MODULI OF VECTOR BUNDLES ON COMPACT RIEMANN SURFACES

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Mathematics & Physics

"The interaction between mathematics and physics is a two-way process, with each of the two subjects drawing from and inspiring the other." [1]

Will now speak about such an interaction between some parts of algebraic & differential geometry on one side, and gauge theory and conformal field theory on the other.

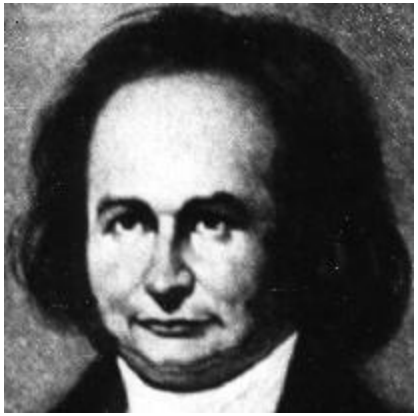
Also related to number theory, Langlands program.

1. E. Frenkel

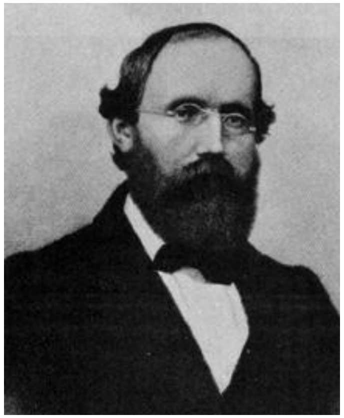
Flat Unitary Bundles on a Compact Riemann Surface

Vast "non-abelian" generalization of work of Abel, Jacobi & Riemann on a compact Riemann surface, envisaged by Andre Weil.

Abel



Jacobi



Riemann

Weil

Classical theory:

Constructed Jacobian, a complex torus, using periods of abelian differentials.

Character group of the first homology group of the surface.

Non-abelian Generalization

- Consider unitary representations of the fundamental group of the surface (instead of the character group).

- These give rise to flat (holomorphic) bundles on the Riemann surface.

- This space of n-dimensional unitary flat bundles $M(n)$, generalizes the Jacobian. (For $n=1$ we get the character group of the homology group).

Non-abelian Generalization

- M is a compact topological space and any such representation gives a holomorphic vector bundle (of Chern class zero) on the Riemann surface; such a bundle is called a flat unitary bundle.

- Using the complex structure on the Riemann surface we can put a complex structure (or even an algebraic structure) on $M(n)$.

- This algebraic variety will be called the moduli space of flat bundles. ($M(1)$ is the Jacobian.)

Stable Bundles & Unitary Flat Bundles

- To construct the algebraic structure one has to have an algebraic characterisation of unitary flat bundles (which are "transcendental objects“).

- This was given by a theorem by Seshadri and myself.

Narasimhan-Seshadri Theorem

DEFINITION: A holomorphic vector bundle of Chern class zero on a compact Riemann surface is said to be stable if the Chern class of every proper holomorphic subbundle has strictly negative Chern class. (Semistablity is defined by replacing strictly negative by ≤ 0).

Informally every subbundle is less positive than the original bundle.

THEOREM: A vector bundle of degree zero is stable if and only if it is a flat unitary bundle arising from an irreducible unitary representation.

Bundles of arbitrary Chern class

- We can also define semistable and stable bundles of arbitrary Chern class and moduli space of these bundles can be constructed.
- A (holomorphic) vector bundle V on X is said to be stable (respectively semi-stable) if for every proper sub-bundle W of V, we have

$$\frac{\deg\ W}{\text{rank}\ W} < \frac{\deg\ V}{\text{rank}\ V} \quad \left(\text{resp. } \frac{\deg\ W}{\text{rank}\ W} \leq \frac{\deg\ V}{\text{rank}\ V}\right)$$

- There is an analogue of the Narasimhan-Seshadri theorem for these bundles too.
- There is a vast literature studying properties of these moduli spaces.

Higgs Bundles

If we look at representations into $GL(n,C)$ (instead of $U(n)$), it was discovered by Hitchin that the corresponding algebro-geometric counterpart is a "stable" pair (E,f) where E is a holomorphic vector bundle and f is a "Higgs Field", i.e., f is a homomorphism from E to the tensor product of E with the canonical bundle of the surface.

Higgs Bundles

These Higgs moduli spaces have been found to be very useful in number theory, e.g., in connection with the proof of the so-called "Fundamental Lemma".

Gauge Theory

- 2-d gauge theory and the associated solutions of Yang-Mills equations (with the unitary group as gauge group) are closely related to the moduli of bundles on Riemann surfaces.

- Atiyah and Bott made a detailed study of Yang-Mills on Riemann surfaces and were instrumental in popularising moduli of stable vector bundles on Riemann surfaces among physicists.

Conformal Field Theory

- Just as classical theta functions are holomorphic sections of line bundles on the Jacobian, it is reasonable to expect a theory of generalised "non-abelian" theta functions, which would be holomorphic sections of line bundles on moduli spaces of bundles.

- Such a theory was developed by algebraic geometers.

Conformal Field Theory

- It turns out that these generalised theta functions are the same as conformal blocks defined by physicists using representations of Kac-Moody algebras.

- The famous Verlinde formula for the dimension of conformal blocks yields the dimension of linear systems on the moduli space of flat bundles.

Geometric Hecke Correspondence

- There is an algebraic correspondence between moduli spaces of vector bundles of different degrees.
- This was introduced by myself and Ramanan and is now quite popular in connection with Geometric Langlands correspondence. This is the key tool for studying the properties of moduli spaces.
- This correspondence associates to a vector bundle on a surface and a point of the corresponding projective bundle, a new vector bundle of degree one less, by using “elementary transformation”.

Derived Categories, Stability & Branes

- Recently, I have been interested in derived categories (of coherent sheaves) on projective varieties, in particular on moduli spaces of vector bundles.
- Physicists are interested in these derived categories and stability conditions in them, apparently since "this category could be obtained as a category of boundary conditions in the B-type topologically twisted sigma model on the variety“.
- Be that as it may, the derived category of coherent sheaves is an interesting object for mathematicians.

Derived Categories, Stability & Branes

- Let me end with a recent theorem of mine:

The bounded derived category of a compact Riemann surface of genus ≥ 4 can be embedded in that of moduli space of rank 2 bundles with fixed determinant of odd degree.

- (In fact the Fourier-Mukai transform defined by a Poincare bundle gives a fully faithful embedding).