p-adic uniformization of locally symmetric spaces

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Motivation

- Let $N \in \mathbb{Z}_{\geq 1}$. Let $f \in S_2(\Gamma_0(N))$ be a Hecke eigenform with Fourier coefficients in $\mathbb{Z}$. Let $p \neq \ell$ be primes such that $(p\ell, N) = 1$.
- **Question :** Does there exist a "p-new" Hecke eigenform $g \in S_2(\Gamma_0(Np))$ with coefficients in $\mathbb{Z}$ such that "$g \cong f(mod\ \ell)$"?
- The congruence "$g \cong f(mod\ \ell)$" can be framed as $a_n(g) \cong a_n(f)$ $\forall (n, N) = 1$. More importantly, it can also be framed in terms of the automorphic/Galois representations associated to f and g.
- **Answer :** (Ribet 1983) There exists g satisfying such a congruence as long as a necessary local condition arising from the representations at p is satisfied, i.e. "level-raising" of modular forms is possible.

Motivation - Why uniformization?

- Ribet's proof uses Ihara's Lemma - a result describing the image of the sum of the two p-degeneracy maps from $J_0(N)$ to $J_0(Np)$ - in a crucial way.
- **Caveat :** For higher rank groups such as GL_n for $n > 2$, analogues of Ihara's lemma are not known in general.
- Thorne(2014) proves new cases of Ihara's lemma, and proves instances of level-raising for automorphic forms on $GL_n(\mathbb{A}_E)$, where E belongs to a certain class of CM fields.
- One of the ingredients in his method is use of p-adic uniformization results, à la Rapoport-Zink, of Kottwitz-Harris-Taylor unitary similitude Shimura varieties.
- In order to generalize his method to all CM fields E, we prove uniformization results for locally symmetric spaces arising from true unitary groups.

What is uniformization?

- Uniformization in general refers to taking quotients of an analytic space under the action of discrete arithmetic groups.
- The interesting cases occur when one can describe complex or p-adic points of an algebraic variety via such a uniformization.

Examples

- Complex points of an elliptic curve E can be described as $E(\mathbb{C}) \cong \mathbb{C}/\Lambda_\tau$, where Λ_τ is the lattice generated by $\{1, \tau\}$.
- Tate curve : $E(\mathbb{C}_p) \cong \mathbb{C}_p^\times / q^{\mathbb{Z}}$, for an elliptic curve $E/\mathbb{Q}_p$ with split multiplicative reduction, is an example of p-adic uniformization.
- (Cerednik 1976) Shimura curves arising from indefinite quaternion algebras B over $\mathbb{Q}$, that are ramified at p, can be p-adically uniformized by the Drinfeld upper half plane $\Omega^1 := \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(\mathbb{Q}_p)$.

Unitary Shimura varieties and p-adic uniformization

- Let E be a CM imaginary field with totally real subfield F with $[F:\mathbb{Q}] = d$. Let D be a central division algebra of dimension n^2 over E. Let $*$ be a positive involution of the second kind on D so that the invariants of $*$ on E is the subfield F.
- We fix a rational prime p. For now, assume p is unramified in E, and that there exists a place ν of F above p that splits in E as $\nu = \omega\omega^c$. We assume that $inv_\omega D = 1/n$, $inv_{\omega^c} D = -1/n$, and that for every other finite place, D is split.
- Let V be a free left D-module of rank 1 and ψ an alternating pairing $\psi : V \times V \to \mathbb{Q}$ satisfying

$$\psi(dv, w) = \psi(v, d^* w)$$

for all $d \in D, v, w \in V$.

- Define an algebraic group $\tilde{G}$ over $\mathbb{Q}$ by its functor of points :

  $$\tilde{G}(R) := \{g \in GL_D(V \otimes R) \mid \psi(gv, gw) = c(g)\psi(v, w), c(g) \in R^{\times}\}.$$

  where R is a $\mathbb{Q}$-algebra. This is a group of unitary similitudes. We assume it has signature $(1, n-1) \times (0, n)^{d-1}$ at ∞.
- **Fact :** There exists a Shimura datum $(\tilde{G}, X)$ associated to $\tilde{G}$.
- (Kottwitz 1992) For $\tilde{U} \subset \tilde{G}(\mathbb{A}^{\infty})$ a neat open compact subgroup, the Shimura varieties $S(\tilde{G}, \tilde{U})(\mathbb{C}) := \tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A}_f) \times X/\tilde{U}$ are smooth projective algebraic varieties over $\mathbb{C}$ and admit canonical models over F, denoted by $S(\tilde{G}, \tilde{U})$.

Unitary Shimura varieties and p-adic uniformization

- (Rapoport-Zink 1996) Assume p is inert in F (+ other hypotheses). Let ν be the unique place of F over p. Then, for each sufficiently small open compact subgroup $\tilde{U}^p \subset \tilde{G}(\mathbb{A}^{p,\infty})$, there exists an integral canonical model $\mathcal{S}$ of $S(\tilde{G}, \tilde{U}^p \tilde{U}_p) \otimes_F F_\nu$ and a canonical isomorphism of formal schemes:

$$\tilde{I}(\mathbb{Q}) \backslash \Omega^n \times \tilde{G}^{p,\infty} / \tilde{U}^p \cong (\mathcal{S} \otimes_{O_{F_\nu}} O_{\mathbb{F}})^\wedge$$

where $\tilde{I}$ is an inner form of $\tilde{G}$ such that $\tilde{G}^{p,\infty} \cong \tilde{I}^{p,\infty}$, Ω^n is the n-dimensional Drinfeld p-adic upper half plane, and $\mathbb{F}$ denotes the completion of the maximal unramified extension of F_ν.

- Let G over $\mathbb{Q}$ be defined by its functor of points :

$$G(R) := \{g \in GL_D(V \otimes R) \mid \psi(gv, gw) = \psi(v, w)\}$$

where R is a $\mathbb{Q}$-algebra. There is an exact sequence of algebraic groups over $\mathbb{Q}$:

$$1 \to G \to \tilde{G} \xrightarrow{c} \mathbb{G}_m \to 1.$$

- Now, we suppose that p is unramified in F, as opposed to the stricter inertness condition before. For $\tilde{U} = \tilde{U}^p \tilde{U}_p$ as above, we set $U := G(\mathbb{A}^\infty) \cap \tilde{U}$. Define

$$S(G, U)(\mathbb{C}) := G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty) \times X / U.$$

Statement of Theorem

- **Theorem :**(K.) For each U satisfying our hypotheses, there exists a canonical model $\mathfrak{S}$ of the complex variety $S(G, U)(\mathbb{C})$ over L where L is a finite extension of F unramified at p. Furthermore, this model admits p-adic uniformization by the Drinfeld p-adic upper half plane in the sense of Rapoport-Zink.
- Application of this result to the problem of level-raising is work in progress.

Thank You!