~~Two~~ LECTURES ON ACTIVE MATTER

EMBO Lecture Course (RRI/NCBS) Bangalore 2017

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Lecture I (a) fundamentals; (b) dry vs wet; (c) Toner-Tu

[Note: this is listed in the programme as Active Matter 3. However, each of my two lectures is self-contained.] EMBO School only

$\boxed{I(a)}$ Fundamentals

(i) Background:

~~I(a) Fundamentals~~ ~~(i)~~ Many books on "Nonequilibrium Statistical Mechanics" are about relaxation to thermal equilibrium (and the response to small perturbations), time correlations in thermal equilibrium, and the intimate relation between these. True nonequilibrium physics is about driven states, including time correlations and response in such states, and it is in this class that Active Matter lies. We treat it as a distinct category of driven system because it consists of constituents that are individually driven by the uptake of energy from an ambient or on-board source and, in the cases of greatest interest, move in a direction set by their own structural asymmetry, not an external field. Clearly living organisms are the pre-eminent example, but so are the autonomously motile components of cells, and so, as we shall see, are many artificial systems. Systems driven by forcing from the boundary (e.g. sheared fluids) or by external electric or gravitational fields are NOT active matter.

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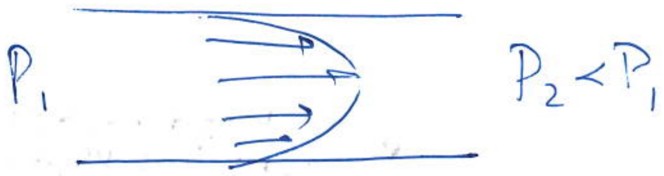
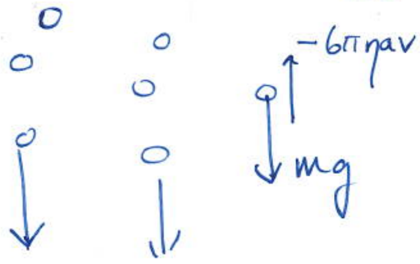


not active matter



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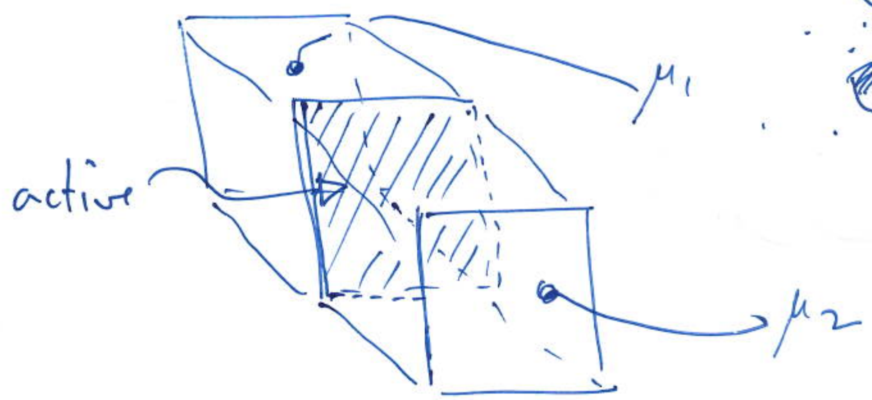
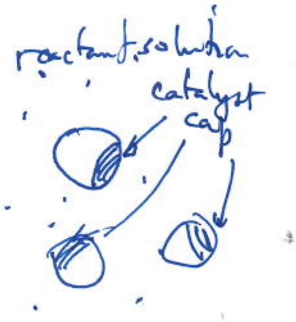
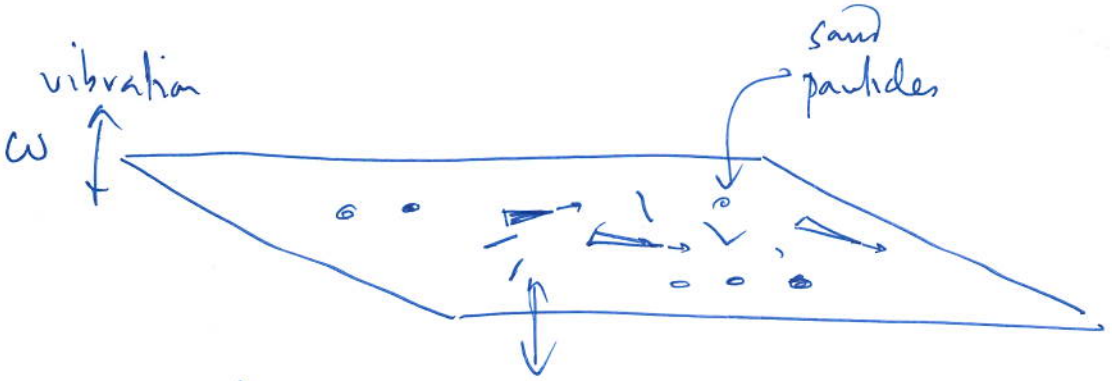




active



active

Examples of active & non-active driven systems.

The previous page shows sketches of non-active & active driven systems. The basic idea is that the effective dynamics of a system, driven in a direction (spatial or chemical) that is in a sense orthogonal to those one is monitoring, is active dynamics. Thus not only is a collection of bacteria with ample nutrient an example of an active system, so is a monolayer of particles lying in the xy plane and agitated in the z direction, and so is a suspension of catalyst-tipped beads in a fluid laden with a compound that the catalyst breaks down.

(ii) <u>Building the equations of motion</u> at <u>equilibrium</u>

1. Identify <u>slow variables</u> (ideally: only conserved, broken-symmetry, and critical); 2. write down all allowed terms in a first-order dynamics (allowed = not ruled out by symmetry), which is a finite task thanks to the gradient-expansion, working only to leading order; 3. include fast degrees of freedom as noise; 4. constrain equations by time-reversal * invariance (hence ~~detailed~~ detailed balance; no currents on average) $\Longrightarrow$ reversible and dissipative parts of dynamics arise ~~from~~ from a generalized force due to effective Hamiltonian ($\sim$ log of steady-state prob)

$\longrightarrow$

and noise covariance $\propto$ symmetric part of kinetic coeff.

(iii) <u>Dynamics in driven state?</u>

* Simply drop requirement # 4 above.
* Will find eq$^{\underline{ns}}$ of motion contain "new" terms, not allowed in equilibrium dynamics.

In practice, how do you "drop requirement # 4"?

<u>BACK TO EQUILIBRIUM DYNAMICS</u>

Write down dynamics of degrees of freedom of interest, (q, p) coupling to an extra degree of freedom, X (= no. of fuel molecules consumed = ~~Klds~~ displacement in the chemical direction). $t \to -t \Rightarrow X \to X$, $q \to q$, $p \to -p$. But assume this is already a coarse-grained description, so eq$^{\underline{n}}$ for q isn't purely kinematic, thus contains dissipative terms.

Let $H(q\,p\,X)$ be the effective Hamiltonian. Then

$$\boxed{\begin{aligned} &\dot{q} + \underset{\text{Relaxation}}{\gamma \partial_q H} = \underset{\text{reversible}}{\partial_p H} + \underset{\text{noise}}{\Theta} && (1a) \\ &\dot{p} + \Gamma_{11} \partial_p H + \Gamma_{12}(q) \dot{X} = -\partial_q H + \eta && (1b) \\ &\Gamma_{21}(q) \partial_p H + \Gamma_{22} \dot{X} = -\partial_X H + \xi && (1c) \end{aligned}}$$

<u>Note</u>: I've ignored inertia for X, and I've written the X-velocity directly as $\dot{X}$, not as derivative of H w.r.t. <u>momentum</u> of X. Also, I've included

an off diagonal damping matrix $[\Gamma_{11}, \Gamma_{12}(q) = \Gamma_{21}(q), \Gamma_{22}]$, ~~If~~ and allowed the off-diag. bits to depend on the coord. q.

I've included (Gaussian zero-mean white) noise sources θ, η, ξ. In that special steady state known as thermal equilibrium we know that

$$\boxed{\begin{aligned} &\langle \theta(0)\theta(t)\rangle = 2k_BT\gamma\,\delta(t), \quad \langle \eta(0)\eta(t)\rangle = 2k_BT\Gamma_{11}\,\delta(t) && 2(a)\\ &\langle \eta(0)\xi(t)\rangle = \langle \xi(0)\eta(t)\rangle = 2k_BT\,\Gamma_{12}(q)\,\delta(t) && (2b)\ \cancel{2(b)}\\ &\langle \xi(0)\xi(t)\rangle = 2k_BT\,\Gamma_{22}\,\delta(t). && (2c)\end{aligned}}$$

The q-dependence of the correlators (2b) does not raise problems of stochastic ambiguity ~~...~~ because we've retained inertia in (1b). Now, still in equilibrium, let me rewrite the dynamics (1), (2), re-expressing $\ddot{X}$ in (1b) in terms of the chemical force ($-\partial_x H$) from (1c). The result

$$\boxed{\begin{aligned} &\dot{q} + \gamma\,\partial_q H = \partial_p H + \theta && (3a)\\ &\dot{p} + \underbrace{\Gamma}_{\text{new damping}}\partial_p H - \frac{\Gamma_{12}(q)\,\partial_x H}{\Gamma_{22}} = -\partial_q H + f \rightarrow (\text{new noise}) && (3b)\\ &\dot{x} + \frac{\Gamma_{12}(q)}{\Gamma_{22}}\partial_p H = -\frac{1}{\Gamma_{22}}\partial_x H + \frac{\xi}{\Gamma_{22}} && (3c)\end{aligned}}$$

[Modified noise $f = \eta - \dfrac{\Gamma_{12}(q)}{\Gamma_{22}}\xi$; modified damping

$$\Gamma = \Gamma_{11} - \frac{\Gamma_{12}^2(q)}{\Gamma_{22}}. \qquad \langle f(0)f(t)\rangle = 2k_BT\,\Gamma\,\delta(t).$$

$\langle f(0)\xi(t)\rangle = 0$ (check for yourself).]

(6)

$\Gamma_{12}(q)$ started life as a dissipative coefficient in (1).

In (3) it enters as a reversible coupling, allowing us to define a Poisson-bracket $[X, p] = \dfrac{\Gamma_{12}(q)}{\Gamma_{22}}$.

Obviously (3) is as good a description of the equilibrium dynamics as (1), because we've just changed some labels.

In either case, the chemo-mechanical machinery enters through the coefficient $\Gamma_{12}(q)$.

NOW FOR ACTIVE DYNAMICS

It's simple. All you do is to hold the chemical force $-\dfrac{\partial H}{\partial X} \equiv +\Delta\mu$ constant, i.e., maintain a fixed chemical potential difference between reactants and products.

$\Delta\mu$ then enters as a new parameter in the equation for p:

$$\boxed{\begin{aligned} \dot{q} + \gamma\,\partial_q H &= \partial_p H + \theta \quad \text{(a)} \\ \dot{p} + \Gamma\,\partial_p H &= \frac{-\Delta\mu}{\Gamma_{22}}\,\Gamma_{12}(q) - \partial_q H + f \quad \text{(b)} \end{aligned}} \qquad (4)$$

The X equation $\left(\dot{X} + \dfrac{\Gamma_{12}(q)}{\Gamma_{22}}\,\partial_p H = +\dfrac{1}{\Gamma_{22}}\Delta\mu + \dfrac{\xi}{\Gamma_{22}}\right)$ (4′)

comes along for the ride, and must be included to compute total entropy production.

As long as X enters the Hamiltonian only in the form $-X\Delta\mu$, ~~so that~~ with $\Delta\mu$ constant, eqn (4) is self-contained, requiring no specification of X. It is a formally complete description of the fluctuating dynamics of q & p at constant $\Delta\mu$. As can be seen from (4'), X itself grows steadily (i.e. fuel molecules are continually consumed) if $\Delta\mu$ is kept at a fixed positive value.

NOTE: the description above is totally general: despite their appearance, the forms (1), (3) or (4) apply just as well if q and p are multicomponent variables, or fields, or both. Only a minor change of notation, replacing ordinary derivatives by variational derivatives etc, is required.

Now look at (4). The result of holding $\Delta\mu$ constant, i.e. maintaining a one-way bias on the chemical reaction, is to introduce a new q-dependent "force" in the equation (4b) for p. You can't absorb it into a redefinition of H, because that would alter (4a), the dynamics of q. Redefining $[q,p]$ doesn't help either.

→

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A further simplification is to ignore inertia altogether and solve 4b for p, setting $\dot{p}$ to 0. The result

$$\dot{q} + (\gamma + \Gamma^{-1})\partial_q H = \frac{-\Delta\mu}{\Gamma\Gamma_{22}}\Gamma_{12}(q) + \Theta + \Gamma^{-1}f \qquad (5)$$

is an effective eq^n of motion for q alone at fixed $\Delta\mu$.

For an ODE for a ~~simple~~ scalar variable q ~~this~~ 5 can always be viewed as dynamics in an effective energy function* and an effective temperature. But if q is multicomponent, or a field, or both, in general $\Gamma_{12}(q)$ won't be a gradient, and the effective Hamiltonian picture won't work.

We will see simple examples of this below.

* There's no guarantee that this effective energy is bounded below, so, e.g. in ratchet problems, the effective dynamics of q could be a steady state with $\langle\dot{q}\rangle \neq 0$, i.e. a nonequilibrium state.

$\longrightarrow$

NOTE: * Nothing in the forms (4) or (5) could not have been guessed by "pure thought" or "whatever is allowed by symmetry". But it's enlightening to see the connection to dynamics at equilibrium. (9)

* If $\Delta\mu >> k_BT$, can't be sure that the noise terms are just the ones that operate at thermal equilibrium. Noise itself could depend on $\Delta\mu$. Recall f in eq. (3b) or (4) combines η and ξ from (1). ξ could change a lot from its equil value if $\Delta\mu$ is large, i.e. X is strongly driven.

* In systems of macroscopic particles, e.g. vibrated granular matter, all the noise is from the driving

EXAMPLES OF ACTIVE TERMS

* If $q \longrightarrow q(\underline{x}, t)$ is the height field of an interface without $q \to -q$ symmetry, and the system is in a driven state, then $\Gamma_{12}(q) = \text{const} + (\nabla q)^2 \dots$

So (5) $\longrightarrow$ interface dynamics + KPZ nonlinearity.

* If local $q \to -q$ symmetry, on interface with height field q, is broken by a local field ψ (an up/down species) then $\Gamma_{12} \sim \psi\,[\text{const} + (\nabla q)^2] \to$ active membrane dynamics.

* If $q \to \underline{q}(\underline{x}, t)$ is a polar vector order parameter for a flock, then (5) will contain ~~Γ_{12} ~ polynomial~~ $\Gamma_{12}(\underline{q})$ ~ vector polynomial in $\underline{q}$ and ∇.

Terms polynomial in $\underline{q}$ can be absorbed into redefined H; Also terms with two ∇ and one $\underline{q}$; also $\underline{q}\,\nabla\cdot\underline{q}$ and $\nabla(\underline{q}\cdot\underline{q})$. But not $\underline{q}\cdot\nabla\underline{q}$, which gives a nonlinearity of "advective" type. This term is crucial to Toner & Tu's proof that flocks have long-range order in $d=2$.

~~* Let $\underline{q} = (q_1, q_2)$.~~

~~* $\underline{q} \to (u, w)$, both even in $t \to -t$.~~

~~* $\underline{q} = (u, w)$, both even in $t \to -t$.~~

~~(5) with for u, with $\Gamma_{12} = \Gamma_{12}(w)$, $\dot{w} + \frac{1}{\tau} w =$~~

* Let $\underline{q} = (u, w)$, both even under $t \to -t$.

Let u obey eq^n like (5), with $\Gamma_{12} \propto w$.

~~Let w " " " " $\Gamma_{12} = 0$~~

* $q \to (u, w)$, both even under $t \to -t$.

Let u obey eq^n like (5), with $\Gamma_{12} = \Gamma_{12}(w)$.

Let w obey eq^n like (5), with $\Gamma_{12} = 0$,

and let H contain a term $\frac{k}{2} w^2$, i.e. w is harmonically bound. Then $\frac{\Delta\mu}{\Gamma_{22}\Gamma} \Gamma_{12}(w)$ is an Ornstein-Uhlenbeck noise in an otherwise "equilibrium" dynamics ~~of~~ for u. This is like Active Ornstein-Uhlenbeck Particles (Fodor et al PRL 2016).

* $\underline{p} \Rightarrow$ momentum density of fluid.

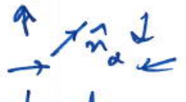
$\underline{\underline{q}} \to$ traceless symmetric orientation tensor

$\Rightarrow$ $\Gamma_{12}(q)$ in p eq^n will contain term

$\sim \nabla \cdot \underline{\underline{q}}$, i.e. active stress $\propto \Delta\mu\, \underline{\underline{q}}$,

equivalently, $[\underline{p}, \overset{\text{chemical coord}}{X}] \propto \nabla \cdot \underline{\underline{q}}$

ASIDE: we'll talk about orientationally ordered systems. For uniaxial achiral case there are two kinds: polar and apolar



p = local average of $\hat{n}_\alpha$

$\underline{\underline{Q}} = \langle \hat{n}_\alpha \hat{n}_\alpha - \frac{1}{d} \underline{\underline{I}} \rangle$

7b Wet vs dry dynamics

Particles in a fluid

coord. $\underline{R}_\alpha(t)$ — vel. field $\underline{u}(\underline{r}\ t)$

mom. $\underline{P}_\alpha(t)$ — density ρ

mass m

$$\dot{\underline{R}}_\alpha = \underline{P}_\alpha / M \qquad (6)$$

$$\cancel{\dot{\underline{P}}_\alpha} + \Gamma\left[\frac{\underline{P}_\alpha}{M} - \underline{u}(\underline{R}_\alpha(t), t)\right] = \cancel{(\text{forces})_\alpha}\ \underline{F}_\alpha(\underline{R}_\alpha)$$

forces

+ noise $\underline{f}_\alpha$

$$\rho\left(\cancel{\dot{\underline{u}} + \underline{u}\cdot\nabla\underline{u}}\right) = \eta \nabla^2 u + \sum_\alpha \delta(\underline{r} - \underline{R}_\alpha(t))\,\underline{F}_\alpha - \nabla P$$

+ noise

$$\nabla\cdot\underline{u} = 0$$

$$\Rightarrow \quad \dot{\underline{R}}_\alpha = \underline{u}(\underline{R}_\alpha(t), t) + \frac{1}{\Gamma}\underline{F}_\alpha \quad + \text{noise} \qquad (7a)\ \cancel{(1)}$$

$$\eta \nabla^2 \underline{u} = \nabla P - \sum_\alpha \delta(\underline{r} - \underline{R}_\alpha)\underline{F}_\alpha\ ; \quad \nabla\cdot\underline{u} = 0 \qquad \cancel{(2)}\ (7b)$$

+ noise

(WET) If $\underline{R}_\alpha$ are coords of monomer, ~~that~~'s ~~Rouse~~ dynamics. 1a ∧ 1b = Zimm "wet"

(DRY) Replace $\underline{u}$ by $\langle\underline{u}\rangle = 0$ in (7a)
Ignore (7b)

} get Rouse dynamics "dry".

In general soft-matter system, active or otherwise, damping provided by fluid medium, momentum given to fluid affects other particles of the system of interest.

~~But~~ That's "wet", "Zimm", "hydrodynamically interacting".

If replace ~~fluid~~ medium by passive momentum sink that's "dry", "Zimm", "hydrodynamically screened".

e.g. ~~for~~ simple ideal polymer chain of length L

has relaxation time $\sim \begin{cases} \Gamma L^2 & \text{Rouse} \\ \dfrac{\eta}{k_BT} \begin{cases} L^{d/2}, & d \leq 4 \\ L^2, & d > 4 \end{cases} & \text{Zimm} \end{cases}$

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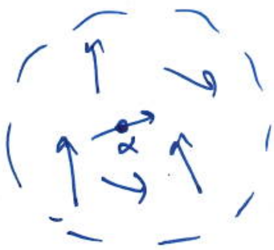
$\to L^{\nu d}, \; d \leq 4$

In active matter too, hydro interaction has dramatic qualitative effects $\longrightarrow$ already saw in JF's lectures.

(14) (Ic) Flocking models

Simplest "living liquid crystal"

Vicsek model $\longrightarrow$ Toner–Tu model of flocking



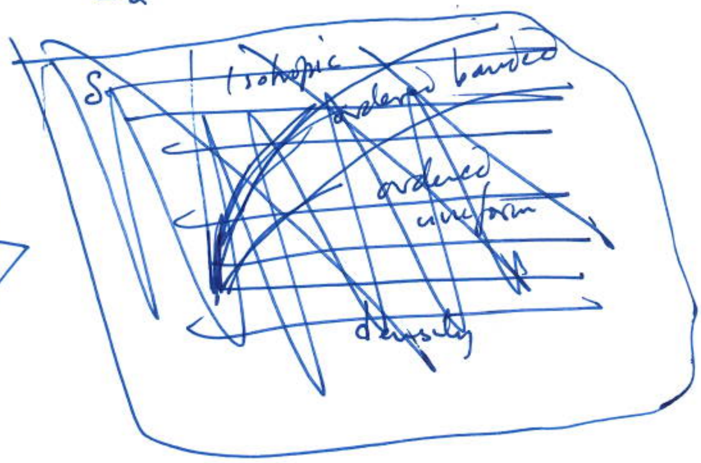
Particle $\alpha \mapsto (\underline{R}_\alpha, \hat{n}_\alpha)$

position orientation

$$\hat{n}_\alpha(t+1) = \underline{\underline{O}}_\delta \langle n_\beta(t) \rangle_{\beta \in \text{nbhd of } \alpha} \quad (9)$$

$\underline{\underline{O}}_\delta$ = rotation through angle drawn uniformly from $(-\delta, \delta)$

$$\underline{R}_\alpha(t+1) = \underline{R}_\alpha(t) + \varepsilon \hat{n}_\alpha(t+1)$$



One self-propelled particle:

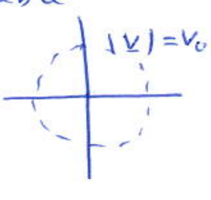
$$m\dot{\underline{v}} + \Gamma_0 \underline{v} = \underline{F}_{ext} + \underline{F}_{propulsion} + \text{noise}$$

$$\underline{F}_{propulsion} = \alpha \underline{v} - \beta \underline{v}\cdot\underline{v}\, \underline{v}$$

$\alpha > \Gamma_0 \Rightarrow \underline{v} = 0$ unstable

$$|\underline{v}| = \sqrt{\frac{\alpha - \Gamma_0}{\beta}} \text{ stable}$$

$\equiv v_0$



Toner - Tu model : Derive from Vicsek or

from collisional (Boltzmann) variant e.g. Bertin, Droz, Gregoire PRE 2006

$\rho = \sum_\alpha \delta(\underline{r} - \underline{R}_\alpha(t))$, $\rho \underline{p} = \sum_\alpha \hat{n}_\alpha \delta(\underline{r} - \underline{R}_\alpha)$

"advection"   local moment   "elastic" "viscous"   "pressure"

$$\partial_t \underline{p} + \lambda \underline{p} \cdot \nabla \underline{p} + \dots = (\alpha - \beta \underline{p} \cdot \underline{p}) \underline{p} + \Gamma \nabla \nabla \underline{p} - \nabla P(\rho, p^2) + \underset{\text{noise}}{\underline{f}} \qquad (10)$$

Coefficients calculable from micro model

Remark: can include diffusion

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \underline{p}) = 0} \qquad (11)$$

→ moving XY / Heisenberg model

Two equil limits

$\underline{p}$ is both orientation & velocity → "pumped" fluid on substrate

$$\boxed{\partial_t \underline{p} + \lambda \underline{p} \cdot \nabla \underline{p} = -\frac{\delta F}{\delta \underline{p}} + \text{noise}} \qquad (12)$$

$$F = \int_{\underline{r}} \left[+ \frac{\alpha}{2} \underline{p} \cdot \underline{p} + \frac{\beta}{4} (\underline{p} \cdot \underline{p})^2 + \frac{\Gamma}{2} \nabla p \nabla p + \underline{p} \cdot \nabla P \right] \qquad (13)$$

in general α, β.... dep on ρ (P depends on ρ and on p^2)

$\beta > 0$. α changes from neg to pos as noise ↓ or mean density ↑. $\alpha < 0 \Rightarrow$

(14) $\langle \underline{p} \rangle = p_0 \hat{z}$, $\underline{p} = \langle \underline{p} \rangle + \delta \underline{p} = \langle \underline{p} \rangle + (\delta \underline{p}_\perp, \delta p_{\parallel})$

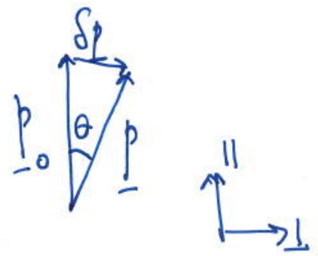
$\rho = \rho_0 + \delta \rho$

→

(16) To leading order in ∇, and for simplicity taking $\lambda = 1$, shift to ref. frame moving with vel $\underline{p}_0$

$$\partial_t \delta\rho + \rho_0 \nabla_\perp \cdot \delta \underline{p}_\perp = 0 \qquad (15a)$$

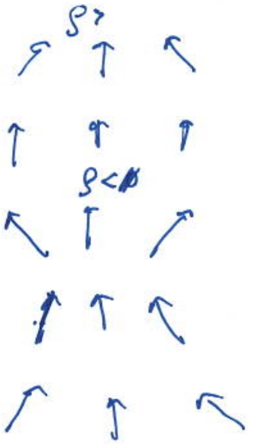
$$\partial_t \delta \underline{p}_\perp = - P'(\rho_0, \underline{p}_0) \nabla_\perp \delta\rho + O(\nabla^2) \qquad (15b)$$



$\delta \underline{p}_\perp \sim \theta$

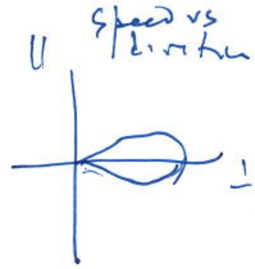
$\delta p_\parallel \sim \theta^2$ ignore

$\delta \underline{p} \approx (\delta \underline{p}_\perp, 0)$



Modes

$e^{i\underline{q}\cdot\underline{r} - i\omega t}$



Splay-density wave "Sound": $\boxed{\omega^2 = c_s^2 q_\perp^2}$ (16a)

$\boxed{c_s^2 = \rho_0 P'(\rho_0)}$ (16b) (fine if $P'(\rho_0) > 0$)

Exercise:
(i) Work this out in the Lab frame
(ii) Include next-to-leading order in ∇

Not like ordinary sound (which relies on momentum conservation)

This ~~waste~~ mode is propagative in the ordered phase only, because $\delta p_\perp$ is a broken-symmetry variable. Nambu-Goldstone....

Will see: (a) $\delta p_\perp$ fluctuations are large (like in any system with spontaneously broken continuous symmetry)

(b) $\delta\rho$ fluctuations also giant (N-G modes invades the density: a nonequilibrium effect)

In fact the density in the ordered phase is wildly different from the density in an ordinary fluid. Include noise and damping:

$$\partial_t \delta p_\perp = +\Gamma \nabla^2 \delta p_\perp - P'(\rho_0) \nabla_\perp \delta\rho + f_\perp \quad (17)$$

zero mean, Gaussian, spatiotemporally white, nonconserving, strength $A_\perp$

Let $\nabla_\perp \cdot \delta p_\perp \equiv \Phi$

$$(\partial_t + \Gamma q_\perp^2)\Phi_q = P'(\rho_0) q_\perp^2 \delta\rho_q + i q_\perp \cdot f_\perp \quad (18a)$$

$$\partial_t \delta\rho_q = -\rho_0 \Phi_q \quad (18b)$$

$$\Rightarrow (\partial_t + \Gamma q_\perp^2)\partial_t \delta\rho_q = -\rho_0 P'(\rho_0) q_\perp^2 \delta\rho_q - i\rho_0 q_\perp \cdot f_{\perp q} \quad (19)$$

~~$\Rightarrow \langle |\delta\rho_{q\omega}|^2 \rangle =$~~

$$\delta\rho_{q\omega} = \frac{-i\rho_0 q_\perp \cdot f_{\perp q\omega}}{-i\omega(-i\omega + \Gamma q^2) + c_s^2 q_\perp^2}$$

$$\Rightarrow \langle \delta\rho\, \delta\rho \rangle_{q\omega} = \frac{\rho_0^2 q_\perp^2 A_\perp}{(\omega^2 - c_s^2 q_\perp^2)^2 + \Gamma^2 q^4 \omega^2} \quad (20)$$

$$\Rightarrow \langle \delta\rho\, \delta\rho \rangle_q \sim \frac{\rho_0^2 \cancel{q_\perp^2} A_\perp \cancel{\Gamma} \cancel{q^2}}{\Gamma^{\cancel{2}} q^{\cancel{4}2} c_s^2 \cancel{q_\perp^2}} \sim \rho_0^2 A \big/ \Gamma c_s^2 q^2 \quad (21)$$

static

Linear theory of giant number fluct^ns in flocks

Remark: nonlinearities change this to $\frac{1}{q^{2-\eta}}, \eta > 0$

Subtle property of ordered phase just past onset

On physical grounds, and in any model with local "metric" interactions, the local ordering tendency should depend on the local density. That is, in (13),

~~$\frac{d\alpha}{d\rho} < 0$. Work at ρ_0 s.t. $\alpha(\rho_0) < 0$, i.e. mean field theory $\Rightarrow$ ordered phase.~~

$\frac{d\alpha}{d\rho} \neq 0$ (in fact < 0, i.e. α more negative, hence stronger tendency to order, if $\rho \uparrow$).

Work at $\langle\rho\rangle = \rho_0$ s.t. $\alpha_0 \equiv \alpha(\rho_0) < 0$ but small.

Define $\alpha'(\rho_0) \equiv \alpha'_0$. $\underline{p} = \underline{p}_0 + \delta\underline{p}$, $\rho = \rho_0 + \delta\rho$

Look at spatial variation along $\underline{p}_0$ only.

~~$$\frac{\partial}{\partial t}\begin{pmatrix}\delta\rho \\ \delta p_{\parallel}\end{pmatrix} = \begin{bmatrix} -i\rho_0 q_{\parallel} & 0 \\ & \end{bmatrix}$$~~

$$\partial_t \delta p = -2|\alpha_0|\,\delta p \;-\; p_0 \alpha_0' \,\delta\rho$$

$$\partial_t \delta\rho = 0 \;-\; iq\,\delta p \qquad (22)$$

Eigenvalues λ:
$$\begin{vmatrix} -2|\alpha_0| - \lambda & -\alpha_0' p_0 \\ -iq & -\lambda \end{vmatrix} = 0 \qquad (23)$$

$$\cancel{(\lambda + 2|\alpha_0|)(\lambda + iq)}$$

$$\lambda(\lambda + 2|\alpha_0|) - i\alpha_0' p_0 q = 0$$

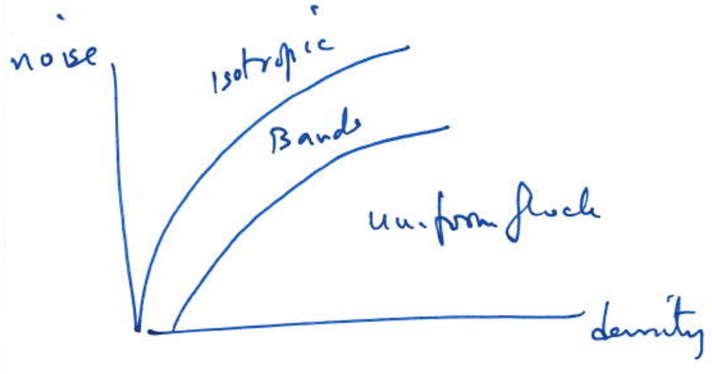
growing mode

$$\lambda \approx \begin{cases} i\dfrac{\alpha_0'}{2\sqrt{|\alpha_0|\beta}}\, q + \dfrac{\alpha_0'^2}{8\beta\alpha_0^2}\, q^2 + \dots \\[2ex] -2|\alpha_0| \end{cases} \qquad (24)$$

Moving Bands aligned $\perp$ to direction of motion

In fact:

Tailleur, Chaté, Solon ...



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Coupled dynamics of orientation & velocity

In principle (based on classification by time-reversal properties) we should distinguish the polar o.p. from the velocity, write down dynamics for both. Only o.p. is slow but nonetheless - - - - .

$$\partial_t \underline{p} = -(a + b\,\underline{p}.\underline{p})\underline{p} + \underset{\text{"Flow alignment"}}{\lambda \underline{v}} + \text{density coupling, gradients etc}$$

$$\partial_t \underline{v} = -\Gamma \underline{v} + \eta \nabla^2 \underline{v} + \underset{\substack{\text{"forcing"}\\ \text{(active)}}}{\alpha \underline{p}} + \quad " \quad " \quad . \; . \; .$$

Eliminate $\underline{v}$ in favour of $\underline{p}$ $\Rightarrow$ effective relax^{n} rate of $\underline{p}$ is $a - \frac{\alpha\lambda}{\Gamma}$ which can turn negative if sgn $\alpha\lambda > 0$ and $\frac{|\alpha\lambda|}{\Gamma}$ big enough.

This mechanism is observed in expts (Kumar et al 2014) on a granular monolayer of motile polar rods and non-motile beads. . . .

Apolar systems:

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Aside: Will return to this later.

Vicsek-style $\hat{n}_\alpha$ $\underline{R}_\alpha$ ~~$\underline{R}_\alpha \to \underline{R}_\alpha + v_0$~~

$R_\alpha \to$ ~~$R_\alpha - R_{(\alpha)}, v_0 \hat{n}_\alpha \Delta t = \pm$, equal probs~~

$= \underline{R}_\alpha + \Delta \underline{R}_\alpha$ ~~$v_0 = \pm \varepsilon$, equal~~ prob.

$\Delta R_\alpha = \pm \varepsilon \hat{n}_\alpha$, equal prob

No advective ~~current~~ current.

BUT: $\rho(\underline{r}, t+\Delta t) - \rho(\underline{r}, t)$

$$= \sum_\alpha \left[\delta(\underline{r} - \underline{R}_\alpha(t+\Delta t)) - \delta(\underline{r} - \underline{R}_\alpha(t)) \right]$$

$$= - \sum_\alpha \Delta R_\alpha \cdot \nabla \delta(\underline{r} - \underline{R}_\alpha)$$

$$+ \frac{1}{2} \sum_\alpha \underbrace{\Delta R_\alpha \Delta R_\alpha : \nabla\nabla \delta(r - R_\alpha)}$$

$\downarrow$

~~$\nabla_i \nabla_j \left[\left(\underline{\underline{Q}}_{ij} + \frac{1}{2} \underline{\underline{I}} \right) \right]$~~

$$\nabla_i \nabla_j \left[\left(Q_{ij} + \frac{1}{2} \delta_{ij} \right) \rho \right]$$

This curvature-induced current is crucial to ~~disclin~~ giant number fluctuations, disclination unbinding, moving formation

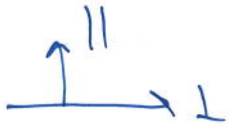
(ultimately closely connected to active instabilities)

giant number fluctuations — disclination —

(22) Toner's notes

LRO in d=2

$\langle \underline{v} \rangle$.



J. Toner

see e.g.

Ginelli
arXiv:1511.01451

$\delta x_\perp$

~~On scale r, $\delta\theta_{RMS}$~~

time t: local error θ_0

spreads (divided) over distance $t^{1/2}$

$\therefore$ each spin deviates $\sim \frac{\theta_0}{t^{d/2}}$

On scale $r \sim t^{1/2}$: r^d spins

In time $t \sim r^2$, each makes $\sim t$ errors

RMS angle error $\sqrt{r_0^d\, t} = \sqrt{r^d\, r^2}$

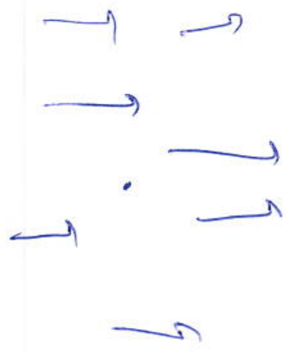
$\therefore$ per spin: $\frac{r^{\frac{2+d}{2}}}{r^d} = r^{1-d/2}$

$= t^{\frac{1}{2} - \frac{d}{4}}$

$\rightarrow \infty,\ d<2$

(in fact log, $d=2$)

(Continued)



Toner's argument for LRO in $d=2$

Ginelli

arXiv: 1511.01451

Error θ_0 at origin spreads distance $\sim t^{1/2}$ in time t through XY model dynamics.

Repeat

In a volume r_0^d, accumulated error in a time t is $(r_0^d t)^{1/2}$, where $r_0 = t^{1/2}$

RMS error per bird $(r_0^d t)^{1/2}/r_0^d = t^{\frac{2-d}{4}} = \delta\theta_{RMS}(t)$

Grows for $d<2$.

But this argument is OK only if bird movement can be ignored. Transverse wandering $\delta x_\perp = ?$

$$\boxed{\begin{array}{l}\delta x_\perp \sim v_\perp^{RMS} t \sim \delta\theta_{RMS} t \sim t^{\frac{2-d}{4}} t = t^{\frac{3}{2}-\frac{d}{4}} \\ \therefore \text{ velocity-based spreading of angle error} \sim t^{\frac{3}{2}-d/4}\end{array}}$$

XY dynamics-based spreading of angle error $\sim t^{1/2}$

velocity-based spreading dominates if

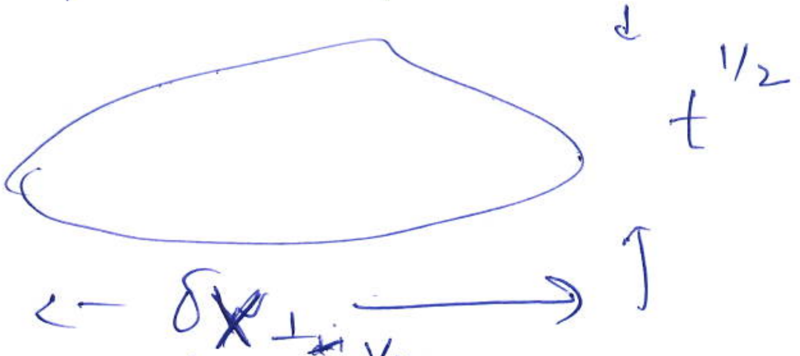
$\frac{3}{2}-\frac{d}{4} > \frac{1}{2} \Rightarrow 1 > \frac{d}{4} \Rightarrow \boxed{d<4}$

This uses the mean-field result for $\delta\theta_{RMS}$ $\rightarrow$ will see later how really to calculate $v_{\perp RMS}$

(24)

Now: do angle fluctuations grow or shrink at long times when velocity-based spreading dominates?

~~Birds spr~~, Errors spread over ~~distance~~ area $\delta \cancel{x}_\perp \times t^{1/2}$



No. of spins: $(\delta x_\perp(t)) t^{1/2}$

Number of errors in time t: $\delta x_\perp(t) t^{1/2} t$.

Sum of all angle errors $\sim \sqrt{\delta x_\perp(t) t^{1/2} t}$

$\sim \cancel{t^{\frac{3}{2}} d}$ $\qquad$ $\cancel{t^{3/2 - \frac{1}{4}} = t^{\frac{2}{2} \cdot \frac{1}{2}} = t}$

$= \sqrt{\delta x_\perp}\, t^{3/4}$

$\therefore$ per spin: $\dfrac{\sqrt{\delta x_\perp}\, t^{3/4}}{\delta x_\perp t^{1/2}} = \dfrac{t^{1/4}}{\sqrt{\delta x_\perp}} \equiv \delta\theta$

$\delta x_\perp \sim t^{5/6}$

$\cancel{\dfrac{t^{1/4}}{t^{1/2}} = t}$

But $\delta\theta\, t = \delta x_\perp$

$\therefore \dfrac{t^{1/4}}{\delta\theta^{1/2} t^{1/2}} = \delta\theta \Rightarrow \delta\theta^{3/2} = t^{-1/4}$

$\rightarrow \delta\theta \sim t^{-1/6}$ LRO

Functional integrals for stochastic dynamical fields

For a general Langevin eq^{n} with spatiotemporally white noise of strength D:

$$\dot{x} = v(x) + \sqrt{2D}\,\eta$$

$$\mathrm{Prob}[x] \propto e^{-\frac{1}{4D}\int(\dot{x}-v)^2 dt}$$

$$= \int[d\hat{x}]\, e^{\int[i\hat{x}(\dot{x}-v) - D\hat{x}^2]dt}$$

So calculate averages using "action"

$$S[x,\hat{x}] = \int[\hat{x}(\dot{x}-v) + i\hat{x}D\hat{x}]$$

$$\langle\theta\rangle = \int[d\hat{x}][dx]\;\theta\; e^{iS[\hat{x},x]}$$

Field Theory of Toner–Tu model

[Exact demonstration of Long Range Order in the case where number conservation is ignored]

Introduce birth & death

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho \vec{S}) + k_+ - k_- \rho$$

$\Rightarrow$ $\delta\rho$ relaxes with rate k_- at zero wavenumber.

Fast. Forget density.

$$\partial_t \underline{S} = (a - b\,\underline{S}\cdot\underline{S})\,\underline{S} - \lambda\,\underline{S}\cdot\nabla\underline{S} + \overbrace{K\nabla^2\underline{S}}^{\text{more complicated}} + \sqrt{2D}\,\underline{f} \quad \left(\begin{smallmatrix}\text{non}\\ \text{conserving}\end{smallmatrix}\right)$$

$$\xrightarrow{\quad} \cancel{K_{\parallel}\nabla_{\parallel}^2 \underline{S} + K_{\perp}\nabla_{\perp}^2 \underline{S}}$$

$$\langle f_i(\underline{0}\,0)\, f_j(\underline{r}\,t)\rangle = \delta_{ij}\,\delta^d(\underline{r})\,\delta(t)$$

Ordered phase: $\underline{S} = S_0\hat{x} + \delta\underline{S}_\perp$

$$\delta\underline{S}_\perp \equiv \underline{p} \qquad \longrightarrow$$

Ordered phase eq^{n} of motion

$$\partial_t \underline{p} + \underbrace{\lambda_1 \underline{p}\cdot\nabla_\perp \underline{p} + \lambda_3 \nabla_\perp \underline{p}\cdot\underline{p}} - \underbrace{\left(\nu_\parallel \nabla_\parallel^2 + \tilde{\nu}_\perp \nabla_\perp^2 + \bar{\nu}\, \nabla_\perp \nabla_\perp \cdot\right)} \underline{p} = \sqrt{2D}\,\eta(x,t)$$

nonconserving spacetime white noise

$d=2$

$$\downarrow \qquad\qquad \downarrow d=2$$

$$\lambda\, p\, \nabla_\perp p \qquad\qquad (\nu_\parallel \nabla_\parallel^2 + \nu_\perp \nabla_\perp^2)\, p$$

$$\lambda = \lambda_1 + 2\lambda_3 \qquad\qquad \nu_\perp = \tilde{\nu}_\perp + \bar{\nu}$$

① "Galilean" invariance, $d=2$

$$x_\perp' = x_\perp - \lambda v t\,,\ t' = t,\ p = p' + v$$

$$\Rightarrow \quad \frac{\partial p}{\partial t} = \frac{\partial p'}{\partial t'} - \boxed{\lambda v \frac{\partial}{\partial x'} p'} \leftarrow \text{cancel}$$

$$\lambda p \frac{\partial p}{\partial x_\perp} = \lambda (p'+v) \frac{\partial (p'+v)}{\partial x_\perp'} = \boxed{\lambda v \frac{\partial p'}{\partial x'}} + \lambda p' \frac{\partial p'}{\partial x'}$$

$$\frac{\partial p}{\partial t} + \lambda p \frac{\partial p}{\partial x_\perp} = \frac{\partial p'}{\partial t'} + \lambda p' \frac{\partial p'}{\partial x'}$$

② $\lambda\, p\, \nabla_\perp p = \frac{\lambda}{2} \nabla_\perp p^2$ vanishes at zero wavenumber

Noise η non-conserving

$\Rightarrow$ ① $\wedge$ ② Neither vertex nor noise acquire corrections upon integrating out degrees of freedom. Only get corrections under rescaling.

③ Vertex $\sim \nabla_\perp$, so $\nu_\parallel$ gets no graphical corrections.

Gaussian fixed pt: Ignore nonlinearity. Then there's no difference between $\parallel$ & $\perp$ directions:

$$x = bx' \,,\; t = b^z t' \,,\; \phi = b^\chi \phi' \,,\; \hat{\phi} = b^{\hat\chi} \hat{\phi}'$$

$$S \to b^{z+d+\hat\chi} \int \hat{\phi}\left(b^{\chi - z} \partial_t \phi + \underbrace{\lambda b^{2\chi - 1} \phi \nabla_\perp \phi}_{\text{ignore for now}} - b^{\chi - 2} \nu \nabla\nabla\phi + i b^{\hat\chi} D \hat{\phi} \right)$$

Choose coeffs so all terms unchanged except λ

$$z + d + \hat\chi + \chi - z = 0 \Rightarrow \hat\chi = -\chi - d$$

$$z + d + \hat\chi + \chi - 2 = 0 \Rightarrow z = 2$$

$$z + d + 2\hat\chi = 0 \Rightarrow \hat\chi = -1 - \frac{d}{2}$$

$$\chi = 1 - \frac{d}{2}$$

now look at λ

$$\Rightarrow \lambda \to b^{2\chi - 1} \lambda = b^{\frac{4-d}{2}} \lambda$$

$\Rightarrow$ Gaussian f.p. unstable for $d < 4$.

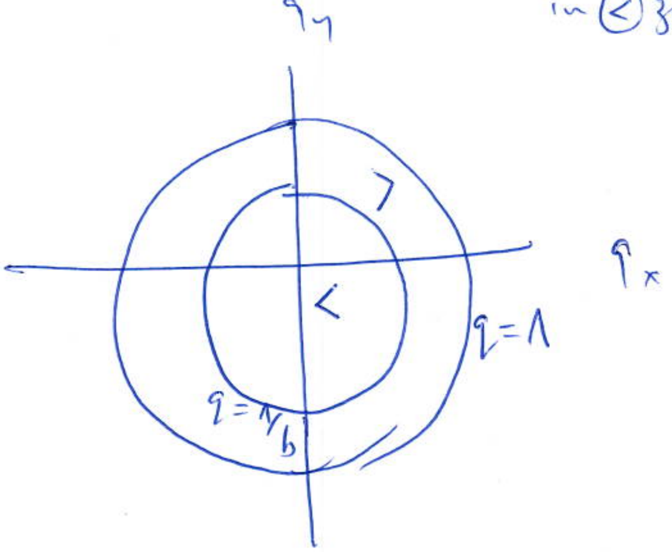
Renormalization-group treatment

Recall: all fields defined with upper cutoff in Fourier space, Λ.
i.e. wavevectors $q < \Lambda$. Define two zones: $\frac{\Lambda}{b} < q < \Lambda$, $q < \frac{\Lambda}{b}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad >\qquad\qquad <$

Write each field as $\varphi = \varphi^{<} + \varphi^{>}$

$\varphi^{<}$: Fourier components in $<$ zone

$\varphi^{>}$: Fourier components in $>$ zone.



$$\int [d\hat{\varphi}][d\varphi]\, e^{iS} = \int [d\varphi d\hat{\varphi}]^{<} [d\varphi d\hat{\varphi}]^{>}\, e^{i(S^{<} + S^{><} + S^{>})}$$

Carry out $\int [d\varphi d\hat{\varphi}]^{>}$, $\quad S^{<} \to S^{<} + \delta S$ (if you're lucky this simply changes the values of existing couplings)

Rescale $x_{\perp} = b x_{\perp}'$, $x_{\parallel} = b^{\zeta} x_{\parallel}'$, $t = b^{z} t'$,

$$\varphi = b^{\chi} \varphi', \quad \hat{\varphi} = b^{\hat{\chi}} \hat{\varphi}'$$

~~Our case~~

$$S' = \int \hat{p}\left(\partial_t p + \lambda p \nabla_\perp p - \nu_{\parallel} \nabla_{\parallel}^2 p - \nu_\perp \nabla_\perp \nabla_\perp p + i D \hat{p}\right) d^d x\, dt$$

Jump to ~~earlier~~ page ~~on left~~ 28

Gaussian fixed point, first

Integrate out degs of freedom leaves λ, D unchanged

$\Downarrow$

$\hat{\chi} = -d - \chi$ (keep $\hat{p}\,\partial_t p\, d^d x\, dt$ unchanged)

$\nu_\perp \to b^{z-2}(\nu_\perp + \Delta\nu)$, $\quad \nu_{\parallel} \to b^{z-2\zeta}(\nu_{\parallel} ~~+\Delta\nu~~)$ $\quad z-2\zeta$

$D \to b^{z-\zeta-2\chi+1-d=} D$

$\lambda \to b^{z+\chi-1}\lambda$

Do RG so as to keep λ fixed. Then $z+\chi=1$

Fixed pt $\qquad$ D fixed, noise strength fixed.

$\Downarrow$

$\boxed{\chi = -1/5}$

$z = 6/5$

$\zeta = 3/5$

$\Rightarrow$ fluctuations in p don't grow with scale, $\therefore$ Long Range Order survives

($\chi = 0 \Rightarrow \log$

$\chi > 0 \Rightarrow$ grow)

~~$\delta\theta(x) \sim x^{-1/5} \sim t$~~

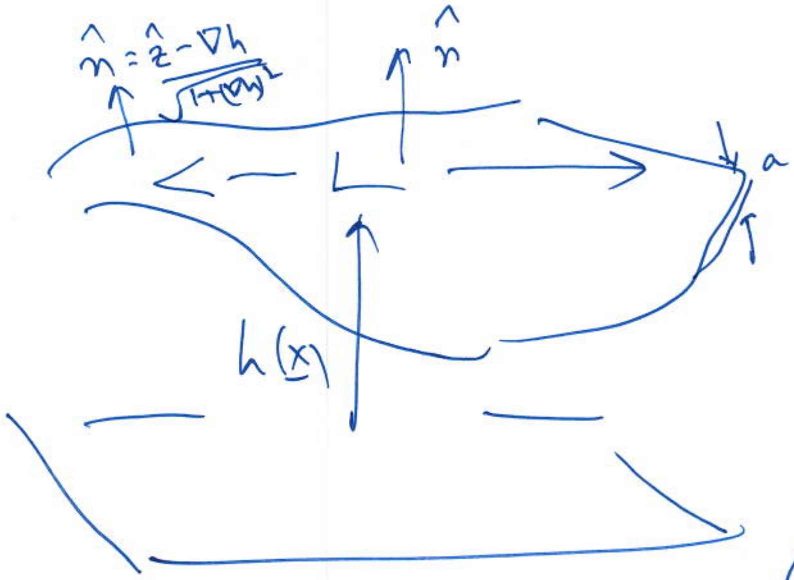
$\delta\theta(r) \sim r^{\chi} = t^{\chi/z} = t^{-\frac{1}{5}\cdot\frac{5}{6}} = t^{-1/6}$ as in the pictorial argument!

[May not cover in these lectures]

Active membranes

BSSP2017 does not include p. 27–35

Recall Pierre Sens's lecture



$L \ggg a$

Energy

$= \frac{\kappa}{2}\int_{\underline{r}} C^2$

C = mean curvature

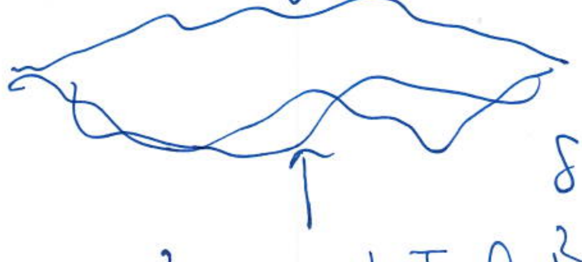
$$\simeq \frac{\kappa}{2}\int_{\underline{r}} (\nabla^2 h)^2 = \frac{\kappa}{2}\int_{\underline{q}} q^4 |h_{\underline{q}}|^2$$

$$h_{\underline{q}} = \int_{\underline{r}} e^{-i\underline{q}\cdot\underline{r}}\, h(\underline{r})$$

$$\frac{\langle |h_q|^2 \rangle}{\text{Area}} = \frac{k_BT}{\kappa q^4}$$

$$\langle h^2 \rangle = \langle h(\underline{x})h(\underline{x})\rangle = \frac{k_BT}{\kappa}\int \frac{d^2q/(2\pi)^2}{q^4} = \frac{k_BT}{\kappa} L^2$$

$\sqrt{T/\kappa}\, L$



$\delta\hat{n} \simeq -\nabla h \Rightarrow \langle|\delta n_q|^2\rangle \sim \frac{k_BT}{\kappa q^2}$

$$\langle(\delta n)^2\rangle \simeq \frac{k_BT}{\kappa}\int \frac{d^2q/(2\pi)^2}{q^2} = \frac{k_BT}{4\pi^2\kappa}\ln\frac{L}{a} = 1 \text{ for } L = l_p$$

$\langle \delta n(\underline{0})\cdot\delta n(\underline{r})\rangle \sim e^{-r/L_p}$ Persistence Length $\Rightarrow$ $L_p = a \exp\frac{\kappa}{(\,)k_BT}$

(28)

Membrane conc. fluctns $\quad c = \dfrac{\text{Area}}{(\text{Area})_{\text{base}}} = \sqrt{1+(\nabla h)^2}$

$$\Rightarrow \delta c \simeq \tfrac{1}{2}\left[(\nabla h)^2 - \langle(\nabla h)^2\rangle\right] \quad \text{Crumpling factor}$$

$$\Rightarrow \langle|\delta c_q|^2\rangle \sim \left(\frac{T}{\kappa}\right)^2 \frac{1}{q^2}$$

with tension? This is cut off at $\lambda = \sqrt{\kappa/\sigma}$

Note: $\langle(\nabla h)^2\rangle_\sigma - \langle(\nabla h)^2\rangle_{\sigma_0}$ itself $\simeq \frac{k_B T}{8\pi\kappa} \ln \sigma/\sigma_0$, the basis of μpipette expts

Membranes with ~~other co~~ more components.

~~Scalar field~~ Components without leaflet preference
(equivalently, without polarity wrt membrane normal)
couple ~~to~~ only to $(\nabla h)^2$ and to K_{Gaussian}.

Components w/ leaflet preference couple to ~~mo~~
local mean curvature. Let's look at that case.

Oh wait, Pierre did already. So Let's study
membrane dynamics.

(1) At equilibrium:

 $\dot{h} = v_z$

$$\eta \nabla^2 \underline{v} = \nabla p - \delta(z-h(x,y))\,\kappa \nabla_\perp^4 h + \text{thermal noise}$$

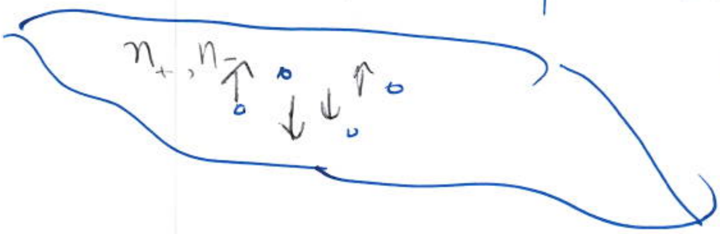
$$\Rightarrow v_z\big|_{\text{membrane}} = -\frac{\kappa}{4\eta} q_\perp^3 h_q + \text{thermal noise}$$

Exercise: work this out, including form of noise,
check $\langle|h_q|^2\rangle = \frac{k_B T}{\kappa q^4}$

~~Now add cell life~~

29

Where's biology? active processes at the membrane.



Ion pumps

actin polymerisation sites

even channels, maintain in presence of gradient

* Active stresses at the membrane

* ~~"push membrane along its normal"~~

* "push" along membrane normal

$n_+ + n_- = \phi, \quad n_+ - n_- = \psi$ ← "polar"

just a density, indep of which is up or down.

Couples to $(\nabla h)^2$

definition linked to ~~the~~ choice of outward normal

Couples to $\nabla^2 h$

(1) ψ moves membrane along normal

$$\mathring{h} = v_0 \left[\psi + \text{---} \right]$$

(2) Moving along normal kinematically causes spreading

$$\mathring{\psi} = v_0 \nabla \cdot \left[\frac{\psi \, \psi \, \nabla h}{1 + (\nabla h)^2} \right] + \text{-- --}$$

(Lateral current of ψ = ψ × lateral velocity

$= \psi v_0 \psi \nabla h$)

(3) Shape and functional polarity (30)

(3a) Shape

Mean curvature is a chemical potential for ψ

Lateral ψ current $\sim c_1 \nabla \nabla^2 h + c_1 \nabla \psi$

$= \Lambda \kappa \bar{H} \nabla \nabla^2 h - \Lambda A \nabla \psi$

3b

Function

Mean curvature affects pump function

$\dot{h} \simeq v_0 l_1 \phi \nabla^2 h$

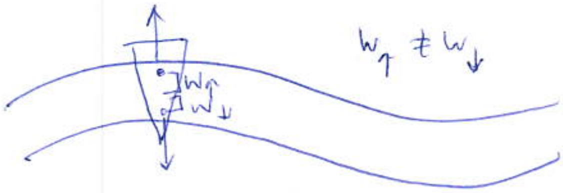
has same effect on normal velocity

(4) Flow carries membrane $\dot{h} = V_z$

(5) Pump stresses affect flow:

$$-\eta \nabla^2 \underline{v} = -\nabla p - \nabla \cdot \underline{\underline{\sigma}}^{pump}$$

$\underline{\underline{\sigma}}^{\text{pump}}$: force dipole not centred at membrane mid plane



$$\nabla\cdot\underline{\underline{\sigma}}^{\text{pump}} = \hat{z}\, F\left[\delta(z-w_\uparrow)-\delta(z+w_\downarrow)\right]\left(\psi+\ell_2\phi\nabla^2 h\right)$$

forget for now

$$\dot{h} = \nu_0\left(\psi+\ell_1\phi\nabla^2 h\right) + v_z - \mu_p\frac{\delta H}{\delta h} \quad + \text{noise}$$

$$\dot{\psi} = \nu_0\nabla\cdot\left[\frac{\psi^2\nabla h}{1+(\nabla h)^2}\right] - \Lambda\kappa\bar{H}\nabla^4 h + \Lambda A\nabla^2\psi$$
+ conserving noise

algebra → ($\sigma^{\text{bare}} = 0$, Ignore v_z) $\Longrightarrow$ if $\ell_2, \bar{H} > 0$ then stable

$$\frac{\langle |h_q|^2\rangle}{\text{area}} = \frac{k_B T}{\frac{\nu_0}{\mu_p}\ell q^2 + \kappa_{eff}q^4} + \frac{\nu_0 a^2}{\Lambda A \ell q^4}$$

↑ !!

$$\ell \simeq \ell_1 + \frac{\kappa}{k_B T}\bar{H}a^2$$

active tension (+ve for now ~~could be -ve, of course~~)

3) l_2 and/or $\bar{H}$ negative enough

$\Rightarrow$ pump clump bump instability.

32

Back to +ve case:

But μp is minute! ~~Need~~ Can't take
almost impermeable
refuge in lowest order ∇ exp^n (need to go to centimetres!)

Include v_2 via Stokes eq^n

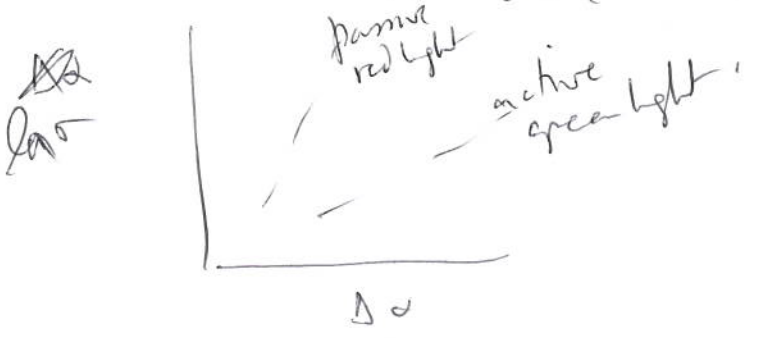
Induces $F|q|\Psi$ in h eq^n

~~eg~~ eventually $\langle |h_q|^2 \rangle \sim \dfrac{T_{eff}}{\kappa q^4}$, → can therefore get this in $\mu p i p$ expts.

T_{eff} similar to earlier analysis, ~ $T_{eff} \sim 2 T_{thermo}$

Comparison to expt, $\boxed{\text{GUV+BR}}$

Numbers: see Manneville, Bassereau, SR, Prost
PRE 64 (2001) 021908



Modes: waves w/ speed $\sim v_0$ ~~fetro~~

Impermeable case: $\omega^2 \sim q^3$, speed very dispersive

but involves v_0 and force dipole strength

Marchetti et al 2014 derived these eqns from ordinary membrane + active fluid

↓

Unstable case: numerical sol^n → self generated waves.

Mietvitu et al 2014

Fluid membrane + csk filaments + solvent

contains species that knows ↑ from ↓ e.g. actin nucleators

active stresses

polar op

~~$h = v + v_0 P_z + \zeta \nabla \cdot \underline{P}\underline{P} - \mu_P \frac{\delta F}{\delta h}$~~

(a) $\underline{v}_{membrane} = v + v_0 \underline{P} + \zeta \nabla \cdot \underline{P}\underline{P} - \frac{\mu_p}{\sqrt{g}} \frac{\delta F}{\delta \underline{R}_{membrane}}$

Treadmilling + active stress + elasticity

$\parallel$ vs $\perp$ anchoring of filaments at membrane.

(b) Tangential polar filaments transported actively along their own $\underline{p}$

coupled eqns for h, c & p

Instabilities leading to tubules or ridges

~~29~~ (C) ~~#~~ 1 + 1 d ~~numer~~ PDE 35

$$\dot{h} = v_0 \psi + h'' - \mu_p u h'''' - \mu_p v_0 \psi''$$

$$\dot{\psi} = v_0 (\psi h')' + D \psi'' - \kappa_0 h''''$$

passive ~~dissipative~~ coupling

Single fourier mode

$$\dot{h}_1 = m_1 \psi_1 - m_2 h_1$$

$$\dot{\psi}_1 = -m_3 \psi_1 \psi_1 h_1 - m_4 \psi_1 - m_5 h_1$$

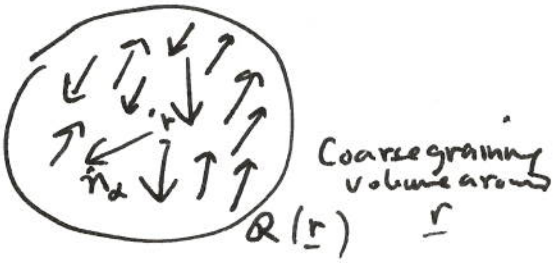
generalized ~~vdP~~ van der Pol osc

has limit cycle.

Full PDE has self generated noisy h waves

Back to flocking models: apolar order

Recall apolar means



$$\langle \hat{n}_\alpha \rangle_{R(\underline{r})} = 0$$

but $\langle \hat{n}_\alpha \hat{n}_\alpha \rangle \neq \frac{1}{d} \underline{\underline{1}}$

MANY EXAMPLES IN CELL AGGREGATES, TISSUE . . . .

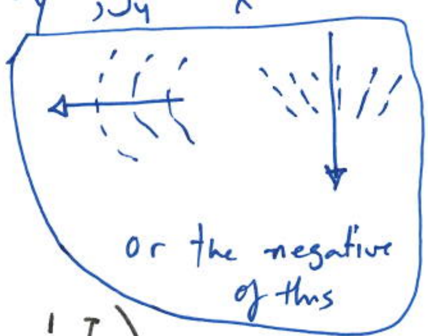
$$\underline{\underline{Q}}(\underline{r}) \equiv \left\langle \underline{\underline{Q}}_\alpha \right\rangle_{R(\underline{r})} \equiv \left\langle \hat{n}_\alpha \hat{n}_\alpha - \frac{1}{d} \underline{\underline{1}} \right\rangle_{R(\underline{r})}$$

Nematic order parameter

Aligned state: $\underline{\underline{Q}} \neq 0$, uniform. No sense of forward or back, only an axis. No net motion.

$\therefore$ current arises only if $\underline{\underline{Q}}$ is non uniform.

Simplest: $\underline{J} \propto \underline{\nabla} \cdot \underline{\underline{Q}} \Rightarrow J_x = " \partial_y \theta, J_y = " \partial_x \theta$



In more detail: Slow variables:

Number density ρ

Order parameter $\underline{\underline{Q}} = S \left(\hat{n}\hat{n} - \frac{1}{d} \underline{\underline{I}} \right)$

Magnitude S is slow if there is a transition with continuous onset. $\longrightarrow$

The axis $\hat{n}$ is slow because global rotation turns $\hat{n}$ with no restoring torque.

Spontaneously broken symmetry, perturbations of $\hat{n}$ are a Nambu-Goldstone mode.

Dynamics (~~[illegible]~~ on general symmetry grounds)

$$\partial_t \underline{\underline{Q}} = \text{polynomial in } \underline{\underline{Q}} + K \nabla\nabla \underline{\underline{Q}}$$

$$+ \text{nonlinearities} + \text{coupling to } \rho$$

$$+ \text{noise}$$

$$= -(a + b\, \underline{\underline{Q}} : \underline{\underline{Q}})\underline{\underline{Q}} + K \nabla^2 \underline{\underline{Q}}$$

$$\left(+ \quad Q \nabla \nabla Q \quad + \quad \nabla Q \nabla Q\right) \Big\{ \text{discuss later}$$

$$+ C \left(\nabla\nabla - \tfrac{1}{d}\underline{\underline{1}} \nabla^2\right) \delta\rho$$

$$+ \underline{\underline{f}}_Q$$ (nonconserving isotropic traceless symmetric tensor noise, spatiotemporally white,

$$\partial_t \rho = D \nabla^2 \rho + \zeta_1 \nabla \cdot (\underline{\underline{Q}} \cdot \nabla \rho) + \zeta_2 \nabla \cdot (\rho \underline{\nabla} \cdot \underline{\underline{Q}})$$

$$+ \nabla \cdot \underline{\eta}$$ (conserving noise)

Where does all this come from?

① Effective equilibrium dynamics

$$F = \int \Big[\frac{a}{2} \underline{\underline{Q}}^2 + \frac{u}{4}(\underline{\underline{Q}}:\underline{\underline{Q}})^2 + \frac{K}{2}(\nabla \underline{\underline{Q}})^2$$
$$+ (\,) \, Q \nabla Q \nabla Q$$
$$+ U(\rho) + \frac{B}{2}(\nabla\rho)^2$$
$$- c \, \underline{\underline{Q}} : \nabla\nabla\delta\rho \Big]$$

Dynamics in equilibrium

$$\partial_t \underline{\underline{Q}} = - \frac{\delta F}{\delta \underline{\underline{Q}}} + \text{nonconserving noise}$$

$$\partial_t \rho = \nabla^2 \frac{\delta F}{\delta \rho} + \text{conserving noise}$$

accounts for all ~~both~~ but the ~~both~~ ζ_2 terms.

ζ_1 is anisotropic diffusion ~~ζ_2 is~~ and must (at equilibrium) be accompanied by suitable $\underline{\underline{Q}}$-dependent noise, will see in later lectures.

ζ_2 is wholly nonequilibrium.

$$\underline{\underline{Q}} = S \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\simeq S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2S \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$$

$$\Rightarrow \quad \partial_t \theta = K \nabla^2 \theta + f_\theta \text{ (non conserving noise)} \quad ⓐ$$

+ unimportant terms

$$\partial_t \delta\rho = D \nabla^2 \delta\rho + \zeta_1 S (\partial_x^2 - \partial_y^2) \delta\rho \quad ⓑ$$

$$+ \boxed{4\zeta_2 S \partial_x \partial_y \theta} + f_\rho \text{ (conserving)}$$

$$ⓐ \Rightarrow \int_{\underline{r}\,t} \langle \theta(\underline{0}\,0)\, \theta(\underline{r}\,t) \rangle e^{-i(\underline{q}\cdot\underline{r} - \omega t)} = \frac{\text{Noise strength}}{\omega^2 + K^2 q^4}$$

$$\Rightarrow \int_{\underline{r}} \langle \theta(0)\, \theta(\underline{r}) \rangle e^{-i\underline{q}\cdot\underline{r}} = \frac{\text{Noise strength}}{K q^2}$$

ζ_2 term in ⓑ $\Rightarrow$ $\langle \delta\rho\, \delta\rho \rangle_q \sim \dfrac{\zeta_2^2 S^2 q_x^2 q_y^2}{K q^6}$

static structure factor

$$\propto S^2 \frac{\sin^2\phi \cos^2\phi}{q^2}$$

$$\Rightarrow (\Delta N)^2 \propto N^{\boxed{2-(\,)}} \longrightarrow > 1$$

System of size L means smallest $q \sim \frac{1}{L}$

Showed that

$$S_q \equiv \int d^d r \langle \delta\rho(0)\delta\rho(\underline{r})\rangle e^{-i\underline{q}\cdot\underline{r}} \sim \frac{q_{\parallel}^2 q_{\perp}^2}{q^6}$$

$$\Rightarrow \underbrace{S_{q=\frac{1}{L}}}_{\downarrow \text{ show this}} \sim L^2 = \left(L^d\right)^{2/d} \propto N^{2/d}$$

number $N \propto L^d$

$$= \frac{\langle(\delta N)^2\rangle}{N} \propto N^{2/d}$$

$$\Rightarrow \delta N_{RMS} \propto N^{\frac{1}{d}+\frac{1}{2}}$$

This is modified in a more careful treatment

but still find $\delta N_{RMS} \propto N^{\frac{1}{2}+a}$, $a > 0$

i.e. ~~δN~~ variance bigger than Poissonian.

$\therefore$ not only Vicsek-Toner-Tu flocks (polar) but ~~non~~ active nematic flocks (apolar) have giant number fluctuations.

The exponent $\frac{1}{2}+a$ remains to be calculated in an improved (beyond linear) theory.

From microscopic stochastic models

to coarse-grained stochastic PDEs

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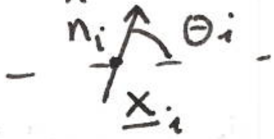
see, e.g., Bertin, Chaté, Ginelli, Mishra, Peshkov, SR

New Jour. Phys 15 (2013) 085032

[Micro model: Chaté, Ginelli, Montagne, PRL 96 (2006) 180602]

Vicsek Model

$$\hat{n}_i^t = (\cos\theta_i^t, \sin\theta_i^t); \qquad \theta_i^{t+\Delta t} = \arg \sum_{k \in nbh(i)} e^{i\theta_k^t} + \eta \xi_i^t$$



$$\underline{x}_i^{t+1} = \underline{x}_i^t + \Delta t \, v_0 \, \hat{n}_i^t$$

$\boxed{nbh(i) \text{ has radius } > v_0 \Delta t}$

ξ_i^t = unit white angular noise

Coarse-graining (Boltzmann eq^n approach)

Bertin, Droz, Grégoire PRE 74 (2006) 022101

$$\rho(\underline{r}\, t) = \int_{-\pi}^{\pi} f(\underline{r}\, \theta\, t)\, d\theta$$

$$\rho(\underline{r}\, t)\, \underline{u}(\underline{r}\, t) = \int_{-\pi}^{\pi} d\theta\, f(\underline{r}\, \theta\, t)\, \hat{n}(\theta)$$

Check: $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$

Define

$$f_k(r\,t) = \int_{-\pi}^{\pi} d\theta\, e^{ik\theta} f(\underline{r}\,\theta\,t)$$

Build PDEs for ρ (already done, identity)

and $\underline{w} = \rho\underline{u}$ from this evolution.

Get Toner-Tu form with coefficients

depending on parameters above

(rotational noise strength η, speed v_0,
mean number in neighbourhood)

~~Details~~ Values depend on collisional versus Vicsek-style updates.

We will do this procedure for APOLAR

particles. What's the difference?

$$\hat{n}_i^t = (\cos\theta_i^t, \sin\theta_i^t)$$

$$\theta_i^{t+\Delta t} = \frac{1}{2} \arg \sum_{j \in nbh(i)} e^{i 2\theta_j^t} + \eta \xi_i^t$$

η: strength ξ_i^t: unit white

$$x_i^{t+\Delta t} = x_i^t + \underbrace{\Delta t\, v_0}_{d_0} \kappa_i^t \hat{n}_i^t$$

d_0: not really connected to a "velocity"

κ_i^t: ± 1, prob $\frac{1}{2}$, IID

i.e. $x_i^{t+\Delta t} = x_i^t \pm d_0 \hat{n}_i^t$

Build evolution eqⁿ for distribution function, get PDEs from moments (also get coarse-grained ~~noise~~ noise)

Let $f(\underline{x}\,\theta\,t) =$ prob density in position and orientation space.

Apolar particles, so $\theta \to \theta + \pi$ symmetry.

$\Rightarrow$ Expand in <u>even</u> Fourier components wrt θ.

sorry to use same symbol for diff't object (this one is apolar)

$$f(\underline{x}\,\theta\,t) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} f_k(\underline{x}\,t)\, e^{-i2k\theta}$$

$$f_k(\underline{x}\,t) = \int_{-\pi/2}^{\pi/2} d\theta\, f(\underline{x}\,\theta\,t)\, e^{i2k\theta}$$

$$\rho(\underline{x}\,t) = \int_{-\pi/2}^{\pi/2} d\theta\, f(\underline{x}\,\theta\,t) = f_0(\underline{x}\,t)$$

$$W_{11}(\underline{x}\,t) = -W_{22}(\underline{x}\,t) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta\, f(\underline{x}\,\theta\,t)\cos 2\theta = \frac{1}{2}\,\mathrm{Re}\, f_1$$

$$W_{12}(\underline{x}\,t) = W_{21}(\underline{x}\,t) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta\, f(\underline{x}\,\theta\,t)\sin 2\theta = \frac{1}{2}\,\mathrm{Im}\, f_1$$

But how does f evolve? $\longrightarrow$

① Move forward or backward along $\hat{n}_i^t$:

$$f(\underline{x}, \theta, t+\Delta t) = \frac{1}{2}\Big[f(x+\hat{n}\, d_0, \theta, t) + f(x-\hat{n}\, d_0, \theta, t)\Big]$$

$$\simeq \frac{d_0^2}{2\,\Delta t}\, \partial_\alpha \partial_\beta \left[\hat{n}_\alpha \hat{n}_\beta f(\underline{x}\ \theta\ t)\right]$$

Ito Calculus

② Rotational diffusion
③ Collisions.

$\Big\}$ Less interesting

Fourier transform wrt θ to get in terms of f_k.

$\longrightarrow$

$$\partial_\alpha \partial_\beta \int_{-\pi/2}^{\pi/2} d\theta\, e^{i2k\theta}\, \hat{n}_\alpha \hat{n}_\beta\, f(\underline{x}\,\theta\, t)$$

$$= \partial_\alpha \partial_\beta \int_{-\pi/2}^{\pi/2} d\theta\, e^{2ik\theta} \left[\left(n_\alpha n_\beta - \frac{1}{2}\delta_{\alpha\beta}\right) + \frac{1}{2}\delta_{\alpha\beta}\right] f$$

$$= \frac{1}{2}\Delta f_k + \frac{1}{4}\left(\nabla^{*2} f_{k+1} + \nabla^2 f_{k-1}\right)$$

$$\Rightarrow \cancel{\partial_t f_0}$$

$$\Rightarrow \partial_t \rho = \partial_t f_0 = \frac{1}{2}\Delta f_0 + \frac{1}{4}\underbrace{\nabla^{*2} f_1}$$

$$(\partial_x - i\partial_y)(\partial_x - i\partial_y)(\mathrm{Re} f_1 + i\,\mathrm{Im} f_1)$$

Use $w_{11} = -w_{22} = \frac{1}{2} \mathrm{Re}\, f_1 = \frac{\rho}{2} \langle \cos 2\theta \rangle_{\underline{x}}$

$w_{21} = w_{12} = \frac{1}{2} \mathrm{Im}\, f_1 = \frac{\rho}{2} \langle \sin 2\theta \rangle_{\underline{x}}$

get

$$\partial_t \rho = \frac{1}{2} \Delta \rho + \frac{1}{2} \hat{\Gamma}_{\alpha\beta} w_{\alpha\beta}$$

$$\hat{\Gamma}_{\alpha\beta} = \begin{pmatrix} \partial_x^2 - \partial_y^2 & 2 \partial_x \partial_y \\ 2 \partial_x \partial_y & \partial_y^2 - \partial_x^2 \end{pmatrix}$$

Get $\nabla_\alpha \nabla_\beta (\rho\, Q_{\alpha\beta})$

which we claimed earlier.

What about noise?

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Blank

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Back to a simpler problem, diffusion of N indep particles

D S Dean 1996

$$\underline{X}_i \qquad \dot{\underline{X}}_i = \eta_i$$

$\hookrightarrow$ white noise, strength 2D

$$\rho(\underline{x}t) = \sum_i \underbrace{\delta(\underline{x} - \underline{X}_i(t))}_{} \equiv \sum_i \rho_i(\underline{x}t)$$

$$f(X_i(t)) = \int dx\, \rho_i(x t)\, f(x)$$

$$\frac{d f(X_i)}{dt} = \int \rho_i(xt)\Big[\eta_i \cdot \nabla f(x) + D\, \underbrace{\nabla^2 f(x)}_{\text{usual}}\Big]$$

$\uparrow$ normally set to 0.

$$= \int dx\, f(x) \Big\{ -\nabla \cdot [\eta_i \rho_i(xt)] + D \nabla^2 \rho_i \Big]$$

$$= \int dx \frac{\partial \rho_i(x,t)}{\partial t} f(x)$$

$$\therefore \quad \partial_t \rho_i = -\nabla \cdot (\rho_i \eta_i) + D \nabla^2 \rho_i$$

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$$\Rightarrow \partial_t \rho = D\nabla^2\rho \underbrace{- \sum_i \nabla \cdot [\rho_i(xt)\eta_i]}$$

how do we deal with that ?

Define $h(xt) = -\sum \nabla \cdot [\rho_i(xt)\eta_i]$

linear combination of gamma noises.

$$\cancel{\langle h(x0)\, h(xt)\rangle = 2D\,\delta(t)\,\nabla}$$

$$\langle h(x0)\, h(x't)\rangle = 2D\,\delta(t)\, \nabla_x \cdot \nabla_{x'} [\rho(xt)\rho(x't)]$$

But $\rho_i(xt)\rho_i(x't) = \delta(x-x')\,\rho_i(xt)$

$$= \delta(x-x')\,\rho_i(x't)$$

$$\Rightarrow 2D\,\delta(t-t')\,\nabla_x \cdot \nabla_{x'} [\delta(x-x')\,\rho(xt)]$$

~~ρ~~ Define $\xi(xt) = \nabla \cdot [\underline{\eta}(xt)\,\rho^{1/2}(xt)]$

where $\underline{\eta}(00)\,\underline{\eta}(xt) = 2D\,\delta(t)\,\delta(\underline{x})\,\underline{\underline{1}}$

Then $$\boxed{\dot{\rho} = -\nabla \cdot (\underline{\eta}\,\rho^{1/2}) + D\nabla^2\rho}$$

Similar but messier algebra for nematic

(Bertin et al NJP 2013)

$$\partial_t \rho = \frac{1}{2} \underline{\underline{\Gamma}} : \underline{\underline{W}} + \frac{1}{2} \Delta \rho + \nabla \cdot (\underline{\underline{K}} \cdot \underline{h})$$

$$\underline{\underline{K}} \cdot \underline{\underline{K}} = \frac{\rho}{2} \underline{\underline{I}} + \underline{\underline{W}} = \frac{\rho}{2} \left(\underline{\underline{I}} + 2 \underline{\underline{Q}} \right)$$

$$\langle \underline{h}(0 0) \underline{h}(\underline{x} t) \rangle = 2 \, \underline{\underline{1}} \, \delta^d(\underline{x}) \, \delta(t)$$

FD rel$^{\underline{n}}$

$$\underline{\underline{\Gamma}} : \underline{\underline{W}} \rightarrow \nabla_i \nabla_j (\rho Q_{ij}) = \nabla_i (Q_{ij} \nabla_j \rho) + \nabla_i (\rho \nabla_j Q_{ij})$$

"Wet" active matter

Recall general framework for coupled dynamics of particles $\{\underline{R}_\alpha\},\{\underline{P}_\alpha\}$ + fluid (velocity $\underline{u}(\underline{r},t)$)

$$\dot{\underline{R}}_\alpha = \underline{P}_\alpha / m$$

$$\dot{\underline{P}}_\alpha + \Gamma\left[\frac{\underline{P}_\alpha}{m} - u(\underline{R}_\alpha(t),t)\right] = \underbrace{\underline{F}_\alpha(\{R_\beta\})}_{\text{External + interaction}} + \underbrace{\underline{f}_\alpha(t)}_{\text{noise}}$$

$$\rho(\dot{\underline{u}} + \underline{u}.\nabla\underline{u}) = \eta\nabla^2\underline{u} + \sum_\alpha \delta(\underline{r}-\underline{R}_\alpha(t))\,\underline{F}_\alpha - \nabla p + \underbrace{\underline{h}(\underline{r},t)}_{\text{noise}}$$

So far: replace $\underline{u}$ by $\langle\underline{u}\rangle = 0$ "Rouse" "Dry"

Now include $\underline{u}$. Not in microscopic theory.

Write down coarse-grained theory directly.

Recall general dynamical equations for coords q, momenta p, coupled to chemistry X.

$$\dot{q} + \gamma \partial_q H = \partial_p H + \Theta,$$

$$\dot{p} + \Gamma \partial_p H \boxed{- \frac{\Gamma_{12}(q)}{\Gamma_{22}} \partial_x H} = -\partial_q H + f,$$

$$\mathring{X} + \frac{\Gamma_{12}(q)}{\Gamma_{22}} \partial_p H = -\frac{1}{\Gamma_{22}} \partial_x H + \frac{\xi}{\Gamma_{22}} ;$$

Aside: to calculate entropy production need all 3

that is,

$$\dot{q} + \gamma \partial_q H = \partial_p H + \Theta$$

$$\dot{p} + \Gamma \partial_p H + \frac{\Delta\mu}{\Gamma_{22}} \Gamma_{12}(q) = -\partial_q H + f$$

Apply this to fluid with orientational degrees of freedom (apolar, but can do polar similarly)

$q \longrightarrow$ traceless symmetric $\underline{\underline{Q}}$

$p \longrightarrow \underline{g} = \rho \underline{u}$ = momentum density

$\Longrightarrow \Gamma_{12}(q) \longrightarrow \gamma_{12} \nabla \cdot \underline{\underline{Q}}$ (lowest-gradient vector you can make from $\underline{\underline{Q}}$)

Hydrodynamic equations

$$\left(\partial_t + \underline{u}\cdot\nabla\right)\underline{\underline{Q}} - [\underline{\underline{\Omega}}, \underline{\underline{Q}}] = \left(\lambda_0 \underline{\underline{\kappa}} + \lambda_1 \underline{\underline{\kappa}}\,\underline{\underline{Q}}\right)_{ST} - \Gamma \frac{\delta H}{\delta \underline{\underline{Q}}} + \underset{\text{noise}}{\underline{\underline{f}}}$$

$$\boxed{\begin{array}{ll} \underline{\underline{\Omega}} = (\nabla \underline{u})_{\text{antisym}} & ST = \text{symmetrized traceless} \\ \underline{\underline{\kappa}} = (\nabla \underline{u})_{\text{sym}} & \end{array}}$$

$$\rho\left(\partial_t + \underline{u}\cdot\nabla\right)\underline{u} = \eta \nabla^2 \underline{u} - \nabla p - \nabla\cdot\underline{\underline{\sigma}}^Q + \underset{\text{noise}}{\underline{\xi}}$$

$$\underline{\underline{\sigma}}^Q = -\left(\lambda_0 \frac{\delta H}{\delta \underline{\underline{Q}}} + \lambda_1 \underline{\underline{Q}} \frac{\delta H}{\delta \underline{\underline{Q}}}\right)_{ST}$$

$$+ \ \sigma_0 \underline{\underline{Q}} \quad \left(+ \sigma_1 \nabla \underline{p} \ \ldots \text{ if polar}\right)$$

$$\sigma_0 = \frac{\gamma_{12}}{\Gamma_{22}} \Delta\mu$$

$H =$ Ginzburg-Landau-deGennes free energy functional for $\underline{\underline{Q}}$, produces nematic phase etc. Discussed last lecture

Local anisotropic structure $\underline{\underline{Q}}$

$\Rightarrow$ local anisotropic stress $\sigma_0 \underline{\underline{Q}}$

$\Rightarrow$ Macroscopic spontaneously aligned state has macroscopic uniaxial stress.

Deeply nonequilibrium effect. Breaks Pascal's Law.

Equilibrium nematic liquid crystal has $\underline{\underline{Q}} \neq 0$ but stress at equilibrium is isotropic like in any liquid.

(a) Consequences?

(b) Microscopic origin?

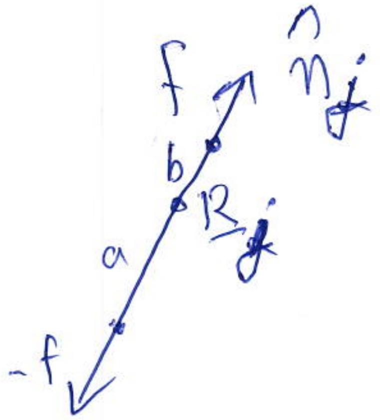
First ~~(b)~~

Microscopic origin of active stress: swimmers

Force-free swimmers.

Newton $\underline{\overline{III}}$ $\longrightarrow$ total force on organism + fluid $= 0$

$\therefore$ force dipoles are the minimal possibility.



$a \neq b$ to get net movement

$$(\text{Force density})_\alpha = f\,\hat{n}_{j\alpha}\left[\delta(\underline{r}-\underline{R}_j-b\hat{n}_j)-\delta(\underline{r}-\underline{R}_j+a\hat{n}_j)\right]$$

Consider a collection of these, construct collective force density $\sum_j (\text{Force density})_j \longrightarrow$

Expand: $\delta(\underline{r}-\underline{R}_j - b\hat{n}_j) - \delta(\underline{r}-\underline{R}_j + a\hat{n}_j)$

$$= -(a+b)\,\hat{n}_j \cdot \nabla\, \delta(\underline{r}-\underline{R}_j) + O(\nabla\nabla)$$

$$\Rightarrow \sum_j (\text{Force density})_j = -(a+b) f\, \nabla\cdot \sum_j \hat{n}_j \hat{n}_j\, \delta(\underline{r}-\underline{R}_j)$$

$$+ \frac{(a+b)(a-b)}{2} f\, \nabla\nabla : \sum n_\alpha n_\alpha n_\alpha\, \delta(\;)$$

$$+ \cdots$$

First term $= -\nabla\cdot \underline{\underline{\sigma}}^{Q} + \nabla(\text{scalar})$

$$\underline{\underline{\sigma}}^{Q} = (a+b) f \sum_j \left(\hat{n}_j \hat{n}_j - \frac{1}{d}\underline{\underline{1}}\right) \delta(\underline{r}-\underline{R}_j)$$

$$= (a+b) f\, c(\underline{r}\,t)\, \underline{\underline{Q}}(\underline{r}\,t).$$

Stress $\propto$ orientation as promised.

Higher orders in ∇ :

$$\sigma^{polar} \sim (a-b)\nabla \underline{p}$$

need this for swimming

Consequences :

* aligned state unstable
* swimming changes viscosity
* Independent swimmers → long-ranged correlations of fluid velocity

etc.

* Interesting effects in confined 2d systems

Instability of aligned state:

Linearize ~~about~~ $\underline{\underline{Q}} = S\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + S'\begin{pmatrix}0 & 2\theta\\ -2\theta & 0\end{pmatrix}$

Linearize $\underline{u}$ eq^n and drop inertia

$$\eta \nabla^2 \underline{u} = \nabla p + \nabla\cdot\underline{\underline{\sigma}}^Q$$

$$= \nabla p + \sigma_0 \nabla\cdot\underline{\underline{Q}} - \underbrace{\nabla\cdot\left(\lambda_0 \frac{\partial H}{\delta \underline{\underline{Q}}} + \lambda_1 \underline{\underline{Q}}\frac{\delta H}{\delta \underline{\underline{Q}}}\right)}_{\text{linearize this too}}$$

Solve for $\underline{u}$ in terms of $\underline{\underline{Q}}$,

Replace $\underline{\underline{\Omega}}$ and $\underline{\underline{\kappa}}$ in $\underline{\underline{Q}}$ eq^n by $\frac{1}{2}(\nabla u - \nabla u^T)$

and $\frac{1}{2}(\nabla u + \nabla u^T)$

Find

sign?

nematic elastic constant

$$\partial_t \theta = \sigma_0 \left(\begin{matrix}\text{factors depending on}\\ \lambda_0, \lambda_1\end{matrix}\right)\cos 2\phi - Kq^2\theta$$

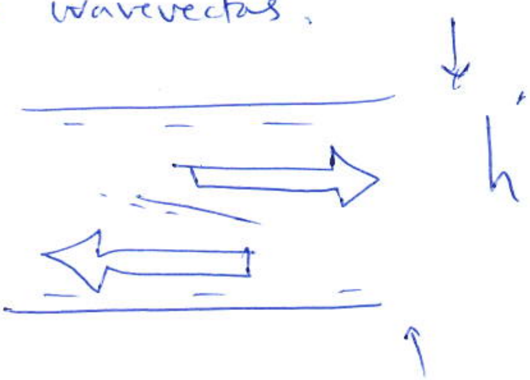
ϕ = angle between mean ordering direction

and wavevector.

Channel geometry

spontaneous flow.

if $h \gtrsim \sqrt{\dfrac{K}{\sigma_0}}$



Viscosity of isotropic active fluid depends on activity

$$\partial_t \underline{\underline{Q}} = \underbrace{-\Gamma a \underline{\underline{Q}}}_{\text{from } -\Gamma \frac{\delta H}{\delta \underline{\underline{Q}}}} + \lambda_0 \underline{\underline{\kappa}}$$

$$-\eta \nabla^2 \underline{u} = -\nabla p - \sigma_0 \nabla \cdot \underline{\underline{Q}} + \cdots ; \nabla \cdot \underline{u} = 0$$

steady state with imposed flow

$$\Rightarrow \quad \underline{\underline{Q}} = \frac{\lambda_0}{\Gamma a} \underline{\underline{\kappa}} = \frac{\lambda_0}{2\Gamma a} (\nabla u + \nabla u^T)$$

$$\Rightarrow \quad \nabla \cdot \underline{\underline{Q}} = \frac{\lambda_0}{2\Gamma a} (\nabla^2 u + \nabla \nabla \cdot u)$$

$$= \frac{\lambda_0}{2\Gamma a} \nabla^2 \underline{u}$$

$$\Rightarrow \quad \eta_{eff} = \eta + \frac{\sigma_0 \lambda_0}{2\Gamma a} = \eta + \sigma_0 \tau$$

relax^{n} time of nematic order.

enhance or reduce viscosity depending on $\mathrm{sgn}(\sigma_0)$.

Lots of experiments now.

Independent "run & tumble" swimmers

with no alignment $\longrightarrow$ long-range velocity correlations

 $\Rightarrow$ fluctuating $\underline{\underline{Q}}$ field: zero space correl, time correl: τ

$$\rho \partial_t u - \eta \nabla^2 \underline{u} = -\nabla p - \sigma_0 \nabla \cdot \underline{\underline{Q}}$$

$$\langle \underline{\underline{Q}}(0\,0)\, \underline{\underline{Q}}(\underline{r}\, t) \rangle = \underline{\underline{C}}\, e^{-|t|/\tau}$$

$$\Rightarrow \quad \langle \underline{u}(0)\, \underline{u}(\underline{r}) \rangle \sim \frac{e^{-r/\xi}}{r}$$

$d = 3$

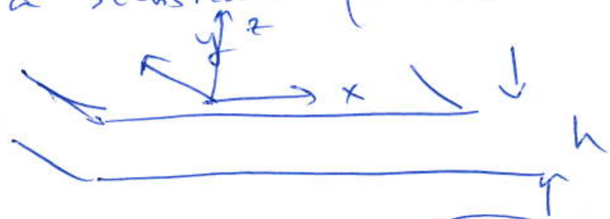
$$\xi = \sqrt{\frac{\eta}{\rho}\tau} \qquad \left(= \infty \text{ in the strict Stokesian limit } \rho = 0 \right)$$

M Graham 2005

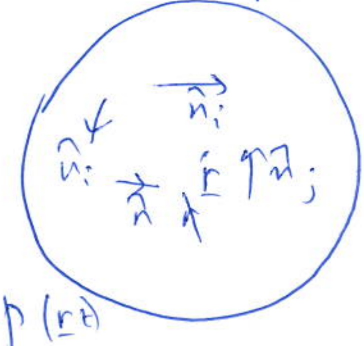
Lau & Lubensky 2008

Dadhichi, Rao, SR unpublished . . .

Active polar fluid on a substrate (or between two plates)



<u>Polar</u> system: order parameter



$\langle \hat{n}_i(t) \rangle_{\underline{r}} = \underline{p}(\underline{r}, t)$

Integrate 3d equations over thin z direction

$$\rho \partial_t \underline{u} = -(\Gamma - \eta \nabla^2)\underline{u} + \alpha \underline{p} + \dots$$

$$\cancel{\rho}\, \partial_t \underline{p} = \lambda \underline{u} - (a - K\nabla^2)\underline{p} + \dots$$

α: local orientation $\Rightarrow$ local forcing

λ: local flow reorients . . .

$\alpha\lambda$ large & positive $\Rightarrow$ spontaneous alignment.