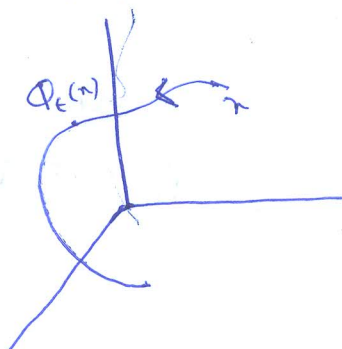


Classical System
(N particles obeying Hamiltonian Dynamics)

Microstates \rightarrow sets $\{ \vec{r}_i, \vec{p}_i \}_{i=1 \text{ to } N}$

dN dimensional phase space. Microstate x , point in space

$\Phi_t(x)$ - trajectory. $\Phi_0(x) = x$



~~Ergodicity~~ Some function (property) f
 $f^+(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\Phi_t(x)) dt$ - infinite time average

Ensemble average

$$\langle f \rangle = \int_{\Omega} f(x) dx / \int_{\Omega} dx ; \Omega = \text{constant energy shell}$$

Ergodic Hypothesis

$f^+(x) = \langle f \rangle$ \forall integrable f and "almost all" x .

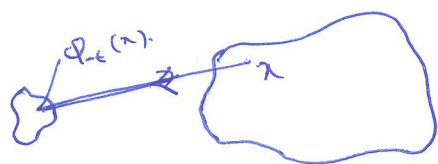
$\int_M dx = 0$, $\forall x \in M$ for which the Hypothesis is not true.

Ergodic Hypothesis: Amount of time spent in a region \propto volume of region

Exercise: 1D Harmonic Oscillator $H = \frac{\omega}{2} (p^2 + q^2)$ is ergodic.

Liouville's theorem: Phase space density

$$p_t(x) = p_0[\Phi_{-t}(x)] \quad \forall t \text{ and } \forall x \in \Omega.$$



Why are f^* and $\langle f \rangle$ useful? $\langle f \rangle$ can be calculated easily

$f^* \rightarrow$ time average. Measurements take finite time

$$f_{\text{obs}}^{(x)} = \frac{1}{\tau} \int_0^{\tau} f(\Phi_t(x)) dt \quad \text{Is } f_{\text{obs}}^{(x)} = f^*(x)? \quad \tau \neq \infty$$

How large should τ be to effectively be ∞ , i.e.

How much time does x take to sample all Ω ?

(Poincaré recurrence time).

N particles $\tau \sim e^N \tau_0$. τ_0 - microscopic time

$N = 10^{23} \Rightarrow \tau \sim e^{10^{23}} \tau_0 \gg$ age of the universe

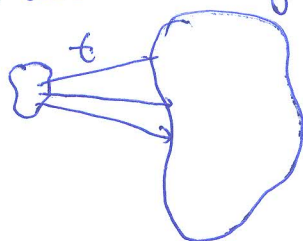
for any reasonable τ_0

We need $\lim_{t \rightarrow \infty} f[\Phi_t(x)] \rightarrow \langle f \rangle \quad \forall$ almost all x .

How can we justify this?

Approach 1: Gibbsian or Ensemble approach

Start with density $p_0(x)$ and not single point



and say " $f[\Phi_t(x)] \rightarrow f^{\text{mc}}$ " where $f^{\text{mc}} = \frac{1}{\int_{\Omega} dx}$

Cannot happen $f^{mc}(x)$ is ~~not~~ "stationary"

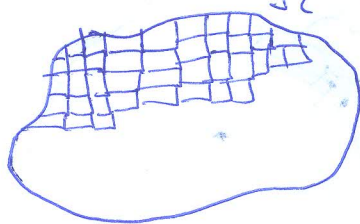
(3)

$\frac{\partial f^{mc}}{\partial t} = 0$, ~~nothing~~ no initial distribution can tend to it.

Solution, say that

~~$f_t = \int_{\Omega} f(x) P_t(x) dx$~~ $f_t = \int_{\Omega} f(x) P_t^{mc}(x) dx \rightarrow \int_{\Omega} f(x) P^{mc}(x) dx$

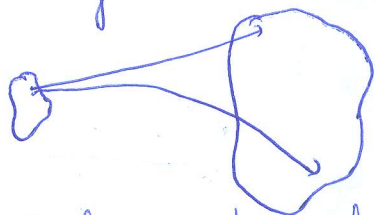
only for $f(x)$ defined over regions larger than some coarse-grained cell. $\Rightarrow \int_{\Omega} f(x) P(x) dx = \sum_i f(x_i) P(x_i)$



Is ergodicity + coarse graining sufficient?

No: Eg 1D Harmonic Oscillator

Another ingredient: Chaos



Exponential separation of ~~big~~ trajectories

Ergodicity + chaos + coarse graining $\Rightarrow f_t \rightarrow \int_{\Omega} f(x) P^{mc}(x) dx$
for $P_0(x)$ is an ensemble.

Approach 2: Boltzmannian or individualist approach

$P_0(x) = \delta(x - x_0)$. Single microstate, not an ensemble.

$P_t(x) = \delta[x - \Phi_t(x_0)]$, Also single microstate.

No sense in which $P_t(x) \rightarrow P^{mc}$.

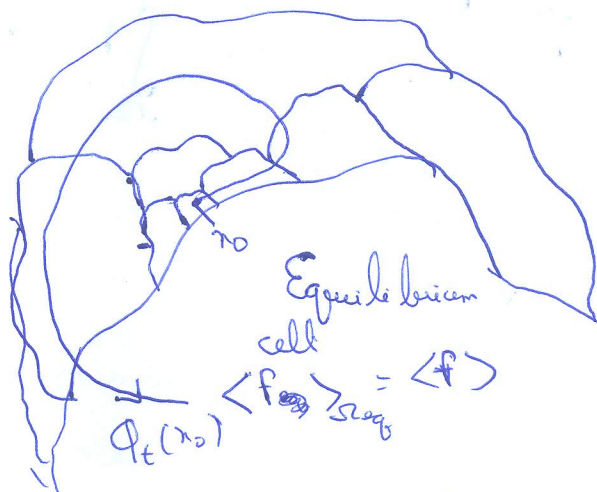
⊕

How about

$$\int_{\Omega} f(x) P_t(x) dx \rightarrow \int f(x) P^{mc} dx ?$$

$$P_t(x) = \delta[x - \Phi_t(x_0)] \Rightarrow f(\Phi_t(x_0)) \rightarrow \int f(x) P^{mc} dx \quad \forall \text{ for most } x_0$$

Typicality

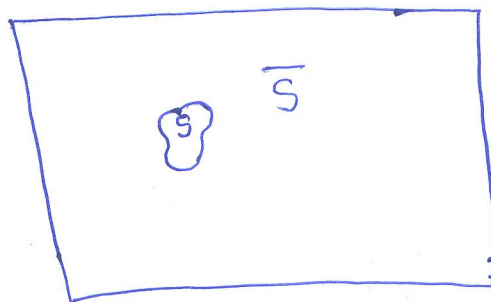


f - "Macroscopic" observables $\Omega_{eq} \approx \Omega$

Macroscopic, involving a large # of degrees of freedom

Are there any differences ~~between the~~ in predictions from the two approaches?

Yes. Look at subsystems.



$S \cup \bar{S}$ - isolated

~~⊗~~

$$P_S(x') = \int_{\Omega} f(x'; x) P(x) dx \quad P(x) \text{ is } P_{SUS}(x)$$

Subsystem - subset of the total set of degrees of freedom.

$$f(x'; x) = \prod_{i=1}^M \delta(x_i - x'_i)$$

$$P_S(x') = \int \prod_{i=1}^M \delta(x_i - x'_i) P(x) dx = \int dx_1 dx_2 \dots dx_N P(x_1, x_2, \dots, x_N)$$

Integrate out the "bath" remaining degrees of freedom.

~~Spd~~ Subsystem in ^{real} space.

~~$f(x'; x) = \prod_{i=1}^M \delta(\vec{p}_i - \vec{p}'_i)$ with $\vec{p}'_i \in V$ in space $\forall i \in M$ and $\vec{p}_i \in V \forall i \in M$~~

~~$f(x'; x)$ in general some product of δ fns.~~

Difference between Gibbs and Boltzmann:

Gibbs: $P_S(x')$ in general a proper density $\forall t$ and x'

Boltzmann: $P_S(x')$ always a δ function ($\forall t$ and x')

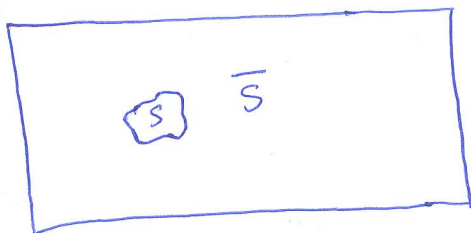
Boltzmann - subsystems do not thermalize (no distribution)

~~Subsystem~~ Subsystem thermalization - Microscopic Thermalization (MITE)

Macroscopic observable thermalization - Macroscopic Thermalization (MATE)

MITE (Classical Ensemble)

6



We can calculate P_S in the limit $\bar{S} \rightarrow \infty$ with $P = P^{mc}$

P_S equivalent to a probability distribution of ~~is~~

$$P_i \propto e^{-\beta(E - E_S^i)}$$

i - labels microstate of $S \Rightarrow$ bath has energy E_S^i

E - energy of $S \cup \bar{S}$ $\beta = \frac{1}{k_B T} = \left. \frac{\partial \ln \Omega}{\partial E} \right|_{S \cup \bar{S}}$

Familiar case (textbook canonical ensemble)

$$E = E_S^i + E_S^i \quad P_i \propto e^{-\beta E_S^i}$$

Other conserved quantities

$$P_i \propto e^{-\beta(E_S^i - \alpha N_S^i - \dots)} \text{ etc.}$$

$$\int f(x; \alpha) P(x) dx \equiv T_{\alpha \bar{S}} P(x) = T_{\alpha \bar{S}} P$$

*

Definition of thermalization of subsystems S

$$\lim_{\substack{t \rightarrow \infty \\ \bar{S} \rightarrow \infty}} P_S(t) \Rightarrow T_{\alpha \bar{S}} P^{mc}$$

$$P_S(t) = T_{\alpha \bar{S}} P(t) = T_{\alpha \bar{S}} P_t(x) \text{ for almost all } t$$

Thermalization $\nexists \equiv$ MITE

Individual isolated classical systems do not thermalize

MITE => MATE

Ergodicity + chaos => MITE => MATE

MITE ^{??} => Ergodicity + chaos

Main difference between Classical & Quantum Systems
Individual isolated quantum systems thermalize.

Quantum Stat. Mech.

$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle$ - Schrödinger

Dynamical equation for $|\psi\rangle$, instead of $\{p_i, q_i\}$ for a classical system.

Phase space = Hilbert space

H - time independent

$H|\psi_i\rangle = E_i|\psi_i\rangle$ - E_i - energy eigenvalues

Phase space density on Hilbert space

~~$|\psi\rangle = \sum_n c_n(\psi) |n\rangle$~~ $|\psi\rangle = \sum_n c_n(\psi) |n\rangle$

$\sum_n |c_n(\psi)|^2 = 1$ $\forall \psi$ normalization

Hermitian Operator A, expectation value

$\langle \psi | A | \psi \rangle = \sum_{nm} c_n^*(\psi) c_m(\psi) A_{nm}$

space average

$$\langle A \rangle = \frac{\int \sum_{nm} c_n^* c_m A_{nm}}{\int \sum_{nm} c_n^* c_m}$$

$$\langle A \rangle = \frac{\int p(\xi, c, c^*) \sum_{nm} c_n^* c_m A_{nm} dc dc^*}{\int p(\xi, c, c^*) dc dc^*}$$

$\sum_{nm} |c_n|^2 = 1$, surface of $2N$ dimensional sphere of unit radius

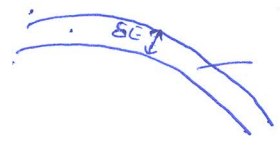
$$\langle A \rangle = \sum_{nm} \overline{c_n^* c_m} A_{nm} \quad \text{(\dots) average over filled space}$$

$p(\xi, c, c^*)$ uniform $\Rightarrow \overline{c_n^* c_m} = \frac{1}{N} \delta_{nm}$

$$\langle A \rangle = \frac{\text{Tr} A}{N} = \text{Tr}(PA); \text{ where } P_{nm} = \frac{\delta_{nm}}{N}$$

density matrix; $\text{Tr}(P) = 1$.

Consider a constant energy shell of width δE



Quantum system - discrete energies, ~~no~~ at constant energy shell may not have a large # of states

i.e., $\langle \psi | H | \psi \rangle \in [E, E + \delta)$



Hilbert space \mathcal{D} is D_E dimensional

$$P^{mc} = \frac{1}{D_E} \mathbb{1}, \quad \mathbb{1} \text{ identity operator.}$$

$$\langle A \rangle_{mc} = \text{Tr}(P^{mc} A)$$

$P^{mc} = \frac{1}{D_E} \mathbb{1}$ is the equivalent of $P^{mc} = \frac{1}{\Omega}$ classically.

What about $P(t)$? P is any general matrix of

the form $P = \sum_{\alpha} P_{\alpha} |\psi_{\alpha}\rangle \langle \psi_{\alpha}|$ $\sum_{\alpha} P_{\alpha} = 1; 0 \leq P_{\alpha} \leq 1$
 $\text{Tr}(P) = 1$

$|\psi_{\alpha}\rangle$ distinct but not necessarily orthogonal states

$$\frac{\partial P}{\partial t} = i\hbar [H, P] - \text{Quantum Liouville equation}$$
$$P(t) = e^{-iHt/\hbar} P(0) e^{iHt/\hbar}$$

P equivalent of $P(x)$.

$\text{Tr}(PA)$ is the equivalent of $\int f(x) P(x) dx$

Pure state $P = |\psi\rangle \langle \psi|$, equivalent of $P(x) = \delta(x-x')$

$P(t) = |\psi(t)\rangle \langle \psi(t)|$, equivalent of $P_t(x) = \delta[x - \psi_t(x)]$

$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi\rangle$ equivalent of $\psi_t(x)$

Subsystems



$|\psi\rangle = \sum_{ij} c_{ij} |\varphi_i^S\rangle |\varphi_j^B\rangle$, where $\{|\varphi_i^S\rangle\}$ & $\{|\varphi_j^B\rangle\}$ are orthogonal states

$$\sum_{ij} |c_{ij}|^2 = 1$$

$$\rho_S = T_{AB} \rho$$

$T_{AB} \rho$ is equivalent to $\int_{\Omega} f(x', x) \rho(x) dx$

Entropy of subsystem

$$S = -T_{AS} (\rho_S \ln \rho_S)$$

For $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi| = \sum_{ijkl} c_{ij} c_{kl}^* |\phi_i^S\rangle |\phi_j^B\rangle \langle\phi_k^S| \langle\phi_l^B|$

$$\rho_S = T_{AB} \rho = \sum_P \langle\phi_P^B| \rho |\phi_P^B\rangle = \sum_{iKP} c_{iP} c_{KP}^* |\phi_i^S\rangle \langle\phi_K^S|$$

which is not of the form for a pure state.

ρ_S - ensemble even when ρ is a pure state.

MITE possible in principle for pure quantum state !!

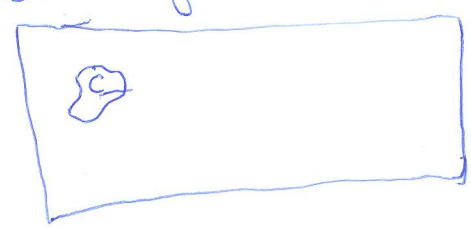
Class for a quantum system

$$|\psi(t)\rangle = e^{-i\frac{Ht}{\hbar}} |\psi(0)\rangle; \langle\psi(t)| = e^{i\frac{Ht}{\hbar}} \langle\psi(0)|$$

$$\Rightarrow \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\psi(0)\rangle$$

Distances and angles preserved. No divergence of trajectories, no chaos!!

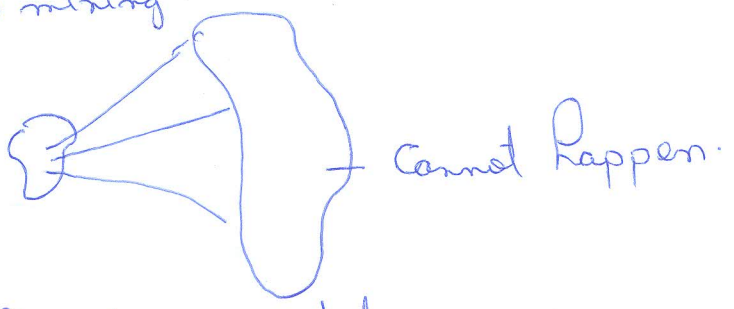
Start system out in some cell of Hilbert space



$$\rho(0) = \sum_{ij} p_{ij} |i\rangle\langle j|, \text{ with } |i\rangle, |j\rangle \in \text{Span}(C)$$

$$\Rightarrow \rho(t) = \sum_{ij} p_{ij}(t) |i\rangle\langle j| \text{ with } |i\rangle, |j\rangle \in \text{Span}(C)$$

No mixing



So, how do we justify

$$\lim_{t \rightarrow \infty} \text{Tr} (P(t) A) = \text{Tr} (P^{mc} A) \text{ after coarse graining?}$$

How can we obtain chaos? What is quantum chaos?

QUANTUM CHAOS

Go back to classical mechanics

~~Integrable~~ Integrable \Rightarrow No chaos

N degrees of freedom (N position coordinates)

$$\text{Integrable} \iff \exists I_i : \{ I_i, H \} = 0 \text{ and } \{ I_i, I_j \} = 0$$

The I_j 's are functionally independent and at least N in number.

$$\Rightarrow (p, q) \rightarrow (I, \theta) \text{ action-angle variables}$$

$$H(p, q) = H(I) \text{ and } \theta_j = \frac{\partial H}{\partial I_j} = 0 \text{ Motion trivial}$$

N-dimensional torus (invariant tori), no chaos

Generally Integrable \iff Non-chaotic

Carry idea over to quantum mechanics

$$\{ \hat{I} \} \rightarrow [\]$$

If $\exists I_i : [I_i, H] = 0$ and $[I_i, I_j] = 0$; functionally independent and "at least N in number".
 # of degrees of freedom not trivial for a quantum system.
~~Confused~~ Counting system not chaotic.

(17)

Integrable = Non-Chaotic

Integrable: ~~of~~ Simple examples

- 1) 1D ~~and~~ energy conserving systems
- 2) Hydrogen atom problem (superintegrable $2N-1$ I_j 's)
- 3) Central potential problems

Classical integrable systems - periodic ~~by~~ trajectories

Can be quantized $\oint p dq = nh$ to obtain quantum energies. What about eigenstates?

Non-integrable systems, no such quantization procedure available in general.

Integrable system

$$E = \sum_j f(n_j) \omega_j$$

~~j labels conserved~~
 j labels different, independent periodic trajectories

Concrete example: Independent harmonic oscillators

$$E = \sum_j n_j \omega_j$$

Large E , $E \in [E, E + \delta E]$ come from very different sets of $\{n_j\}$. E can be treated as random numbers with Poisson statistics. Probability on n levels in $[E, E + \delta E]$ is

$$P_n = \frac{\lambda^n}{n!} e^{-\lambda}, \quad \lambda, \text{ average \# of levels}$$

in the interval.

Probability $P(s)$ of successive levels separated by energy s (in units of mean level spacing)

$P(s) = e^{-s}$ - Poissonian level spacing distribution

Classically integrable + quantization \Rightarrow Poissonian statistics
(Berry - Tabor conjecture)

In general quantum integrable also believed to have Poissonian statistics.

Poissonian statistics \Rightarrow Lack of chaos

What about systems with chaos? ~~exp~~

Random Matrix Theory (RMT)

Wigner - Dyson. In energy window $[E, E + \delta]$ non-integrable Hamiltonian looks random in "generic" basis. Simplest example of random Hamiltonian $(R \times R)$

$$H = \begin{bmatrix} E_1 & \frac{V}{\sqrt{R}} \\ \frac{V^*}{\sqrt{R}} & E_2 \end{bmatrix}$$

E_1, E_2 & V are drawn from a Gaussian distribution of width σ . V assumed to be real.

$P(H) \propto e^{-\frac{(E_1^2 + E_2^2 + V^2)}{\sigma^2}} = e^{-\frac{\text{Tr}(H^2)}{\sigma^2}}$, the same for unitary from orthogonal transformations of H

$P(H) \propto e^{-\text{Tr}(H^2)/\sigma^2}$ - Gaussian Orthogonal Ensemble (GOE); H -real symmetric
 (Systems with time reversal symmetry)

Level spacing statistics

$\Delta E = \sqrt{(E_i - E_j)^2 + 2V^2}$

~~$P(s) = \frac{\pi s}{2} e^{-\frac{\pi s^2}{4}}$ - signature of GOE, to be contrasted~~

~~with $P(w)$ -~~
 $P(s) = \frac{\pi s}{2} e^{-\frac{\pi s^2}{4}}$ - signature of GOE.

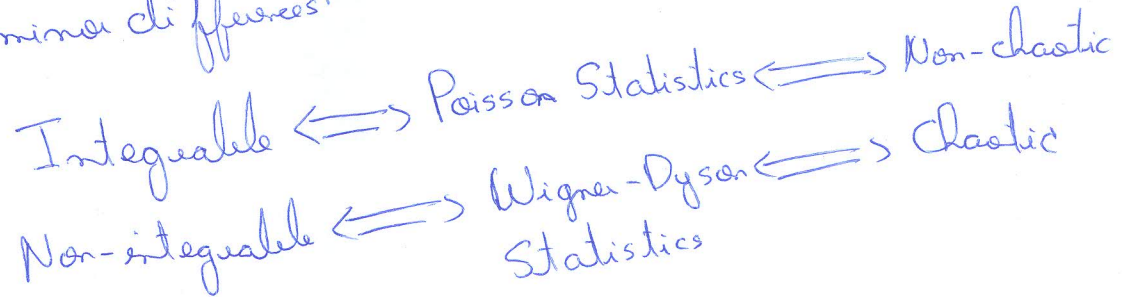
Systems without time reversal symmetry

$P(H) \propto e^{-\text{Tr}(H^2)/\sigma^2}$ - Gaussian Unitary Ensemble (GUE); H complex Hermitian

$P(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4s^2}{\pi}}$ - signature of GUE

~~Poissonian Statistics~~

Holds for $N \times N$ matrices as well but with minor differences.



Classical chaotic system \implies Quantum Counterpart has Wigner-Dyson Statistics (Bohigas, Giannoni and Schmit) BGS

Quantum Energy eigenstates of integrable models
=> periodic orbits in the classical limit.

Energy eigenstates of chaotic systems => no sensible classical limit.

RMT => energy eigenstates in a "generic" basis look random

Matrix elements of operators

Operator

$$O = \sum_i O_i |i\rangle\langle i|$$

Energy eigenstates $|m\rangle$ and $|n\rangle$

$$\langle m|O|n\rangle = O_{mn} = \sum_i O_i (\psi_i^m)^* \psi_i^n$$

$$\psi_i^m = \langle i|m\rangle$$

Average $\overline{O_{mn}}$ over random eigenstates gives

$$\overline{O_{mn}} = \sum_i O_i \overline{(\psi_i^m)^* \psi_i^n} = \left(\frac{1}{D} \sum_i O_i\right) \delta_{mn}$$

$$\therefore \overline{(\psi_i^m)^* \psi_j^n} = \frac{1}{D} \delta_{mn} \delta_{ij} \quad D\text{-dimension}$$

of Hilbert space

$$\begin{aligned} \text{Fluctuation in } O_{mn} &: \overline{(\Delta O_{mn})^2} = \overline{O_{mn}^2} - \overline{O_{mn}}^2 \\ &= \sum_{i,j} O_i O_j \overline{(\psi_i^m)^* \psi_i^m (\psi_j^m)^* \psi_j^m} - \left(\sum_{i,j} O_i O_j \overline{(\psi_i^m)^* \psi_i^m (\psi_j^m)^* \psi_j^m}\right) \\ &= \sum_i O_i^2 (\overline{|\psi_i^m|^4} - \overline{|\psi_i^m|^2}^2) \end{aligned}$$

$$\overline{(\Psi_i^m)^* \Psi_i^m (\Psi_j^m)^* \Psi_j^m} = \overline{(\Psi_i^m)^* \Psi_i^m} \overline{(\Psi_j^m)^* \Psi_j^m} \text{ when } i \neq j$$

$$\begin{aligned} (\Delta O_{mm})^2 &= \frac{2}{D^2} \bar{O}^2 \text{ for GOE} \\ &= \frac{1}{D^2} \bar{O}^2 \text{ for GUE} \end{aligned}$$

$$\bar{O}^2 = \frac{1}{D} \sum_i O_i^2$$

$$(\Delta O_{mm})^2 = \sum_i O_i^2 \overline{|\Psi_i^m|^2 |\Psi_i^m|^2} = \frac{1}{D} \bar{O}^2$$

$$O_{mm} \approx \bar{O} \delta_{mm} + \sqrt{\frac{\bar{O}^2}{D}} R_{mm}$$

R_{mm} - real GUE
- complex GUE

$$\langle R_{mm} \rangle = 0 \text{ and } \langle R_{mm}^2 \rangle = 1$$

Average over ensemble of Hamiltonians

$D \rightarrow \infty$ fluctuations small, so $O_{mm} \approx \bar{O} \delta_{mm} + \sqrt{\frac{\bar{O}^2}{D}} R_{mm}$

can also be used for a single Hamiltonian

Examples of random matrix systems (slide)

- 1) Nuclear levels
- 2) Single particle in a cavity

$$H = \sum_{j=1}^L \left[-J (f_j^+ f_{j+1} + h.c.) + V (n_j - 1/2) (n_{j+1} - 1/2) - J' (f_j^+ f_{j+2} + h.c.) + V' (n_j - 1/2) (n_{j+2} - 1/2) \right]$$

Many-body eigenstates. Are they truly random?

No, they have more structure.

Information (Shannon) entropy

$$S_m = - \sum_i |C_m^i|^2 \log |C_m^i|^2$$

$$|m\rangle = \sum_i C_m^i |i\rangle, \quad |m\rangle - \text{energy eigenstate}$$

$|i\rangle - \text{some fixed basis}$

$$\text{GOE} - S_{\text{GOE}} = \ln(0.48D) + O(1/D)$$

Show slide of $S_m / \ln(0.48D)$ vs E_m / L

States close to the tail seem to not look random. (Emergent symmetries??)

In the bulk of the spectrum, they look random.

Random Matrix Theory and Thermalization

MATE

$$O(t) = \langle \Psi(t) | O | \Psi(t) \rangle = \sum_{m,n} C_m^* C_n e^{i(E_m - E_n)t} O_{mn}$$
$$= \sum_m |C_m|^2 O_{mm} + \sum_{n \neq m} C_m^* C_n e^{i(E_m - E_n)t} O_{mn}$$

E_m, E_n in energy window of the RMT

$$O_{mn} \approx \bar{O} \delta_{mn} + \sqrt{\frac{\bar{O}^2}{D}} R_{mn}$$

$$\Rightarrow O(t) = \bar{O} + O\left(\frac{1}{\sqrt{D}}\right) \quad D \sim e^{-N}$$

$O(t) = \bar{O}$ - microcanonical

In RMT Microcanonical ensemble has no energy dependence.

Also

$$\sum_{m \neq n} C_m^* C_n e^{i(E_m - E_n)t/\hbar} O_{mn}$$

$$O_{mn} \sim \frac{1}{\sqrt{D}} \quad \sum_{m \neq n} \text{has } D \text{ terms} \quad \sum_{m \neq n} \text{random} \sim \sqrt{D}$$

Destructing phase coherence between a finite fraction may be enough to cause exponential decay.

Another way to see this.

Consider $\lim_{t_0 \rightarrow \infty} \frac{1}{t_0} \int_0^{t_0} [O(t) - \bar{O}]^2 dt$

$$= \lim_{t_0 \rightarrow \infty} \frac{1}{t_0} \int_0^{t_0} \sum_{m, n, p, q} O_{mn} O_{pq} C_m^* C_p^* C_n C_q e^{i(E_m + E_p - E_n - E_q)t} dt$$

$$= \sum_{n \neq m} |C_m|^2 |C_n|^2 |O_{mn}|^2 \leq \max |O_{mn}|^2 \sum_{n \neq m} |C_n|^2 |C_n|^2$$

$$= \max |O_{mn}|^2 \propto e^{-N}$$

We have assumed that $E_m + E_p = E_n + E_q$
 $E_m - E_n = -E_p + E_q \Rightarrow m = q \text{ and } n = p$
or $m = n \text{ and } p = q$

No resonances, non-degenerate energy

differences. So, $O(t)$ is close to \bar{O} and stays close for almost all times

$$\sigma_0^2 = \lim_{t_0 \rightarrow \infty} \frac{1}{t_0} \int_0^{t_0} [\langle \psi(t) | O | \psi(t) \rangle - \bar{O}]^2 dt$$

not to be confused with

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} \frac{1}{t_0} \int_0^{t_0} \langle \psi(t) | (O - \bar{O})^2 | \psi(t) \rangle dt &= \overline{\delta O^2} \\ &= \sum_m |c_m|^2 (O^2)_{mm} - \bar{O}^2 \approx \overline{O^2} - \bar{O}^2 = \delta O_{ME}^2 \end{aligned}$$

which need not be small.

~~RMT~~ RMT - Thermalization (We have shown MATE but also MITE)

Thermalization due to nature of energy eigenstates.

RMT $\Rightarrow \bar{O}$ independent of energy. Not true in real systems.

Energy dependence of \bar{O} - Eigenstate Thermalization Hypothesis (ETH) (MATE version)

$$\begin{aligned} \text{RMT: } O_{mn} &= \bar{O} \delta_{mn} + \sqrt{\frac{\overline{\delta O^2}}{D}} R_{mn} \\ \text{ETH: } O_{mn} &= \bar{O}(\bar{E}) \delta_{mn} + e^{-S(\bar{E})} f_0(\bar{E}, \omega) R_{mn} \end{aligned}$$

$\bar{E} \equiv \frac{E_m + E_n}{2}$; $\omega = E_n - E_m$; $S(E)$ is the thermodynamic entropy at energy E . $\bar{O}(E)$ and $f_0(E, \omega)$ are smooth functions of E and ω

$$\bar{O} = \lim_{t_0 \rightarrow -\infty} \frac{1}{t_0} \int_0^{t_0} dt O(t) = \sum_m |C_m|^2 O_m = \text{Tr}(P_{DE} O) \quad (20)$$

$$P_{DE} = \sum_m |C_m|^2 |m\rangle \langle m|$$

$$O = H \Rightarrow \bar{H} = E = \sum_m |C_m|^2 E_m$$

$$\delta E = \sqrt{\sum_m |C_m|^2 E_m^2 - \left(\sum_m |C_m|^2 E_m\right)^2} \text{ has to be small}$$

δE has to be subextensive, ~~rest~~ restriction on C_m 's. True for any conserved quantity M

δM has to be subextensive \Rightarrow No Schrödinger cat states.

~~Both are~~

$$\text{Let } \langle \Psi_I | H | \Psi_I \rangle = \langle E \rangle$$

$$O_{mm} = O(\langle E \rangle) + (E_m - \langle E \rangle) \frac{dO}{dE} \Big|_{\langle E \rangle}$$

$$+ \frac{1}{2} (E_m - \langle E \rangle)^2 \frac{d^2 O}{dE^2} \Big|_{\langle E \rangle} + \dots \quad \left[\begin{array}{l} O_{mm}(E) \\ = O(E_m) \\ \text{is a smooth} \\ \text{function of } E \end{array} \right]$$

$$\bar{O} = O(\langle E \rangle) + \frac{1}{2} \frac{d^2 O}{dE^2} \Big|_{\langle E \rangle} \sum_m |C_m|^2 (E_m - \langle E \rangle)^2 + \dots$$

If $|C_m|^2$ are very small outside $[\langle E \rangle, \langle E \rangle + \delta]$

$$O(\langle E \rangle) = O(\langle E \rangle) \sum_m |C_m|^2 \approx \sum_m |C_m|^2 O_{mm}$$

$$= O_{mc} \quad (\text{mc - microcanonical})$$

$$\bar{O} \approx O_{mc} + \frac{1}{R} \left[(\delta E)^2 - (\delta E_{mc})^2 \right] O''(\langle E \rangle)$$

δE_{mc} are subextensive and δE also subextensive

$\Rightarrow \bar{O} \approx O_{mc} + \text{subextensive}$

Only assume C_m is narrow. No need to assume smooth energy distributions like in classical stat. mech.

Narrowest $C_m = C_m = 1$ for $|m\rangle$,
0 otherwise

Single energy eigenstate.

$$\bar{O} = \langle m | O | m \rangle = O_{mc}$$

Single eigenstate ensemble.

Individual energy eigenstates are thermal!!

Any Restrictions on O .

O has to be local or the sum of local operators.

Operator O , that does not thermalize

$P_m = |m\rangle\langle m|$, projection operator onto eigenstate $|m\rangle$.

P_m typically highly non-local.

$$\sigma_O^R = \max |O_{mn}|^R \text{ (in ETH as for RMT.)}$$

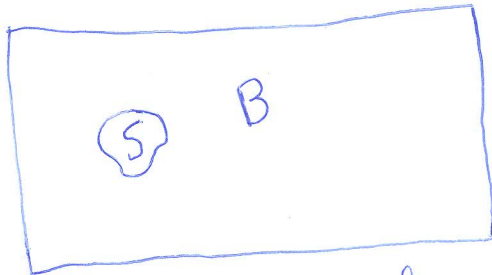
$\propto e^{-S(\bar{E})}$ - exponentially small.

~~ETH = MATE & MITE versions~~

~~MAT~~ Focus on energy eigenstates

Quantum systems & MITE
Canonical Typicality

(2)



$|\psi\rangle$ state of SUB
 $\mathcal{H} = \mathcal{H}_S \cup \mathcal{H}_B$

~~Claim~~ $H = H_S + H_B$ (weak coupling)

Claim: For randomly chosen $|\psi\rangle$ in the limit $B \rightarrow \infty$

$$P_S = \text{Tr}_B |\psi\rangle\langle\psi| = \frac{1}{Z} e^{-\beta H_S} \quad \beta = \frac{dS}{dE}$$

$$S = \lim_{N \rightarrow \infty} [S(E)/N]$$

$$e = \lim_{N \rightarrow \infty} E/N$$

$|\psi\rangle$ chosen randomly from energy interval $[E, E+\delta]$

$$P_S = \frac{1}{\dim(\mathcal{H}_{[E, E+\delta]})} \sum_i \dim(\mathcal{H}_i^B) |E_i^S\rangle\langle E_i^S| = \text{Tr}_B(P_{mc})$$

$$\mathcal{H}_i^B = \mathcal{H}^B [E - E_i^S, E - E_i^S + \delta]$$

$$P_{mc} = \frac{1}{\dim(\mathcal{H}_{[E, E+\delta]})} \sum_{E \in [E, E+\delta]} |E\rangle\langle E|$$

~~As~~ As $B \rightarrow \infty$ $\dim(\mathcal{H}_i^B) \rightarrow e^{-\beta E_i^S}$

Consider $\rho = |\psi\rangle\langle\psi|$

$\overline{|\psi\rangle} = 0$ and covariance = Identity on $\mathcal{H}_{[E, E+\delta]}$

$$|\psi\rangle = \frac{|\phi\rangle}{\|\phi\|}$$

$$|\phi\rangle = \sum_{ij} c_{ij} |E_i^S\rangle |E_j^B\rangle$$

A typical state \Rightarrow One with random c_{ij} $\langle c_{ij} \rangle = 0$

$$H_S |E_i^S\rangle = E_i^S |E_i^S\rangle$$

$$\langle |c_{ij}|^2 \rangle = 1$$

$$H_B |E_j^B\rangle = E_j^B |E_j^B\rangle$$

$$E_i^S + E_j^B \in [E, E + \delta]$$

$$|\Phi\rangle = \sum_i |E_i^S\rangle |\Phi_i\rangle$$

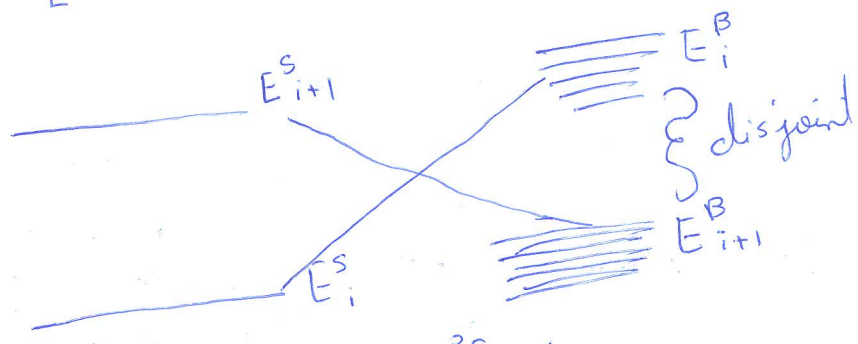
$$|\Phi_i\rangle = \sum_j c_{ij} |E_j^B\rangle$$

$$P^\Psi = T_{AB} |\Psi\rangle \langle \Psi| = \frac{1}{\|\Phi\|^2} \sum_{i,j,i'} \langle \Phi_i | \Phi_{i'} \rangle |E_i^S\rangle \langle E_{i'}^S|$$

Suppose $E_{i+1}^S - E_i^S > \delta \forall i$

$$E_B^i \in [E - E_i^S, E - E_i^S + \delta]$$

$$E_B^{i+1} \in [E - E_{i+1}^S, E - E_{i+1}^S + \delta] \text{ but } E - E_{i+1}^S + \delta < E - E_i^S$$



$$\Rightarrow \langle \Phi_i | \Phi_{i'} \rangle = \|\Phi_i\|^2 \delta_{ii'}$$

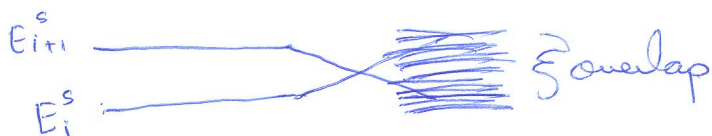
$$P^\Psi = \frac{1}{\|\Phi\|^2} \sum_i \|\Phi_i\|^2 |E_i^S\rangle \langle E_i^S|$$

$$= \frac{1}{\dim[\mathcal{H}_{[E, E+\delta]}]} \sum_i \dim(\mathcal{H}_i^B) |E_i^S\rangle \langle E_i^S|$$

$$\|\Phi\|^2 = \dim[\mathcal{H}_{[E, E+\delta]}]$$

$$\|\Phi_i\|^2 = \dim[\mathcal{H}_i^B]$$

If $E_{i+1}^S - E_i^S \not\gg \delta \forall i$



$\langle \Phi_i | \Phi_{i'} \rangle$ are still orthogonal when $i \neq i'$ because these are independent random vectors.

$$\frac{\langle \Phi_i | \Phi_{i'} \rangle}{\|\Phi\|^2} \rightarrow \frac{\dim(\mathcal{H}_i^B)}{\dim(\mathcal{H}_{[E, E+\delta]})} \delta_{ii'} \text{ as } B \rightarrow \infty$$

$$\langle \Phi_i | \Phi_{i'} \rangle \rightarrow \dim(\mathcal{H}_i^B) \delta_{ii'} \text{ as } B \rightarrow \infty$$

so, once again $\rho^\Psi = \rho^S$

Typical states are random, hence typical and hence thermal (in the sense of MITE). Also, holds for energy eigenstates if they are random.

Atypical State $|\Psi\rangle = |\Psi^S\rangle |\Psi^B\rangle$ - no entanglement.

Typicality \Rightarrow MITE due to entanglement.

MITE holds up to a length scale l_0 .

If l_0 is large $E_{i+1}^S - E_i^S > \delta$ ($\because E_{i+1}^S - E_i^S \sim e^{-L_S}$)

$\langle \Phi_i | \Phi_{i'} \rangle$ are not going to be orthogonal if $\dim(\mathcal{H}_i^B)$ is not large.

Maximum length scale for MITE = $l_0 \approx \frac{L}{\sqrt{2}}$ (crude estimate)

MITE \Rightarrow MATE

Macroscopic observables over a length scale $< l_0$ are thermal (MATE) if the system is in MITE.

Typicality (Entanglement) \Rightarrow MITE \Rightarrow MATE

for quantum systems

Analogous to

Ergodicity + Chaos \Rightarrow MITE \Rightarrow MATE
for classical systems

More on subsystems

$$P_s = T_A B P$$

Entanglement entropy = Thermal Entropy
 $S = -T_A (P_s \ln P_s)$ - Von-Neumann entropy

For a thermal system in a thermal state.

$$S \propto L_s^d \quad (\text{Volume law})$$

$$S = s(T) L_s^d \quad s(T) \text{ corresponding specific}$$

Entropy.

If $S \not\propto L_s^d$, state of the system is not thermal (i.e. it is not in MITE)

Strongest definition of thermalization is MITE.

A bath is something a system gets entangled with

Many-Body Localization (MBL)

MBL systems generally do not thermalize.
ETH fails (specifically MITE ETH fails)

Consider a state

$|\psi\rangle = |\uparrow \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow \uparrow \dots\rangle$ product state of random
up and down spins

~~$\langle S_j^z \rangle = 0$~~ $0 = \sum_j S_j^z$ for some ^{macroscopic # of} values of j

~~$\langle \psi | O | \psi \rangle = 0$~~

State in MITE ETH at $T = \infty$ for short ranged

Hamiltonian

But any single spin (small subsystem) in definite ^(pure) state
(Not $T = \infty$ state)

Not in MITE

MBL systems are those with energy eigenstates
that look like $|\psi\rangle$

Spins \rightarrow Spinless fermions in 1D

$|\uparrow\rangle = |1\rangle, |\downarrow\rangle = |0\rangle$

Example

$H = f(\sum_j n_j \epsilon_j)$; where $n_j = c_j^\dagger c_j$

Specifically

$H = \sum_j \epsilon_j n_j + \sum_{j,k} V_{jk} n_j n_k + \sum_{j,k,l} W_{jkl} n_j n_k n_l + \dots$

~~H is not~~ H is an MBL system (All eigenstates are violate MITE ETH)

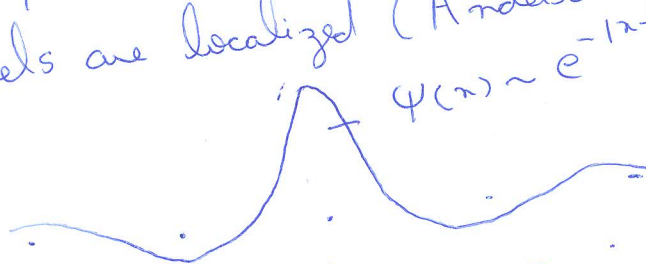
Not generic, no hopping

What happens with hopping?

Consider

$$H = -t \sum_j c_j^\dagger c_{j+1} + h.c. + \sum_j \epsilon_j n_j - \epsilon_j - \text{random (disorder)}$$

One particle Hamiltonian, all single particle energy levels are localized (Anderson localization)

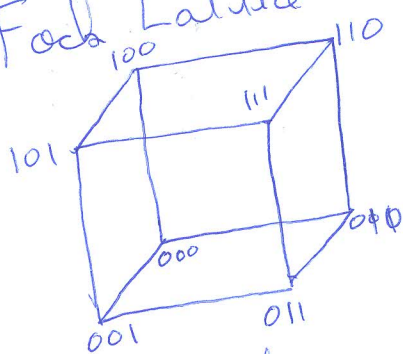


Perfect localization $\xi \rightarrow 0$

Many-body states;

Perfect localization - all eigenstates product states = 0 entanglement entropy

Fock Lattice



Perfect localization real space \Rightarrow Perfect localization in Fock space.

Anderson localization \Rightarrow Fock space Anderson localization. Many-body wavefunction falls off exponentially with Hamming distance.

Energy eigenstates not product states but weakly entangled. Entanglement entropy $\sim L^0$ and not L .

Eigenstates are not thermal.

For MBL systems, generically in eigenstates

$S \sim L^0$ (area law entanglement)

$H = -t \sum_j c_j^\dagger c_{j+1} + h.c. + \sum_j \epsilon_j n_j$ not generic enough.

No-interactions.

What happens with interactions?

$H = -t \sum_j c_j^\dagger c_{j+1} + h.c. + \sum_j \epsilon_j n_j + V \sum_j n_j n_{j+1}$

MBL persists perturbatively to all orders in V !!
(Basko, Aleiner & Altshuler)

Most (perhaps all) eigenstates violate ETH for $V < V_c$.

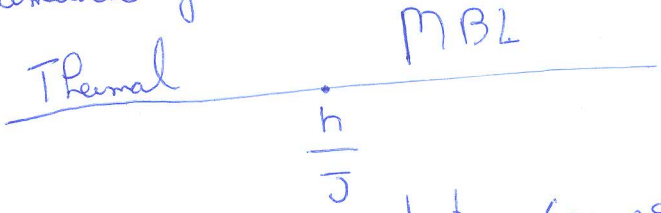
MBL \rightarrow Thermal transition at $V = V_c$.

Dynamical transition.

$V = 2t \quad H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \sum_i h_i S_i^z$

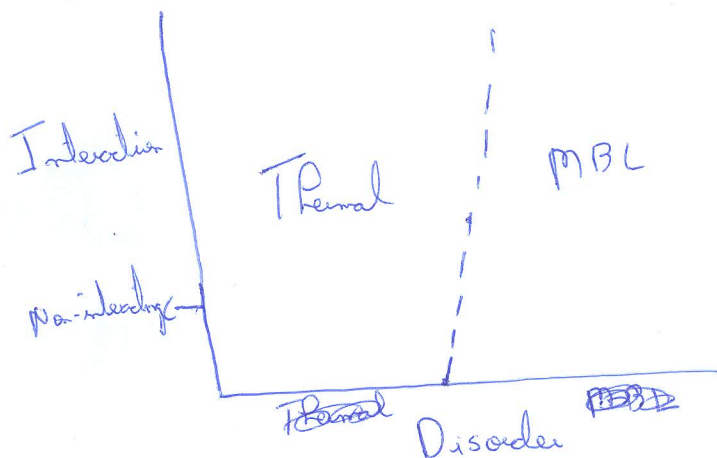
$h_i \in [-h, h]$ random

Numerically



Very high energy eigenstate (corresponding to $T = \infty$ of the system thermalized!!)

Generally (not in 1D)



Local integrals of motion

MBL occurs because the Hamiltonian in the MBL phase can be written as

$$H_{diag} = \sum_i h_i \tau_i^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \sum_{n=1}^{\infty} \sum_{i_1 \neq i_2 \neq \dots \neq i_n} K_{i_1 i_2 \dots i_n} \tau_{i_1}^z \tau_{i_2}^z \dots \tau_{i_n}^z$$

$$[\tau_i^z, \tau_j^z] = 0$$

$$J_{eff} = J_{12} + \sum_{n \neq k \neq j} K_{i_1 i_2 \dots i_n} \tau_{i_1}^z \tau_{i_2}^z \dots \tau_{i_n}^z$$

$$J_{eff}(r) \sim J_0 e^{-r/\xi}$$

$$H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \sum_i h_i S_i^z$$

Going from $\{\vec{S}_i\}$ to τ_i^z ~~Part 2~~

Any Hamiltonian in terms of $\{\vec{S}_i\}$ has 2^N eigenstates
 H_{diag} has 2^N eigenstates each a simultaneous eigenstate of all the τ^z . $2^N!$ ways of making this assignment.
 $\{\vec{S}_i\}$ - physical bits p-bits

We want τ_i^z to be local for the system to be MBL.

Q2

Small J local τ_i^z can be constructed

$\tau_i^z = \sigma_i^z + \text{"dressing"}$ $T \rightarrow$ local bits, l -bits
dressing $\rightarrow 0$ ~~exponent~~ exponentially decaying with distance.

H local interactions of the $\sigma^s \Rightarrow H_{diag}$ exponentially decaying.

$\tau_i^z = \sigma_i^z + \text{"dressing"}$

\Rightarrow $[\overset{entangled}{| \tau_1 \rangle | \tau_2 \rangle \dots | \tau_n \rangle} = \text{short ranged (area law)}$
entangled state in terms of $| \sigma_1 \rangle, | \sigma_2 \rangle \dots | \sigma_n \rangle$
and vice-versa]

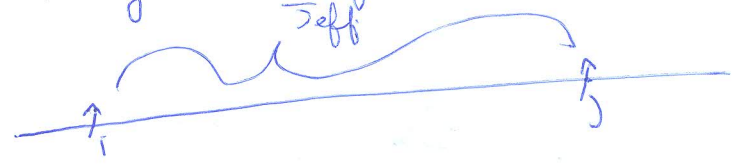
Eigenstates of H_{diag} are product states \Rightarrow
area law entanglement in terms of the $| \sigma \rangle$'s.
 \Rightarrow violation of MITE-ETH

Dynamics

No diffusion \Rightarrow Zero DC conductivity, perfect insulators at all temperatures!!

Spreading of entanglement. Start from initial product states $S \sim \log t$

Reason: $H_{diag} = \text{Sum of } J_{eff} \text{ Ising terms}$



Consider $H_{ij} = J_{ij} \tau_i^z \tau_j^z$ two spin Ising model

Initial state $|\uparrow\rangle_i |\uparrow\rangle_j$ - unentangled

$$= \frac{1}{2} (|\uparrow\rangle_i + |\downarrow\rangle_i) (|\uparrow\rangle_j + |\downarrow\rangle_j)$$

$$S(t) = -\text{Tr}(\rho(t) \ln \rho(t))$$

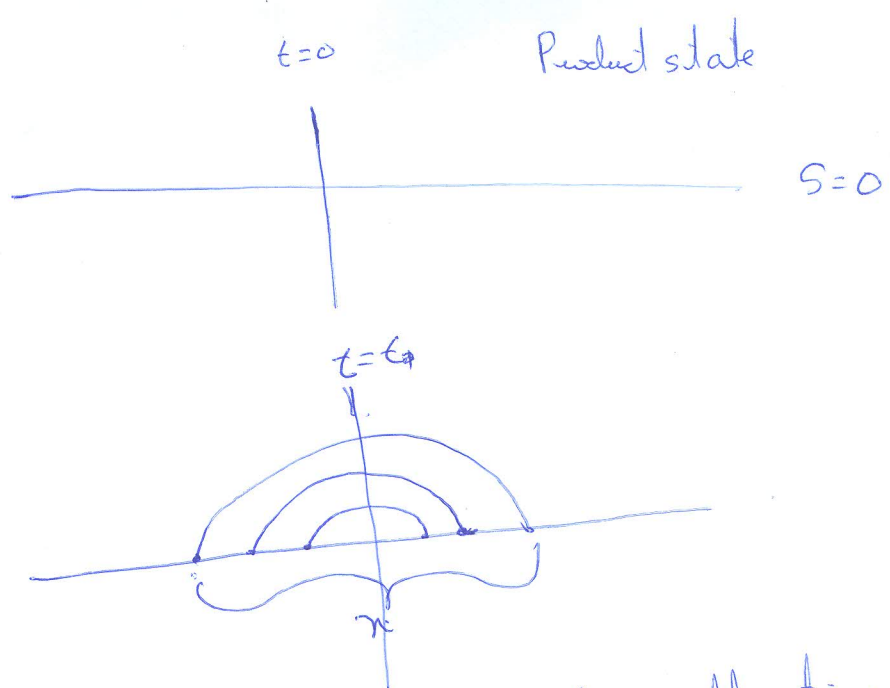
$$S(t) = \frac{1 + \cos \omega t}{2} \ln \left(\frac{1 + \cos \omega t}{2} \right) + \frac{1 - \cos \omega t}{2} \ln \left(\frac{1 - \cos \omega t}{2} \right) \quad \omega = 2 J_{ij}^{eff}$$

Entanglement generated ~~over a~~

Two ~~2~~-bits entangled only through direct ^{effective} interaction.

Time scale for ~~interaction~~ entanglement $t \sim \frac{1}{J_{ij}^{eff}}$

Many pairs entangled, so $S(t)$ is not periodic (incommensurate frequencies)



length scale of entanglement ξ after time t is given by $J_{\text{eff}}(\xi)t \approx 1 \Rightarrow \int_0^{\xi} e^{-\lambda|x|} dx \approx 1$

$\Rightarrow \xi \sim \log t$

Entanglement entropy $S \sim \xi$ (# of entangled pairs across the cut)

$\Rightarrow S(t) \sim \log t$

Thermal system: $J_{\text{eff}}(\xi)$ falls off much slower

$S \sim t^{1/z}$ (typically $z=1$)

Typically $z=1$; $S \sim t$ (Lieb-Robinson bound)

Integrable system $S \sim t$ (ballistic transport)

Systems with conformal invariance (John Cardy's talks)

Thermal

Single particle Localized

MBL

MITE-ETH

No MITE-ETH

No MITE-ETH

$$\sigma_{DC}(\omega=0) \neq 0$$

$$\sigma_{DC}(\omega=0) = 0$$

$$\sigma_{DC}(\omega=0) = 0$$

$$S \sim L$$

$$S \sim L^0$$

$$S \sim L^0$$

$$S(t) \sim t$$

$$S(t) \sim t^0$$

$$S(t) \sim \log t$$

Dissipation
& dephasing

No dissipation
No dephasing
(completely local
dynamics)

No dissipation
Slow dephasing