K-groups over Number Fields

Haiyan Zhou

School of Mathematical Science, Nanjing Normal University, China

Dec. 17, 2016

Contents

Tame Kernels

Let F be a number field, $\mathcal{O}_F$ its ring of integers. The problem of computing the higher K-groups of a number field F, and of its rings of integers $\mathcal{O}_F$, has a rich history (See [1], [4], [5] and [17]). Quillen [17] proved that for all $n > 0$ the K-theory groups $K_n(\mathcal{O}_F)$ are finitely generated. There are various conjectures about their torsion subgroups. One of them, due to Lichtenbaum, says: **Lichtenbaum Conjecture:** For all $n \geq 2$,

$$\zeta_F^*(1-n) = \pm \frac{|K_{2n-2}(\mathcal{O}_F)|}{|K_{2n-1}(\mathcal{O}_F)_{\mathrm{tors}}|} R_n^B(F)$$

up to powers of 2, where $R_n^B(F)$ is the Borel regulator, $\zeta_F^*(1-n)$ is the first non-vanishing coefficient in the Taylor-expansion of zeta-function $\zeta_F(s)$ at $s = 1-n$.

Tame Kernels

Let F be a number field, $\mathcal{O}_F$ its ring of integers. The problem of computing the higher K-groups of a number field F, and of its rings of integers $\mathcal{O}_F$, has a rich history (See [1], [4], [5] and [17]). Quillen [17] proved that for all $n > 0$ the K-theory groups $K_n(\mathcal{O}_F)$ are finitely generated. There are various conjectures about their torsion subgroups. One of them, due to Lichtenbaum, says: **Lichtenbaum Conjecture:** For all $n \geq 2$,

$$\zeta_F^*(1-n) = \pm \frac{|K_{2n-2}(\mathcal{O}_F)|}{|K_{2n-1}(\mathcal{O}_F)_{\text{tors}}|} R_n^B(F)$$

up to powers of 2, where $R_n^B(F)$ is the Borel regulator, $\zeta_F^*(1-n)$ is the first non-vanishing coefficient in the Taylor-expansion of zeta-function $\zeta_F(s)$ at $s = 1 - n$.

Tame Kernels

For $n = 1$, there is an exact sequence as follows (See [18]):

$$0 \to K_2(\mathcal{O}_F) \to K_2(F) \overset{\oplus\tau_{\mathcal{P}}}{\to} \oplus_{\mathcal{P} \text{ finite}}(k_{\mathcal{P}})^* \to 0.$$

The map $\oplus\tau_{\mathcal{P}}$ is explicitly given by the so-called tame symbol: For each prime ideal $\mathcal{P}$ of $\mathcal{O}_F$, the map

$$\tau_{\mathcal{P}}: \ K_2(F) \to k_{\mathcal{P}}^*$$

defined by

$$\tau_{\mathcal{P}}(a,\ b) = (-1)^{v_{\mathcal{P}}(a)v_{\mathcal{P}}(b)a^{v_{\mathcal{P}}(b)}b^{-v_{\mathcal{P}}(a)}}\mathrm{mod}\mathcal{P},$$

where $v_{\mathcal{P}}$ denotes the $\mathcal{P}$-adic valuation. Therefore $K_2\mathcal{O}_F = \ker(\oplus\tau_{\mathcal{P}})$, and hence $K_2\mathcal{O}_F$ is also called the **tame kernel**.

Tame Kernels

In the classical algebraic number theory, the study of the ideal class group is one of the fundamental problems. The analog of the class group in the realm of K2-functor is the tame kernel, which is much more complicated than the class group. One of the fundamental problems in this areas is the understanding of p-rank of K_2O_F of number fields F. By the 2-rank K_2O_F formula due to J. Tate, Browkin and Schinzel states that for $F = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ square-free, the subgroup of K_2O_F consisting of all elements with order 2 can be generated $\{-1, m\}, m|d (m > 0$ if $d > 0)$ together with $\{-1, u + \sqrt{d}\}$ if $d = u^2 - 2w^2$ with $u, w \in \mathbb{Z}$.

Tame Kernels

In the classical algebraic number theory, the study of the ideal class group is one of the fundamental problems. The analog of the class group in the realm of K2-functor is the tame kernel, which is much more complicated than the class group. One of the fundamental problems in this areas is the understanding of p-rank of K_2O_F of number fields F. By the 2-rank K_2O_F formula due to J. Tate, Browkin and Schinzel states that for $F = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ square-free, the subgroup of K_2O_F consisting of all elements with order 2 can be generated $\{-1, m\}, m|d(m > 0$ if $d > 0)$ together with $\{-1, u + \sqrt{d}\}$ if $d = u^2 - 2w^2$ with $u, w \in \mathbb{Z}$.

Tame Kernels

Qin's following theorem determines the 4-rank of $K_2 O_F$.

Theorem 1.1[10] [11]

Let $F = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ square-free, $S(d) = \{\pm 1, \pm 2\}$ if $d > 0$ or $\{1, 2\}$ if $d < 0$. Suppose that $m|d$ ($m > 0$ if $d > 0$) and write $d = u^2 - 2w^2$ with $u, w \in \mathbb{Z}$ (we take $u > 0$ if $d > 0$) if $2 \in NF$. Then $\{-1, m\} \in K_2 O_F^2$ if and only if one can find an $\epsilon \in S(d)$ such that
(i) $(\frac{dm^{-1}}{p}) = (\frac{\epsilon}{p})$, for any odd prime $p|m$,
(ii) $(\frac{m}{p}) = (\frac{\epsilon}{p})$, for any odd prime $p|dm^{-1}$;
and $\{-1, m(u + \sqrt{d})\} \in K_2 O_F^2$ if and only if one can find a $\delta \in S(d)$ such that
(iii) $(\frac{dm^{-1}}{p}) = (\frac{\delta(u+w)}{p})$, for any odd prime $p|m$,
(iv) $(\frac{m}{p}) = (\frac{\delta(u+w)}{p})$,, for any odd prime $p|dm^{-1}$.

Tame Kernels

Indeed, Qin introduced sign matrices via Legendre symbols to determine the 4-rank of K_2O_F where F is the quadratic extension in Theorem 1.1 above, see [19], [20], [12]. In the case of relative quadratic extension , the sign matrices defined via local Hilbert symbols to compute the 4-rank of the tame kernel was the work of Hurrelbrink and Kolster. In [14], [13], Qin determined the 8-rank of K_2O_F completely.

Tame Kernels

For any odd prime p, we shall consider the following short exact sequence

$$0 \to (\mu_p \oplus CL(O_E[1/p]))^G \to K_2O_F/p \to \oplus_{p \in S'} \mu_p \to 0,$$

where $E = F(\zeta_p)$, $G = Gal(E/F)$ acts on $\mu_p \oplus CL(O_E[1/p])$ by the formula

$$(\zeta \oplus x)^g = \zeta^g \oplus x^g, \text{for } \zeta \in \mu_p,\ g \in G,\ x \in CL(O_E[1/p]),$$

and S' is the set of prime ideals of F which divide p and split completely in E(See [7]). Based on appropriate reflection theorems and on the above exact sequence , some authors proved several results connecting the p-rank K_2O_F with the p-rank of $(CL(O_K))$, where K is an appropriate subfiled of $F(\zeta_p)$. In the case of quadratic fields and cubic fields, $p = 3,\ 5$ the results are explicit(See [2], [21], [16],[15]).

Tame Kernels

For any odd prime p, we shall consider the following short exact sequence

$$0 \to (\mu_p \oplus CL(O_E[1/p]))^G \to K_2O_F/p \to \oplus_{p\in S'}\mu_p \to 0,$$

where $E = F(\zeta_p)$, $G = Gal(E/F)$ acts on $\mu_p \oplus CL(O_E[1/p])$ by the formula

$$(\zeta \oplus x)^g = \zeta^g \oplus x^g, \text{for } \zeta \in \mu_p,\ g \in G,\ x \in CL(O_E[1/p]),$$

and S' is the set of prime ideals of F which divide p and split completely in E(See [7]). Based on appropriate reflection theorems and on the above exact sequence , some authors proved several results connecting the p-rank K_2O_F with the p-rank of $(CL(O_K))$, where K is an appropriate subfiled of $F(\zeta_p)$. In the case of quadratic fields and cubic fields, $p = 3,\ 5$ the results are explicit(See [2], [21], [16],[15]).

Tame Kernels

For any odd prime p, we shall consider the following short exact sequence

$$0 \to (\mu_p \oplus CL(O_E[1/p]))^G \to K_2 O_F/p \to \oplus_{p \in S'} \mu_p \to 0,$$

where $E = F(\zeta_p)$, $G = Gal(E/F)$ acts on $\mu_p \oplus CL(O_E[1/p])$ by the formula

$$(\zeta \oplus x)^g = \zeta^g \oplus x^g, \text{for } \zeta \in \mu_p,\ g \in G,\ x \in CL(O_E[1/p]),$$

and S' is the set of prime ideals of F which divide p and split completely in E(See [7]). Based on appropriate reflection theorems and on the above exact sequence , some authors proved several results connecting the p-rank $K_2 O_F$ with the p-rank of $(CL(O_K))$, where K is an appropriate subfiled of $F(\zeta_p)$. In the case of quadratic fields and cubic fields, $p = 3,\ 5$ the results are explicit(See [2], [21], [16],[15]).

Tame Kernels

For any odd prime p, we shall consider the following short exact sequence

$$0 \to (\mu_p \oplus CL(\mathcal{O}_E[1/p]))^G \to K_2\mathcal{O}_F/p \to \oplus_{p\in S'}\mu_p \to 0,$$

where $E = F(\zeta_p)$, $G = Gal(E/F)$ acts on $\mu_p \oplus CL(\mathcal{O}_E[1/p])$ by the formula

$$(\zeta \oplus x)^g = \zeta^g \oplus x^g, \text{for } \zeta \in \mu_p,\ g \in G,\ x \in CL(\mathcal{O}_E[1/p]),$$

and S' is the set of prime ideals of F which divide p and split completely in E(See [7]). Based on appropriate reflection theorems and on the above exact sequence , some authors proved several results connecting the p-rank $K_2\mathcal{O}_F$ with the p-rank of $(CL(\mathcal{O}_K))$, where K is an appropriate subfiled of $F(\zeta_p)$. In the case of quadratic fields and cubic fields, $p = 3,\ 5$ the results are explicit(See [2], [21], [16],[15]).

Tame Kernels

Qin [15] proved a reflection theorem which for p=3 is Scholz Reflection Theorem. He obtained some formulae for the p^n-rank of K_2O_F. With the help of this reflection theorem, he established relations between the p^n -divisibility of the order of K_2O_F and the p^n-divisibility of the class number of some algebraic number fields.

Tame Kernels

Let F/K be a Galois extension of a number field of degree n. We studied the structure of the odd part of the tame kernel K_2O_F of F by using the intermediate fields of F/K.

Theorem 1.2[22]

Let F/K be an abelian extension with Galois group G of order n and $p \nmid n$. Then $(K_2O_F)_p = \sum(K_2O_E)_p$, where E runs over all intermediate fields cyclic over K.

Proposition 1.3[22]

Let $F = \mathbb{Q}(\sqrt{d_1}, \cdots, \sqrt{d_n})$, where $d_1, \cdots, d_n$ are square-free integers and $G = Gal(F/\mathbb{Q})$. If p is an odd prime, then $(K_2O_F)_p = \bigoplus(K_2O_E)_p$, where E runs over all quadratic subfields of F.

Tame Kernels

Let F/K be a Galois extension of a number field of degree n. We studied the structure of the odd part of the tame kernel K_2O_F of F by using the intermediate fields of F/K.

Theorem 1.2[22]

Let F/K be an abelian extension with Galois group G of order n and $p \nmid n$. Then $(K_2O_F)_p = \sum(K_2O_E)_p$, where E runs over all intermediate fields cyclic over K.

Proposition 1.3[22]

Let $F = \mathbb{Q}(\sqrt{d_1}, \cdots, \sqrt{d_n})$, where $d_1, \cdots, d_n$ are square-free integers and $G = Gal(F/\mathbb{Q})$. If p is an odd prime, then $(K_2O_F)_p = \bigoplus(K_2O_E)_p$, where E runs over all quadratic subfields of F.

Tame Kernels

Recall the well-known Birch-Tate conjecture:

$$\#K_2\mathcal{O}_F = w_2(F)|\zeta_F(-1)|$$

for every totally real number field F, where ζ_F is the Dedekind zeta function of F, and $w_2(F)$ is the maximal order of the roots of unity belonging to the compositum of all quadratic extensions of F. The conjecture is known to be true when F is abelian over $\mathbb{Q}$ and is known to be true in general up to a power of 2. By Proposition 1.3, the order of the group K_2O_F can be computed for a real multiquadratic field.

Tame Kernels

Recall the well-known Birch-Tate conjecture:

$$\#K_2\mathcal{O}_F = w_2(F)|\zeta_F(-1)|$$

for every totally real number field F, where ζ_F is the Dedekind zeta function of F, and $w_2(F)$ is the maximal order of the roots of unity belonging to the compositum of all quadratic extensions of F. The conjecture is known to be true when F is abelian over $\mathbb{Q}$ and is known to be true in general up to a power of 2. By Proposition 1.3, the order of the group $K_2\mathcal{O}_F$ can be computed for a real multiquadratic field.

Tame Kernels

Let E/F be a finite extension of number fields.The transfer $\mathrm{tr}_{E/F}$ was defined which is a group homomorphism $\mathrm{tr}_{E/F} : K_2(E) \to K_2(F)$. Using some basic properties of the transfer mapping in K-theory, we proved some relations among orders of odd parts of tame kernels of some subfields of E/F. We denote by $K_2(E/F)$ the kernel of the map $\mathrm{tr}_{E/F} : K_2O_E \to K_2O_F$.

Tame Kernels

Proposition 1.3[22]

Let l be an odd prime number and E/F be a dihedral extension with Galois group D_l, k its quadratic subfield fixed by $< \sigma >$, K the fixed field of τ and $K^{'}$ the fixed field of $\sigma\tau$. Then for every odd prime $p \neq l$, we have

$$K_2(E/k)(p) \cong K_2(K/F)(p) \times K_2(K^{'}/F)(p),$$

and

$$|K_2(O_E)||K_2(O_F)|^2 =_p |K_2(O_K)|^2|K_2(O_k)|.$$

Higher class numbers in extensions of number fields

It is known that one has the analytic class number formula

$$\zeta_F^*(0) = -\frac{R_1(F)h_F}{w_1(F)},$$

where $w_1(F)$ is the number of roots of unity in F, h_F is the class number of F, $R_1(F)$ is the first regulator of F and $\zeta_F^*(0)$ is the first non-vanishing coefficient in the Taylor-expansion of the zeta-function $\zeta_F(s)$ around $s = 0$.

There are conjectural analogues of the formula when 0 is replaced by negative integers.

Higher class numbers in extensions of number fields

Motivic formulation of the Lichtenbaum Conjecture.

For any number field F and for any integer $n \geq 2$,

$$\zeta_F^*(1-n) = \pm \frac{R_n^M(F) h_n(F)}{w_n(F)},$$

where $h_n(F)$ is the order of the motivic cohomology group $H^2_M(\mathcal{O}_F, \mathbb{Z}(n))$, $w_n(F)$ is the order of the torsion subgroup of the motivic cohomology group $H^1_M(\mathcal{O}_F, \mathbb{Z}(n))$ and $R_n^M(F)$ is the motivic regulator of $H^1_M(\mathcal{O}_F, \mathbb{Z}(n))$.

Higher class numbers in extensions of number fields

We also note that for all $n \geq 2$, the motivic groups $H^2_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Z}(n))$ are finite, $H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Z}(n))$ are finitely generated $\mathbb{Z}$-modules, and

$$d_n = \mathrm{rk}_{\mathbb{Z}}(H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Z}(n))) = \begin{cases} r_1 + r_2, & \text{if } n \text{ is odd,} \\ r_2, & \text{if } n \text{ is even,} \end{cases}$$

where r_1 and r_2 are respectively the numbers of real and complex places of F.

The relationship between motivic cohomology, étale cohomology and K-theory is described via Chern characters

Higher class numbers in extensions of number fields

We also note that for all $n \geq 2$, the motivic groups $H^2_M(\mathcal{O}_F, \mathbb{Z}(n))$ are finite, $H^1_M(\mathcal{O}_F, \mathbb{Z}(n))$ are finitely generated $\mathbb{Z}$-modules, and

$$d_n = \mathrm{rk}_{\mathbb{Z}}(H^1_M(\mathcal{O}_F, \mathbb{Z}(n))) = \begin{cases} r_1 + r_2, & \text{if } n \text{ is odd,} \\ r_2, & \text{if } n \text{ is even,} \end{cases}$$

where r_1 and r_2 are respectively the numbers of real and complex places of F.
The relationship between motivic cohomology, *é*tale cohomology and K-theory is described via Chern characters

Higher class numbers in extensions of number fields

Lemma 2.1([8])

Let $\mathcal{O}_F$ be the ring of integers in a number field F with r_1 real embeddings, and let $n \geq 2$. Then for $i = 1, 2$,
(i) The Chern character

$$K_{2n-i}(\mathcal{O}_F) \to H^i_M(\mathcal{O}_F, \mathbb{Z}(n))$$

is an isomorphism if $2n - i \equiv 0, 1, 2, 7 \pmod 8$, injective with cokernel $\cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$ if $2n - i \equiv 6 \pmod 8$, surjective with kernel $\cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$ if $2n - i \equiv 3 \pmod 8$.

Higher class numbers in extensions of number fields

In the remaining cases ($n \equiv 3 \pmod 4$) there is an exact sequence

$$0 \to K_{2n-2}(\mathcal{O}_F) \to H^2(\mathcal{O}_F, \mathbb{Z}(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \to K_{2n-1}(\mathcal{O}_F)$$

$$\to H^1(\mathcal{O}_F, \mathbb{Z}(n)) \to 0.$$

(ii) $H^i_M(\mathcal{O}_F, \mathbb{Z}(n)) \otimes \mathbb{Z}_p \cong H^i_{ét}(\mathcal{O}_F[\frac{1}{p}], \mathbb{Z}_p(n))$, for all primes p.

Higher class numbers in extensions of number fields

Let F be a finite Galois extension of a number field k with the Galois group G. R. Brauer (in 1951) and S. Kuroda (in 1950) proved independently some multiplicative relations between the Dedekind zeta functions of some subfields of F. For every cyclic subgroup H of G,

$$c_G(H) := \frac{1}{(G:H)} \sum_{H^* \text{cyclic } H \subseteq H^* \subseteq G} \mu((H^* : H)),$$

where μ is the Möbius function. Then

$$\zeta_k(s) = \prod_{H \text{ cyclic } H \subseteq G} \zeta_{F^H}(s)^{c_G(H)},$$

where F^H is the subfield of F fixed by H.

Higher class numbers in extensions of number fields

Let l be a prime number and D the dihedral group of order $2l$. Let $F/\mathbb{Q}$ be a complex Galois extension with Galois group G, where $G = V_4$ or D. When n is even, we gave the Brauer-Kuroda formulae for higher class numbers by an index of the first Motivic cohomology groups using the Brauer-Kuroda relations about zeta-functions and the Motivic formulation of the Lichtenbaum Conjecture.

Higher class numbers in extensions of number fields

Theorem 2.1 [24]

Let F be a complex biquadratic extension of $\mathbb{Q}$ with quadratic subfields F_0, F_1 and F_2, where F_0 is real. Then for $n > 1$ we have

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{\prod_{i=0}^{2} h_n(F_i)} = 1 \text{ or } 2 \text{ or } \frac{1}{2}.$$

Higher class numbers in extensions of number fields

Theorem 2.2 [24]

Let F be a complex Galois extension of $\mathbb{Q}$ with the dihedral Galois group D_l. Let k be the unique quadratic subfield of F and K (resp. K') the subfield of F fixed by $< \sigma >$ (resp. by $< \tau^2\sigma >$). Then if $l = 3$, we have

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{h_n(k)h_n(K)^2} = \frac{1}{3} \text{ or } 1 \text{ or } 3 \text{ or } 9.$$

Higher class numbers in extensions of number fields

Corollary [24]

Let $K = \mathbb{Q}(\sqrt[3]{m})$ and $F = K(\zeta_3)$, where m is a cubefree integers not equal 1 and -1, ζ_3 is a primitive cube root of unity. Then

$$|K_2(O_F)| = \frac{1}{12}|K_2(O_K)|^2 \text{ or } \frac{1}{4}|K_2(O_K)|^2 \text{ or } \frac{3}{4}|K_2(O_K)|^2 \text{ or } \frac{9}{4}|K_2(O_K)|^2.$$

1. A. Borel, **Stable real cohomology of arithmetic groups**, Ann. Sci. École Norm. Sup. 7(1977), 613-636.
2. J. Browkin, **On the p-rank of the tame kernel of algebraic number fields**, J. Reine Angew. Math., 432(1992), 135-149.
3. J. Browkin, **Tame kernels of cubic cyclic fields**, Math. Comp. 74(2005), 967–999.
4. J. Coates, **On K_2 and some classical conjectures in algebraic number theory**, Ann. of Math. 95(1972), 99-116.
5. H. Garland, **A finiteness theorem for K_2 of a number field**, Ann. of Math. 94(1971), 534-548.
6. B. Kahn, **Descente galoisienne et K_2 des corps de nombres**, K-theory 7(1993), 55-100.
7. F. Keune, **On the Structure of the K_2 of the Ring of Integers in a Number Field**, K-Theory 2(1989),625-645.
8. M. Kolster, **K-theory and arithmetic**, Contemporary developments in algebraic K-theory, M. Karoubi, A.O. Kuku, C. Pedrini, ICTP Lecture Notes, 15, 2003, 191-258.
9. J. Milnor, **Introduction to Algebraic K-Theory**, Annals of Math. Studies 72, Princeton University Press, Princeton, 1971.

10. H.R. Qin, **The 2-Sylow subgroups of the tame kernel of imaginary quadratic fields**, Acta Arith. 69(1995), 153-169.

11. H.R. Qin, **The 4-rank of K_2O_F for real quadratic fields**, Acta Arith. 72(1995), 323-333.

12. H.R. Qin, **The structure of the tame kernels of quadratic number fields (I)**, Acta Arith. 113(2004), 203-240.

13. H.R. Qin, **The 2-Sylow subgroup of K_2O_F for number fields** F, J., Algebra 28(2005), 494-519.

14. H.R. Qin, **Tame kernels and Tate kernels of quadratic number fields**, J. Reine Angew. Math. 530(2001), 105-144.

15 H.R. Qin, **Reflection theorems and the p-Sylow subgroup of K_2O_F for a number field** F, J. Pure Appl. Algebra, 214(2010), 1181-1192.

16. H.R. Qin, H.Y. Zhou, **The 3-Sylow subgroup of the tame kernel of real number fields**, J. Pure and Applied Algebra 209(2007), 245-253.

17. D. Quillen, **Finite generation of the groups K_i of rings of algebraic integers**, Lecture Notes in Math. 341, Springer Verlag, 1973, 179-198.

- 18. C. Soulé, **Groupes de Chow at K-théorie de variétés sur un corps fini**, Math. Ann. 268(1984), 317-345.
- 19. X.B. Yin, H.R. Qin, Q.S. Zhu, **The structure of the tame kernels of quadratic number fields (II)**, Acta Arith. 116(2004), 217-262.
- 20. X.B. Yin, H.R. Qin, Q.S. Zhu, **The structure of the tame kernels of quadratic number fields (III)**, Comm. Algebra 36(3)(2008), 1012-1033.
- 21. H. Zhou, **Tame kernels of cubic cyclic fields**, Acta Arith 124(2006), 293-313.
- 22. H.Y. Zhou, **The tame kernel of multiquadratic number fields**, Comm. Algebra 37(2009), 630-638.
- 23. H.Y. Zhou, **Odd parts of Tame kernels of Dihedral extensions**, Acta Arith. 156 (4) (2012), 341-349.
- 24. H.Y. Zhou, **Higher class numbers in extensions of number fields**, Bull. Malays. Math. Sci. Soc., 38(2015), 897-907.

Thank you!