1-F)\cdot\frac{\partial_{xx}F}{(\partial_x F)^2} \quad \Bigg| \quad \begin{array}{l} F(0) = \rho_a \\ F(1) = \rho_b \end{array}$$

⊙ $$\frac{\delta\phi}{\delta\gamma} = \ln\frac{\gamma(1-F)}{(1-\gamma)F} = \ln\left(1 + \frac{\partial_{xx}F}{(\partial_x F)^2 - F\cdot\partial_{xx}F}\right)$$

(substitute

⊙ $$H = -2\cdot\int_0^1 dx\cdot\frac{d}{dx}\left[\frac{\partial_{xx}F}{\partial_x F}\right] = 2\left.\frac{\partial_{xx}F}{\partial_x F}\right|_{x=0} - 2\left.\frac{\partial_{xx}F}{\partial_x F}\right|_{x=1}$$

$$= 2 \cdot \frac{[\gamma(x) - F(x)]\, \partial_x F}{F(x)\,(1 - F(x))}\Bigg|_{x=1}^{x=0}$$

$= 0$ because at boundary $\gamma(x) = F(x)$

Ref: Bertini, De sole, Gabrielli, Jona-Lasinio and Landim 2001, PRL, 87, 40601

* ~~solving for general is difficult and not known yet~~ 2002, JSP, 107, 635

Correlations

~~For~~

$\boxed{\text{Remark}}$: Solving for general system is difficult and an open problem!

Other known model: $\boxed{\text{KMP}}$: $D = 1$; $\sigma = 2\rho^2$

Ref: ~~Bertini~~

Bertini, Gabrielli and Lebowitz (2005)

JSP, 121, 843.

* A general solution and correlation!

Correlation

⊛ Cumulant generating function: "Recap of earlier lecture"

$$G[h] = \max_{\rho(x)} \left[\int dx\, h(x)\rho(x) - \phi[\rho(x)] \right]$$ Legendre transform

cumulants
$$\langle \rho(x_1)\cdots\rho(x_k)\rangle_c = \frac{1}{L^{k-1}} \frac{\delta^k G}{\delta h(x_1)\cdots\delta h(x_k)}\Big|_{h=0}$$

⊙ What is $G[h]$ for NI particles:

$$\phi[\rho] = \int dx \cdot \left\{ \rho \ln \frac{\rho}{P(x)} - \rho + P \right\}$$

Legendre transform

$$\Rightarrow \boxed{G[h] = \int dx \cdot P(x)\left(e^{h(x)} - 1\right)}$$

$$= \int dx\, h(x)\cdot P(x) + \frac{1}{2!}\int dx\, h^2(x) P(x) + \cdots$$

⇒ Average
$$\langle \rho(x)\rangle = P(x)$$

$$\langle \rho(x)\rho(y)\rangle = \frac{1}{L} P(x)\,\delta(x-y)$$ [Short-range correlation]

⊙ Symmetric exclusion process:

$$G[h] = \int dx \left[\log\left(1 - F + F e^{h}\right) - \ln \frac{\partial_x F}{\rho_b - \rho_a} \right] \equiv \boxed{\int dx \left[\ln\left(1 + F(e^{h}-1)\right) - \ln \frac{F'}{\rho_b - \rho_a} \right]}$$

with
$$F'' + \frac{(F')^2(1-e^{h})}{1 - F + F e^{h}} = 0 \equiv \boxed{F'' = \frac{(F')^2(e^{h}-1)}{1 + F(e^{h}-1)}}$$

Expanding

$$G[h] = \int dx\, h(x)\, \rho(x) - \frac{1}{2}\int dx \int dy \cdot \left[x(1-y)\,\Theta(y-x) + y(1-x)\,\Theta(x-y) \right] h(x)\, h(y) \cdot (\rho_a - \rho_b)^2$$

$$+ \frac{1}{2}\int dx\; \rho(x)(1-\rho(x))\, h^2(x) + \cdots$$

Average:

$$\langle \tau(x) \rangle = \rho(x) = \rho_a (1-x) + \rho_b x$$

$$\langle \tau(x)\tau(y) \rangle = \frac{1}{L}\Big\{ \rho(x)(1-\rho(x))\,\delta(x-y) - \left[x(1-y)\,\Theta(y-x) + y(1-x)\,\Theta(x-y) \right](\rho_a-\rho_b)^2 \Big\}$$

(Long-range correlation)

Vanishes for $\rho_a = \rho_b$

$\boxed{\text{~~How to derive for general case~~}}$?

$\boxed{\text{Back ground}}$: long-range correlation at generic parameter value.

Mention other references.
"Sriram Ramaswami"
People from Bangalore ?

Ref: Garrido, Lebowitz, Maes, Spohn: PRA 42, 1954 (1990)

Grinstein, Lee, Sachdev PRL 64, 1927 (1990)

Dorfman, Kirkpatrick, Sengers Ann. Rev. Physics. chem 45, 213 (1994)

Garrido

Correlation for general case : A perturbative solution of H-J equation.

$$H\left[\frac{\delta\phi}{\delta\gamma},\gamma\right]=0$$

$$\Rightarrow \quad \int dx \left\{ \frac{\sigma(\gamma)}{2}\cdot\left(\partial_x \frac{\delta\phi}{\delta\gamma}\right)^2 - \partial_x\gamma\cdot D(\gamma)\cdot\left(\partial_x\frac{\delta\phi}{\delta\gamma}\right)\right\}=0$$

The cumulant generating function

$$G[h] = \int dx\cdot h(x)\,\gamma(x) \quad - \phi[\gamma(x)]$$

where $\quad h(x) = \dfrac{\delta\phi}{\delta\gamma(x)}$ $\quad$ (To be consistent $h=0$ at $x=0$ and 1

Also $\quad \gamma(x) = \dfrac{\delta G}{\delta h(x)}$ $\quad$ (* Legendre transform is invertible * why?)

leads to

$$\boxed{\int dx\cdot\left\{\frac{1}{2}\cdot\sigma\left(\frac{\delta G}{\delta h}\right)\left(\partial_x h\right)^2 - \partial_x\left(\frac{\delta G}{\delta h}\right)\cdot D\cdot\left(\frac{\delta G}{\delta h}\right)\cdot(\partial_x h)\right\}=0}$$

H-J equation for $G[h]$

Cumulants :

$$G[h] = \int dx\, h(x)\,\langle\gamma(x)\rangle + \frac{1}{2!}\int dx\int dy\, \underbrace{\overbrace{\langle \cancel{\gamma(x)\gamma(y)}\rangle_c}^{h(x)h(y)}}_{C(x,y)} + \frac{1}{3!}\int dx_1\, dx_2\, dx_3 \cdots \underbrace{h(x_1)\,h(x_2)\,h(x_3)\cdots}_{C(x_1,x_2,x_3)}$$

~~Expand in powers of h~~

$$\Rightarrow \frac{\delta G}{\delta h(x)} = \underbrace{\langle\gamma(x)\rangle}_{P(x)} + \int dy\, h(y)\,\underbrace{\langle\cancel{\gamma(x)\gamma(y)}\rangle_c}_{C(x,y)} + \frac{1}{2}\int dy\int dz\, h(y)\,h(z)\,\underbrace{\langle\cancel{\gamma(x)\gamma(y)\gamma(z)}\rangle_c}_{C(x,y,z)} + \cdots$$

Substitute

$$\sigma\left(\frac{\delta G}{\delta h}\right) = \sigma(\rho(x)) + \sigma'(\rho)\cdot\int dy\, h(y)\, c(x,y) + \cdots$$

Similar for $D\left(\frac{\delta G}{\delta h}\right)$

The H-J equation

$$\int dx \cdot \Big\{ \frac{1}{2}\cdot\left[\sigma(\rho) + \sigma'(\rho)\cdot\int dy\, h(y)\, c(x,y) + \cdots\right](\partial_x h)^2$$

$$- \left[\partial_x \rho + \int dy\, h(y)\, \partial_x c(x,y) + \cdots\right]\left[D(\rho) + D'(\rho)\cdot\int dy\, h(y)\, c(x,y) + \cdots\right]\partial_x h \Big\} = 0$$

$\Rightarrow$ linear order

$$\boxed{-\int dx \cdot \partial_x \rho \cdot D(\rho)\cdot \partial_x h = 0} \Rightarrow \int dx\, h(x)\cdot \partial_x\left[D(\rho)\,\partial_x \rho\right] = 0$$

$\Rightarrow$ quadratic order

$$\int_0^1 dx \cdot \Big\{ \frac{\sigma(\rho)}{2}\cdot(\partial_x h)^2 - (\partial_x \rho)\cdot D'(\rho)\cdot \int_0^1 dy\cdot h(y)\, c(x,y)\cdot \partial_x h(x)$$

$$- D(\rho)\cdot \int_0^1 dy\, h(y)\, \partial_x c(x,y)\cdot \partial_x h \Big\} = 0$$

$$\Rightarrow \int dx\, dy \cdot \Big\{ \frac{\sigma(\rho(x))}{2}\cdot \delta(x-y)\cdot \partial_x h\cdot \partial_y h(y) - \partial_x\big(D(\rho)\cdot c(x,y)\big)\cdot \partial_x h(x)\cdot h(y) \Big\} = 0$$

$$\Rightarrow \boxed{\int dx\, dy\cdot h(x)\, h(y)\cdot \Big\{ \frac{d^2}{dx\, dy}\left[\frac{\sigma(\rho(x))\cdot \delta(x-y)}{2}\right] + \frac{d^2}{dx^2}\cdot\left[D(\rho)\, c(x,y)\right] \Big\} = 0}$$

Because $h(x)$ is an arbitrary function:

① $$\partial_x [D(\rho)\partial_x \rho] = 0 \quad \Rightarrow \quad D(\rho)\partial_x\rho = \langle J \rangle = \text{constant in } x$$

determines the average

② Symmetrize the equation and get

$$\frac{d^2}{dx^2}\left[D(\rho(x))\, c(x,y)\right] + \frac{d^2}{dy^2}\left[D(\rho(y))\, \underbrace{c(y,x)}_{c(x,y)}\right] = -\frac{1}{2}\,\partial_x\partial_y\left[\sigma(\rho(x))\,\delta(x-y) + \sigma(\rho(y))\,\delta(y-x)\right]$$

$$= \partial_x\left[\sigma(\rho(x))\,\delta'(x-y)\right]$$

Poisson-Type eq^n

③ $$\mathcal{L}^{(3)}\, c(x,y,z) = -\sum_{\substack{\text{Perm}\\(x,y,z)}} \left\{ \partial_x\left[\sigma'(x)\,\delta'(x-y)\, c(x,y)\right] + \partial_x^2\left[D'(x)\, c(x,y)\, c(x,z)\right]\right\}$$

General structure:

$$\boxed{\mathcal{L}^{(n)}\, c(x_1 \cdots x_n) = \Lambda\,(\text{lower point correlations})}$$

Longrange nature:

x dependence!

long-range part

$$c(x,y) = \underbrace{\frac{\sigma(\rho(x))}{2D(\rho(x))}\,\delta(x-y)}_{\substack{\text{equilibrium}\\ \text{part}}} + B(x,y)$$

$c_{eq}(x,y)$ "Short range" $\left[\begin{array}{c} \sim e^{-\frac{X}{\xi}} \\ \ell \gg \xi \end{array}\right]$

$$\boxed{\partial_x^2\left[D(\rho(x))\,B(x,y)\right] + \partial_y^2\left[D(\rho(y))\,B(x,y)\right] = \delta(x-y)\underbrace{\left[\langle J\rangle\cdot\frac{d}{dx}\left(\frac{\sigma'(\rho)}{2D(\rho)}\right)\right]}_{\text{charge.}}}$$

(can be generalized for bulk-driven case.)

In equilibrium: $\langle J\rangle = 0 \Rightarrow B(x,y) = 0$

Non-interacting ptls: $\dfrac{\sigma(\rho)}{2D} = \rho \Rightarrow \dfrac{\sigma'(\rho)}{2D} = 1$

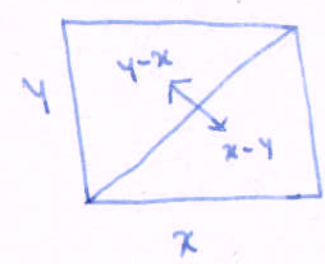
$\Rightarrow \partial_x[\;\;] = 0 \Rightarrow B = 0$

Zero-range process: $\sigma'(\rho) = 2D \Rightarrow B = 0$

CAN BE GENERALIZED TO ANY DIMENSION

When $D =$ constant: $B(x,y) \equiv$ Potential due to line localized charge.

$B(x,y) \sim 1 - |x-y|$ [Dimension=1]



* In general d

$B(x,y) \sim \dfrac{1}{r^{d-2}}$ ~~potential due to quadrupolar charge~~ Inverse Laplacian.

[① Spohn 1983 J.P.A 16, 4275

② Garrido, ... PRA (1990)]

* In general depends on multipole moments

[Sadhu, Majumdar, Mukamel 2014 PRE 90, 012109]

Exercise: verify for SEP $(D=1, \sigma = 2\rho(1-\rho))$

$$\boxed{B(x,y) = -(\rho_a-\rho_b)^2 \left[x(1-y)\,\theta(y-x) + y(1-x)\,\theta(x-y) \right]}$$

What about higher point correlation & General solution?

Green's function:

$$\partial_t g(x,y,t) = \partial_x^2 \left[D(\rho(x))\, g(x,y,t) \right]$$

with $g(x,y,0) = \delta(x-y)$

Any correlation can be written in terms of $g(x,y,t)$

Claim:

$$\boxed{C(x_1\cdots x_n) = -\int_0^\infty dt \int_0^1 dy_1 \cdots \int_0^1 dy_n\, g(x_1,y_1,t)\cdots g(x_n,y_n,t)\, \Lambda(y_1,\cdots,y_n)}$$

Proof: $\partial_{x_1}^2 D(x_1)\, C(x_1\cdots x_n) + \cdots = -\int_0^\infty dt \cdots \cdot \frac{\partial}{\partial t}\left[g(x_1,y_1,t) \cdots \right] \Lambda(y_1,\cdots y_n)$

$$= +\Lambda(x_1\cdots x_n)$$

① For $D=1$: $\quad G = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin(n\pi x)\sin(n\pi y)$

Is there any compact formula for general system?

① verify for SEP. $\quad \partial_t F(x,t) = \partial_x^2 F(x,t)$

④ F in terms of g.

② What happens for non-interacting particles?

③ In-equilibrium?

⊙ The correlation is order $\frac{1}{L}$.

Can it be significant in any practical case.

Example: ~~Fluctuation~~ Fluctuation of total number of particles.

$$N = L \int dx \cdot r(x)$$

For SEP

- $\langle N \rangle = L \cdot \frac{\rho_a + \rho_b}{2}$

- $\langle N^2 \rangle - \langle N \rangle^2 = L \left[\int dx \cdot \rho(x)(1-\rho(x)) - 2(\rho_a - \rho_b)^2 \cdot \int_0^1 dx \int_x^1 dy \, x(1-y) \right]$

~~comp~~

significant difference due to this

⊙ Any example where the correlation ~~is of order 1~~? does not vanish at $L \to \infty$

* correlation is $\propto \langle J \rangle$

⊛ Bulk-driven model, system with localized drive.

Statistics of current

* Current is a characteristic of Non-equilibrium system.

* Green-Kubo formula

$$\text{Conductivity} = \frac{1}{2\rho^2 \chi}\int_{-\infty}^{\infty} dt \, \langle J(0)\, J(t)\rangle_{eq.}$$

* Gallavotti-Cohen symmetry

$$\boxed{\frac{P\left(\frac{Q_t}{t}=-J\right)}{P\left(\frac{Q_t}{t}=J\right)} = e^{-t\cdot J\cdot(\mu_a-\mu_b)}}$$

* Keeping average density fixed ≡ fixed chemical potential μ

$\Rightarrow G = \mu N - F$

$\Rightarrow \mu = \frac{dF}{dN} = f'(\rho)$

[Original statement in terms of entropy production]

Ref: ① Gallavotti and Cohen, J. Stat. Phy. 80, 931 (1995)

② Evans and Searles, Adv. Phys. 51, 1529 (2002)

③ Kurchan J. Phy. A 31, 3719 (1998)]

* Transport properties, heat conduction,

① A. Dhar. Adv. Phys. . . .

Large deviation of integrated current:

$$Q_{\tilde T} = \frac{1}{L}\cdot \int_0^{L} dX \int_0^{\tilde T} d\mathsf{T}\; \mathcal{J}(X,\mathsf{T})$$

$$(t,x) \equiv \left(\frac{T}{L^2}, \frac{X}{L}\right)$$

$$\simeq \frac{1}{L}\cdot L \cdot L^2 \int_0^1 dx \int_0^{\tau} dt\; \underbrace{\dot{j}(x,t)}_{L} \qquad \tau \to \frac{\tilde T}{L^2}$$

$$\simeq L \cdot \int_0^1 dx \int_0^{\tau} dt\; \dot{j}(x,t) \xrightarrow{\text{large } \tau} L\tau \dot{j} + \text{subleading}$$

$$\hookrightarrow \frac{\tilde T}{L}\dot{j}$$

$$\boxed{P\left(\frac{Q_{\tilde T}}{\tilde T} = \frac{\dot j}{L}\right) \asymp e^{-\tau L \phi(\dot j)}} \qquad \text{what is } \phi(\dot j)\,?$$

$$\equiv e^{-\frac{\tilde T}{L}\phi(\dot j)}$$

~~Proof and analysis~~:

How does cumulants scale:

$$\ln\left\langle e^{\lambda Q_{\tilde T}}\right\rangle = \lambda\langle Q_{\tilde T}\rangle + \frac{\lambda^2}{2}\langle Q_{\tilde T}^2\rangle_c + \cdots$$

$$\downarrow$$

$$\ln \int dj\, e^{\lambda\cdot\tau L j - \tau L\phi(j)}$$

$$\downarrow$$

$$\tau L \max_j\{\lambda j - \phi(j)\} = \tau L\, a(\lambda) = \lambda\cdot\tau\cdot L\cdot a(0) + \lambda\cdot\tau L\cdot a'(0) + \cdots$$

Comparing: $\langle Q_{\tilde T}\rangle = \tau L \equiv \frac{\tilde T}{L}\cdot a(0)$ ↑ all cumulants $\sim \frac{\tilde T}{L}$

$\langle Q_{\tilde T}^2\rangle = \frac{\tilde T}{L}\, a'(0)$ ↓

we are ignoring subleading term $\sim \tilde T/L^2$

⊙ There is an expression of $\phi(\dot{j})$ for general D and σ

"Additivity principle"

Proof: $$P\left(\frac{Q_\tau}{\tau}=\frac{\dot{\mathcal{J}}}{L}\right) \simeq \int \mathcal{D}[J,q]\; e^{-L\int_0^\tau dt\int_0^1 dx\, \frac{[J+D(q)\partial_x q]^2}{2\sigma(q)}}\; \delta\left(\int_0^\tau dt\int dx\, J(x,t)=\dot{\mathcal{J}}\tau\right)$$

$$\xrightarrow[\text{history} := (J,\,q(x,t)) = (\dot{\mathcal{J}},\,q(x))]{\text{for large } \tau \text{ the optimal}} e^{-L\cdot\tau\cdot\min_{q(x)}\int_0^1 dx\,\frac{(\dot{\mathcal{J}}+D(q)\partial_x q)^2}{2\sigma(q)}}$$

Time independent

$$\Rightarrow \boxed{\phi(\dot{\mathcal{J}}) = \min_{q(x)} \int_0^1 dx\, \frac{(\dot{\mathcal{J}}+D(q)\partial_x q)^2}{2\sigma(q)}}$$

Solution:

$$= \min_q \int_0^1 dx \left\{ \dot{\mathcal{J}}^2\cdot\left(\frac{1}{2\sigma}\right) + \dot{\mathcal{J}}\cdot\frac{D\,\partial_x q}{\sigma} + \frac{D^2}{2\sigma}\cdot(\partial_x q)^2 \right\}$$

At minimum $\frac{\delta\phi}{\delta q}=0$

$$\Rightarrow \dot{\mathcal{J}}^2\cdot\left(\frac{1}{2\sigma}\right)' + \dot{\mathcal{J}}\left(\frac{D}{\sigma}\right)'\cdot\partial_x q + \left(\frac{D^2}{2\sigma}\right)'(\partial_x q)^2 - \dot{\mathcal{J}}\,\partial_x\left(\frac{D}{\sigma}\right) - \partial_x\left(2\,\partial_x q\cdot\frac{D^2}{2\sigma}\right)$$

cancells

$= 0$

$$-2\cdot\partial_{xx} q\cdot\left(\frac{D^2}{2\sigma}\right) - 2\cdot\left(\frac{D^2}{2\sigma}\right)'\cdot(\partial_x q)^2$$

$$\Rightarrow \boxed{\dot{\gamma}^2 \left(\frac{1}{2\sigma}\right)' - \left(\frac{D^2}{2\sigma}\right)'(\partial_x q)^2 - 2\left(\frac{D^2}{2\sigma}\right)\partial_{xx} q = 0}$$

Multiply $\partial_x q$

$$\frac{d}{dx}\left[\dot{\gamma}^2 \left(\frac{1}{2\sigma}\right)^{\bullet}\right] - \frac{d}{dx}\left[\frac{D^2}{2\sigma}\right](\partial_x q)^2 - \left(\frac{D^2}{2\sigma}\right)\cdot \frac{d}{dx}(\partial_x q)^2 = 0$$

$$\Rightarrow \boxed{\frac{d}{dx}\left[\dot{\gamma}^2 \left(\frac{1}{2\sigma}\right)^{\bullet} - \frac{D^2}{2\sigma}\cdot(\partial_x q)^2\right] = 0}$$

Integrate

a choice (constant)

$$\dot{\gamma}^2 \left(\frac{1}{2\sigma}\right)^{\bullet} - \frac{D^2}{2\sigma}\cdot(\partial_x q)^2 = K\cdot\dot{\gamma}^2$$

integration constant

$$\boxed{D^2(\partial_x q)^2 = \dot{\gamma}^2\left[1 + 2K\sigma(q)\right]}$$

$$\Rightarrow \boxed{D\,\partial_x q = -\dot{\gamma}\sqrt{1+2K\sigma(q)}}$$

optimal profile

why -ve (for positive $\dot{\gamma}$, $\partial_x q < 0$)

~~optimal profile~~

$$\dot{\gamma} = -\int_0^1 dx\, \frac{D\,\partial_x q}{\sqrt{1+2K\sigma}}$$

$$\Rightarrow \boxed{\dot{\gamma} = \int_{p_b}^{p_a} dq\, \frac{D(q)}{\sqrt{1+2K\sigma}}}$$

← ~~optimal profile~~

← K is determined from this!

What is $\phi(\dot{\gamma})$?

$$\phi(\dot{\gamma}) = \int dx \cdot \frac{\dot{\gamma}^2}{2\sigma} \cdot \left(1 - \sqrt{1+2\kappa\sigma} \right)^2$$

$$= \dot{\gamma} \int dx \cdot \frac{\dot{\gamma}}{2\sigma} \left[1 + 1 + 2\kappa\sigma - 2\sqrt{1+2\kappa\sigma} \right]$$

$$= \dot{\gamma} \int dx \cdot \frac{\dot{\gamma}}{\sigma} \cdot \left[1 + \kappa\sigma - \sqrt{1+2\kappa\sigma} \right]$$

$$= -\dot{\gamma} \int dx \cdot \frac{j\sqrt{1+2\kappa\sigma}}{\sigma} \left[1 - \frac{1+\kappa\sigma}{\sqrt{1+2\kappa\sigma}} \right]$$

$$= \dot{\gamma} \int dx \cdot \frac{D \cdot \partial_x q}{\sigma} \left[1 - \frac{1+\kappa\sigma}{\sqrt{1+2\kappa\sigma}} \right]$$

$$\Rightarrow \boxed{\phi(\dot{\gamma}) = \dot{\gamma} \int_{\rho_a}^{\rho_b} dq \cdot \frac{D}{\sigma} \cdot \left[1 - \frac{1+\kappa^{(\dot{\gamma})}\sigma}{\sqrt{1+2\kappa\sigma}_{\,q(\sigma)}} \right]}$$

where κ is given by

$$\boxed{\dot{\gamma} = \int_{\rho_b}^{\rho_a} dq \cdot \frac{D}{\sqrt{1+2\kappa\sigma}_{\,q(\sigma)}}}$$

Parametric solution

Remark 1

Gallavotti-Cohen symmetry: For $\dot{\gamma} \to -\dot{\gamma}$

$$-\dot{\gamma} = \int_{\rho_b}^{\rho_a} dq \frac{D(q)}{\underbrace{\sqrt{1+2\kappa^*\sigma}}_{f(\kappa^*)}} = -\int_{\rho_b}^{\rho_a} dq \cdot \frac{D(q)}{\sqrt{1+2\kappa\sigma}}$$

A simple choice $\dfrac{1}{\sqrt{1+2\kappa^*\sigma}} = -\dfrac{1}{\sqrt{1+2\kappa\sigma}}$

⊛ $\kappa^* = \kappa$ if one squares; it is a choice of which sign to take

$$\Rightarrow \quad \phi(-j) = -j\int_{\rho_a}^{\rho_b} dq\, \frac{D}{\sigma}\cdot\left[1 - \frac{1+K^*\sigma}{\sqrt{1+2K^*\sigma}}\right]$$

$$= -j\int_{\rho_a}^{\rho_b} dq\cdot \frac{D}{\sigma}\left[1 + \frac{1+K\sigma}{\sqrt{1+2K\sigma}}\right]$$

$$\Rightarrow \quad \phi(j) - \phi(-j) = j\int_{\rho_a}^{\rho_b} dq\cdot\frac{2D}{\sigma} = j\int_{\rho_a}^{\rho_b} dq\cdot f''(q) = j\left[f'(\rho_b) - f'(\rho_a)\right]$$

$$= j\left(\mu_b - \mu_a\right)$$

$$\Rightarrow \quad \boxed{\phi(j) - \phi(-j) = -j\left(\mu_a - \mu_b\right)}$$

$$\equiv \quad \frac{P(-j)}{P(j)} = e^{-\tau\cdot L\cdot[\phi(-j) - \phi(j)]} = e^{-\tau\cdot L\cdot j(\mu_a - \mu_b)}$$

$$= e^{-\frac{T}{L}\cdot j\cdot(\mu_a - \mu_b)}$$

<u>Remark 2:</u> $a(\lambda) = \max_j\left\{\lambda j - \phi(j)\right\}$

$$= -K\left[\int_{\rho_b}^{\rho_a} dq\, \frac{D(q)}{\sqrt{1+2K\sigma}}\right]^2$$

with

$$\lambda = \int_{\rho_b}^{\rho_a} dq\cdot\frac{D}{\sigma}\cdot\left[\frac{1}{\sqrt{1+2K\sigma}} - 1\right]$$

<u>Check that:</u>

$$\frac{\langle Q_T\rangle}{T} = \frac{1}{L}\int^{\rho_a} dq\, D(q) \simeq \frac{1}{L} D(\rho)(\rho_a - \rho_b) \quad\Bigg|\quad \frac{\langle Q_T^2\rangle_c}{T} \simeq \frac{1}{L}\,\frac{\int_{\rho_b}^{\rho_a} dq\,\frac{D}{\sigma}}{\int^{\rho_a} dq\, D} = \frac{\sigma(\rho)}{L}$$

$\boxed{\text{Remark 3}}$: For Non-Interacting particles: $D=1$; $\sigma = 2q$ ~~(...)~~

$$Q(\lambda) = \rho_a(e^{\lambda}-1) + \rho_b(e^{-\lambda}-1)$$

[check this]

For SEP: $D=1$; $\sigma = 2q(1-q)$

$$Q(\lambda) = \left[\ln\left(\sqrt{1+\omega} + \sqrt{\omega}\right)\right]^2$$

where $\boxed{\omega = \rho_a(e^{\lambda}-1) + \rho_b(e^{-\lambda}-1) - \rho_a\rho_b(e^{\lambda}-1)(e^{-\lambda}-1)}$

⊛ This feature is found for other cases. and has a symmetry reason.

⊛ Same found in transport of weakly interacting Fermion between two baths

~~Remark 4:~~

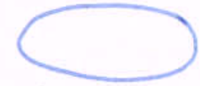
Ref: Lee, Levitov, Yakovets PRB 51, 4079 (1995)

Beenakker and Büttiker PRB 46, 1889 (1992)

Remark 4: ~~...~~

The distribution is Non-Gaussian even at equilibrium. $(\rho_a = \rho_b = \rho)$

* Only in equilibrium at $\rho = \rho^*$ where $\sigma(\rho^*)$ is maximum it is Gaussian, ie., all cumulant $\langle Q^n \rangle_c$ with $n > 2$ $= 0$.

Remark 5: On periodic boundary 

Equilibrium

$$\frac{\ln\langle e^{\lambda Q_T}\rangle}{T} = a(\lambda) = \frac{\sigma(\rho)}{L}\cdot\frac{\lambda^2}{2} + \frac{D}{L^2}\cdot\mathcal{F}\left(\frac{\sigma\sigma'}{16D^2}\cdot(\lambda\sigma)^2\right)$$

$$\downarrow$$

↗ Bernoulli number

$$\mathcal{F}(x) = \sum_{K\geq 2}\frac{B_{2K-2}}{\Gamma(K)\Gamma(K+1)}(-2x)^K$$

$$\left[\ B_0 = 1,\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30}\ \right]$$

...

Cumulants

$$\frac{\langle Q^2\rangle}{T} \sim \frac{1}{L}\ ;\quad \frac{\langle Q^4\rangle}{T} \sim L^2$$

$$\frac{\langle Q^6\rangle}{T} \sim L^4$$

* More fluctuations

* Can have traveling wave

[* ~~Fl~~ Fluctuations depend on Boundary condition

Non surprising even in equilibrium

 $\Phi[\gamma] = \int dx\,\{f(\gamma) - f(\rho) - f'(\rho)(\gamma-\rho)\}$

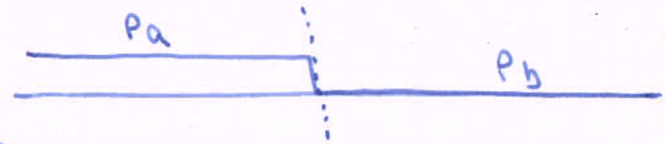
 $\Phi[\gamma] = \int dx\,\{f(\gamma) - f(\rho)\}$

Refs
① Appert-Rolland, Derrida, Lecomte, van Wijland
PRE 78, 021122 (2008)

② Lecomte, Imparato, van Wijland
Prog. Theo. Phy. Suppl. 184, 276 (2010)

Tutorial : ~~Verify the result~~

Remark 6: On an infinite line



$$P\left(\frac{Q_T}{\sqrt{T}} = j\right) \asymp e^{-\sqrt{T}\,\phi(j)}$$

* A different regime !!

Equivalently

$$\ln \langle e^{\lambda Q_T} \rangle \sim \sqrt{T} \cdot \mu(\lambda)$$

$$\Rightarrow \boxed{\langle Q^n \rangle_c \sim \sqrt{T}}$$

Will do an explicit calculation

Briefly discuss joint distribution

(Joint distribution of current and density)

* The density profile ~~is~~ sensitive to where one measures current.

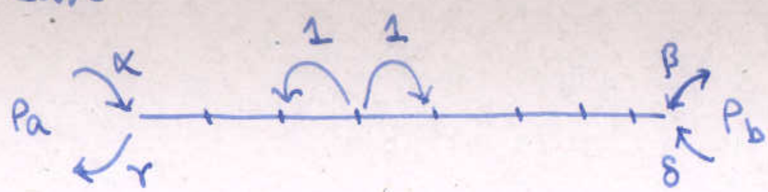
$$\cancel{\frac{1}{T}\left[\int_0^L dx\, A(x)\, Q_T(x)\right] \simeq \frac{q_0}{L}\left[\int_0^L dx\, A(x)\right]}$$

~~Parameter function~~

~~Joint probability~~ Joint Probability

$$P[q_0, n(x)] = ?$$

Result: A solvable case



$$\sqcup \quad \alpha = \gamma \rho_a \Rightarrow \rho_a = \frac{\alpha}{\gamma}$$

$$\log \left\langle e^{\sum_{i=1}^{L-1} A_i Q_i(T) + \sum_{i=1}^{L} h_i n_i(T)} \right\rangle$$

$$\xrightarrow[\substack{\text{Large } T \gg 1 \\ L \gg}]{} T\, \mu(A_1, \cdots A_{L-1}) + \mathcal{X}(A_1 \cdots A_{L-1}; h_1 \cdots h_L)$$

where $$\mu = \frac{1}{L}\left[\rho_a\left(e^{\sum_i A_i} - 1\right) + \rho_b\left(e^{-\sum_i A_i} - 1\right)\right]$$

$$\mathcal{X} = \sum_i \left(e^{h_i} - 1\right)\left[\rho_a\left(1 - \frac{i}{L+1}\right) e^{\sum_{k=1}^{i-1} \lambda_k} + \rho_b \frac{i}{L+1} e^{-\sum_{k=i}^{L-1} \lambda_k}\right]$$

<u>Hydrodynamic limit:</u>

$$i \to x L$$
$$T \to t L^2$$
$$n_i(T) \to \rho(x,t)$$
$$Q_i(T) \to L \cdot \int_0^{\tau} dt\, \dot{J}(x,t)$$
$$A_i \to \alpha(x)/L$$
$$h_i \to h(x)$$

$$\mu \to \frac{1}{L} G\left(\int_0^1 dx\, \alpha(x)\right) = \frac{1}{L}\left[\rho_a\left(e^{\int_0^1 dx\, \alpha(x)} - 1\right) + \rho_b\left(e^{-\int_0^1 dx\, \alpha(x)} - 1\right)\right]$$

$$\chi \to L \int_0^1 dx \cdot \left(e^{h(x)} - 1\right)\left[\rho_a (1-x)\, e^{\int_0^x dy\, \alpha(y)} + \rho_b \cdot x \cdot e^{-\int_x^1 dy\, \alpha(y)}\right]$$
$$= L\, \Psi[h(x), \alpha(x)]$$

$$\Rightarrow \ln\left\langle e^{\sum A_i Q_i + \sum h_i n_i}\right\rangle \longrightarrow \ln\left\langle e^{L \int_0^1 dx\, \alpha(x) \int_0^{\tau} \dot{J}(x,t)\, dt + L \int dx\, h(x) \rho(x)}\right\rangle$$
$$= e^{\tau \cdot L\, G\left(\int dx\, \alpha(x)\right) + L\, \Psi(h, \alpha)}$$

<u>Remark</u>: for $h=0$, one gets back ~~our~~ additivity result

for $\alpha = 0$, one gets back density LDF result.

<u>Remark</u>: Taking Legendre transform

$$\frac{\frac{1}{\tau}\int_0^1 dx\, \alpha(x) \int_0^\tau dt\, j(x,t)}{\int_0^1 dx\, \alpha(x)} = q_0$$

$$\boxed{P[q_0, \gamma(x)] \asymp e^{-\tau \cdot L \cdot \phi(q_0) - L \cdot \Psi[\gamma(x), q_0, \alpha(x)]}}$$

~~Conditionally~~

Conditional probability

$$P[q_0, \gamma(x)] = \mathbb{P}[\gamma(x) | q_0]\, \mathbb{P}(q_0)$$

$$\hookrightarrow e^{-\tau L \phi(q_0)}$$

$$\Rightarrow \mathbb{P}[\gamma(x)|q_0] \asymp e^{-L\Psi[\gamma(x), q_0, \alpha(x)]}$$

Large-deviation function condition on an emperical measure!

<u>Remark</u>: one can write H-J eq^n for $\Psi[\ ]$

$$\int_0^1 dx \left\{ \frac{\sigma(\gamma)}{2}\left(\partial_x \frac{\delta\Psi}{\delta\gamma} + \frac{\alpha(x)}{\int dx\, \alpha} \cdot \phi'(q_0)\right)^2 - D(\gamma) \cdot \partial_x \gamma \cdot \left(\partial_x \frac{\delta\Psi}{\delta\gamma} + \frac{\alpha(x)}{\int dx\, \alpha} \cdot \phi'(q_0)\right)\right\}$$

$$= q_0 \cdot \phi'(q_0) - \phi(q_0)$$

* Verify that NI satisties
* Verify at $\phi(q_0)=0$ one gets back H-J equation for density.