INTRODUCTION TO THE CONJECTURES OF BIRCH AND SWINNERTON-DYER.

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The aim of these lectures is to introduce the Birch and Swinnerton-Dyer conjectures in its entirety. One part of these conjectures predicts the equality of two different 'ranks' associated to an elliptic curve defined over a number field. These are the so called algebraic and analytic ranks. The other part of the conjectures is an exact formula expressing the leading coefficient of a certain power series associated to the elliptic curve in terms of various important and mysterious arithmetic invariants. The approach we shall take is to define and provide a brief introduction to these arithmetic invariants, thereby providing a compact introduction to this conjecture. We omit the details, referring the interested reader to Silverman's book [Si]. Other excellent references to the theme of this article are [D], [W1]. We stress that this is an expository article and is based on the lectures given at the conferences on the 'Theoretical and Computational Aspects of the BSD Conjectures' held at BICMR in December 2014.

Weierstrass equation of elliptic curves. An elliptic curve over a field K is a projective non-singular curve of genus one defined over K with a specified base point (see [Si, Chapter I, section 2] and [Si, Chapter II, section 5]). Recall that any such curve E has a (Weierstrass) equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \tag{1}$$

in $\mathbb{P}^2$, the projective space of dimension two, with $a_1 \cdots a_6 \in K$ for $1 \leq i \leq 6$. Here, $O = [0, 1, 0]$ is the base point (called "the point at infinity"). By dehomogenising (i.e. taking $x = X/Z$ and $y = Y/Z$) the above equation can be expressed as

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{2}$$

Thus $E \subset \mathbb{P}^2$ consists of the points $P = (x, y)$ satisfying the above Weierstrass equation along with the base point O. If $char(K) \neq 2$, then we can simplify the equation by the change of coordinate

$$y \mapsto 1/2(y - a_1x - a_3)$$

which gives an equation of the form

$$E : y^2 = f(x) \tag{3}$$

where

$$f(x) = x^3 + b_2x^2 + 2b_4x + b_6,$$

and

$$b_2 = a_1^2 + 4a_4,\ b_4 = 2a_4 + a_1a_3,\ b_6 = a_3^2 + 4a_6.$$

Associated to the above equation we define the following quantities

$$\begin{aligned}
b_8 &= a_1^2 a_6 + 4a_2 a_6 - a_2 a_3^2 - a_4^2, \\
c_4 &= b_2^2 - 2b_4, \\
c_6 &= -b_2^3 + 36 b_2 b_4 - 216 b_6, \\
\Delta &= b_2^2 b_8 - 8b_4^3 + 9 b_2 b_4 b_6, \\
j &= c_4^3/\Delta, \\
\omega &= dx/(2y + a_1 x + a_3) = dy/(3x^2 + 2a_2 x + a_4 - a_1 y).
\end{aligned}$$

An easy verification shows that,

$$4b_8 = b_2 b_6 - b_4^2 \text{ and } 1728\Delta = c_4^3 - c_6^2.$$

If $char(K)$ is different from 2 and 3, then using the change of coordinates

$$(x, y) \mapsto (x - 3b_2/36, y/108)$$

equation (2) can further be expressed as

$$y^2 = x^3 - 27c_4 x - 54c_6.$$

Substituting $A = 27c_4$ and $B = -54c_6$ we get that the equation

$$y^2 = x^3 + Ax + B$$

which is usually called the *short Weierstrass form.* Put $\Delta = -16(4A^3 + 27B^2)$. The quantity $\Delta \in K$ is an important invariant associated to the curve E, called the *discriminant* of E over K. Further, the non-singularity of E implies that $\Delta \neq 0$ and the cubic $f(x) = x^3 + Ax + B$ has distinct roots. The quantity j is called the $j-invariant$ of the elliptic curve, and ω is the *invariant differential* associated to the Weierstrass equation.

For a field extension L/K we define the set

$$E(L) := \{(x, y) \in L^2 |\, y^2 + a_1 xy + a_3 y = f(x)\} \cup \{O\}.$$

It is well known that for any field extension L/K there exists an abelian group structure on $E(L)$ such that

(1) O is the identity element with respect to this group structure.

(2) If $L_1 \overset{\phi_{L_1,L_2}}{\longrightarrow} L_2$ is a homomorphism of field extensions of K then there exists a corresponding group homomorphism

$$E(L_1) \overset{\phi^\star_{L_1,L_2}}{\longrightarrow} E(L_2)$$

defined as

$$\phi^\star_{L_1,L_2}(x, y) = (\phi_{L_1,L_2}(x), \phi_{L_1,L_2}(y))$$

satisfying $\phi^\star_{L_1,L_2}\phi^\star_{K,L_2} = \phi^\star_{K,L_2}$.

In particular, if L/K is a Galois extension then $E(L)$ is a $\mathrm{Gal}(L/K)$-module. It is well known that the above group can be explicitly described by chord and tangent method (see [ST, Chapter I]).

Elliptic curves over the complex numbers.

Definition 1. A *lattice* in the field of complex numbers $\mathbb{C}$ is a discrete group of the form $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where ω_1 and ω_2 are linearly independent over the real numbers $\mathbb{R}$. Two lattices Λ and Λ' are said to be *equivalent* if there exists $\lambda \in \mathbb{C} - \{0\}$ with $\lambda\Lambda = \Lambda'$. A complex torus T is a quotient $\mathbb{C}/\Lambda$ of the complex plane $\mathbb{C}$ by a lattice with projection denoted by $p : \mathbb{C} \longrightarrow T = \mathbb{C}/\Lambda$.

Remark 2. If $\lambda \in \mathbb{C} - \{0\}$ such that $\lambda\Lambda \subset \Lambda'$ for lattices Λ and Λ', then it induces a homomorphism $\lambda : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda'$. Such a map is called a *homothety* induced by λ. A homothety $\lambda : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda'$ is an isomorphism if $\lambda\Lambda = \Lambda'$. In fact, it can be shown that every complex analytic isomorphism between two tori is associated to a homothety.

Definition 3. *An elliptic function f with respect to a lattice Λ* is a meromorphic function on $\mathbb{C}$ such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$ and $w \in \Lambda$.

The *Weierstrass $\wp$-function* associated to a lattice Λ is given by the infinite sum

$$\wp(z; \Lambda) = \wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} [\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}]. \tag{4}$$

The *Eisenstein series* of weight $2k$ associated to a lattice Λ is the series

$$G_{2k} = G_{2k}(\Lambda) = \sum_{\omega \in \Lambda - \{0\}} \omega^{-2k}. \tag{5}$$

Theorem 4. *(*[Si, Theorem 3.1, Theorem 3.5, Chapter VI.3]*)*
(a) The Eisenstein series $G_{2k}(\Lambda)$ is absolutely convergent for all $k > 1$.
(b) The series defining the Weierstrass $\wp$-function converges absolutely and uniformly on every compact subset of $\mathbb{C}\backslash\Lambda$. The series defines a meromorphic function on $\mathbb{C}$ having a double pole with residue 0 at each lattice point and no other poles.
(c) The Weierstrass $\wp$-function is an even elliptic function.

The Laurent series for $\wp(z)$ around $z = 0$ is given by

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}.$$

Furthermore, we have

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots \\
\wp'(z) &= \frac{-2}{z^3} + 6G_2 z + 20G_3 z^3 + \cdots \\
\wp'(z)^2 &= \frac{4}{z^4} - \frac{24G_2}{z^2} - 80G_3 + \cdots \\
4\wp(z)^3 &= 4\wp(z)(\frac{1}{z^4} + 6G_2 + 10G_3 z^2 + \cdots) \\
60G_2\wp(z) &= \frac{60G_2}{z^2} + 180G_2^2 z^2 + \cdots .
\end{aligned}$$

Comparing the first few terms of the above expressions, one sees that for all $z \in \mathbb{C} \setminus \Lambda$, the Weierstrass $\wp$-function and its derivative satisfy the relation

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_2\wp(z) - 140G_3$$

Put $g_2 = g_2(\Lambda) = 60G_4(\Lambda)$ and $g_3 = g_3(\Lambda) = 14G_6(\Lambda)$.

Proposition 5. *(*[Si, Proposition 3.6]*) Let $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ be the quantities associated to a lattice $\Lambda \subset \mathbb{C}$ as above.*
(a) The polynomial

$$f(x) = 4x^2 - g_2x - g_3$$

has distinct roots, so its discriminant

$$\Delta(\Lambda) = g_2^3 - 27g_3^3$$

is non-zero.
(b) Let $E/\mathbb{C}$ be the curve $E : y^2 = x^3 - g_2x - g_3$ which from (a) is an elliptic curve. Then the map

$$\phi : \mathbb{C}/\Lambda \longrightarrow E(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$$

$$z \mapsto [\wp(z), \wp'(z), 1],$$

is a complex analytic isomorphism of complex Lie groups, i.e. it is an isomorphism of Riemann surfaces which is a group homomorphism.

For an elliptic curve E over $\mathbb{C}$, let $E[m]$ denote the subgroup of m-torsion points of E defined over $\mathbb{C}$.

Corollary 6. *For every integer $m \geq 1$, there is an isomorphism of abelian groups $E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.*

Corollary 7. *Let E_1 and E_2 be two elliptic curves corresponding to lattices Λ_1 and Λ_2 as in the above proposition. Then E_1 and E_2 are isomorphic over $\mathbb{C}$ if and only if Λ_1 and Λ_2 are homothetic.*

An important theorem in the theory of elliptic curves over $\mathbb{C}$ is the *Unifomization Theorem* which asserts that every elliptic curve E defined over $\mathbb{C}$ corresponds to a lattice Λ_E as in the above proposition, i.e. there exists a lattice Λ_E uniquely determined up to homothety such that

$$\phi : \mathbb{C}/\Lambda \longrightarrow E(\mathbb{C})$$

$$\phi(z) = [\wp(z, \Lambda), \wp'(z, \Lambda), 1]$$

is an isomorphism of complex Lie groups and E has a Weierstrass equation given by

$$y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda).$$

The lattice Λ_E for an elliptic curve E given by a Weierstrass equation as in (2) is, in fact, the set of periods

$$\int_\gamma \omega_E$$

where γ runs over all closed paths in $E(\mathbb{C})$ and ω_E is the associated invariant differential $\frac{dx}{2y+a_1x+a_3}$. Equivalently, Λ_E is the image under the homomorphism

$$H_1(E(\mathbb{C}),\mathbb{Z}) \longrightarrow \mathbb{C}$$

$$\gamma \mapsto \int_\gamma \omega_E$$

where $H_1(E(\mathbb{C}),\mathbb{Z})$ denotes the homology group of $E(\mathbb{C})$ with coefficients in $\mathbb{Z}$. The lattice Λ_E is also called the *period lattice* associated to the curve E. A generating set for Λ_E can be obtained by integrating ω_E over a basis $\{\gamma_1,\gamma_2\}$ of $H_1(E(\mathbb{C}),\mathbb{Z})$ (see [Si, Chapter VI,Proposition 5.2] for more details).

Elliptic curves over local fields. Let K be a perfect local field, complete with respect to a discrete valuation v and R be the ring of integers of K. Let $\mathfrak{m}$ be the maximal ideal of R, π be a uniformizer of K, i.e, a generator of $\mathfrak{m}$ and denote the residue field of R at $\mathfrak{m}$ by k. Further, we normalize the valuation v such that $v(\pi)=1$.

Let E/K be an elliptic curve, and let

$$E : y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6 \tag{6}$$

be a Weierstrass equation for E/K. By substituting $(x,y)\mapsto(\pi^{-2t}x,\pi^{-3t}y)$ for a sufficiently large integer t, we may assume that the $v(a_i)\geq 0$ for the coefficients a_i as in (6). In particular, this implies that the valuation $v(\Delta)$ of the discriminant Δ associated to the above equation is ≥ 0. Further, since v is discrete, we can choose a Weierstrass equation defined over R which minimizes the value $v(\Delta)$. Such a Weierstrass equation of E is called a *minimal* $(Weierstrass)$ *equation* for E. The minimal value of $v(\Delta)$ is called the *minimal discriminant* of E at v.

Proposition 8. *(*[Si, Proposition 1.3, Chapter VII]*) (a) Every elliptic curve E/K has a minimal Weirstrass equation unique up to a change of coordinates*

$$x = u^2x'+r, \quad y = u^3y'+u^2sx'+t$$

with $u\in R^\times$ and $r,s,t\in R$.
(b) The invariant differential,

$$\omega = \frac{dx}{2y+a_1x+a_3}$$

associated to a minimal equation is unique up to multiplication by an element of $R^\times$.

Given a minimal Weierstrass equation of the form (6), we can reduce its coefficients modulo π to obtain a curve over k given by

$$\tilde{E} : y^2+\tilde{a_1}xy+\tilde{a_3}y = x^3+\tilde{a_2}x^2+\tilde{a_4}x+\tilde{a_6} \tag{7}$$

The curve $\tilde{E}$ is called the *reduction of E modulo π*. The restriction of the reduction map from

$$\mathbb{P}^2(K) \longrightarrow \mathbb{P}^2(k)$$

induces a map

$$E(K) \longrightarrow \tilde{E}(k)$$

which is again called the *reduction map*. If the curve $\tilde{E}$ is non singular, then E is said to have *good reduction* over K, otherwise E is said to have *bad reduction* over K. Let $\tilde{E}_{ns}(k)$ denote the set of non singular points of $\tilde{E}(k)$. In particular, if E has good reduction over K then $\tilde{E}(k) = \tilde{E}_{ns}(k)$. If $\tilde{E}$ has bad reduction over K then the following situations occur:
(i) If $\tilde{E}$ is a cuspidal cubic , then $\tilde{E}_{ns} \cong \mathbb{G}_a$, and E is said to have *additive reduction over* K.
(ii) If $\tilde{E}$ is a nodal cubic, then $\tilde{E}_{ns} \cong \mathbb{G}_m$, and E is said to have *multiplicative reduction*. Two further sub-cases occur in this situation which we mention now. If the tangent directions at the node of $\tilde{E}$ are defined over k then E is said to have *split multiplicative reduction over* K, otherwise it has *non-split multiplicative reduction over* K.

Put

$$E_0(K) = \{P \in E(K) | \tilde{P} \in \tilde{E}_{ns}(k)\}$$

$$E_1(K) = \{P \in E(K) | \tilde{P} = O.\}$$

Proposition 9. *The sets $E_0(K)$, $E_1(K)$ and $\tilde{E}_{ns}(k)$ have a group structure such that $E_0(K)$ and $E_1(K)$ are subgroups of $E(K)$. Further, we have the exact sequence*

$$0 \longrightarrow E_1(K) \longrightarrow E_0(K) \longrightarrow \tilde{E}_{ns}(k) \longrightarrow 0.$$

Here, the right hand map is the reduction map modulo π.

Definition 10. The *Tamagawa number* of E over K is defined as the index $c_K(E) := [E(K) : E_0(K)]$.

Theorem 11 (Kodaira, Néron)**.** *Let E/K be an elliptic curve. If E has split multiplicative reduction over K, then $E(K)/E_0(K)$ is a cyclic group of order $v(\Delta) = -v(j)$. In all other cases, the group $E(K)/E_0(K)$ is finite and has order at most 4.*

If E has split multiplicative reduction over K, then by a theorem of Tate

$$E(K) \cong K^{\times}/q_E^{\mathbb{Z}}$$

where q_E is called the *Tate period* of E over K, and is related to the j-invariant $j(E)$ of E via the equation

$$j(E) = j(q_E) = q_E^{-1} + 744 + 196884q_E + 21493760q_E^2 + \cdots .$$

Here, j denotes the modular j-function (see [T] for more details, see also [Si1, Chapter V, Theorem 3.1(b)]). In this case the Tamagawa number $c_K(E) = ord_v(q_E)$ (see [Si, Corollary 15.2.1]). If E has good reduction over K then $c_K(E) = 1$.

Elliptic curves over number fields. If K is number field and E/K is an elliptic curve then we have the following celebrated "Mordell-Weil theorem"(see [Si, Chapter VIII]).

Theorem 12. *If K is a number field and E/K is an elliptic curve defined over K, then $E(K)$ is a finitely generated as an abelian group.*

In particular

$$E(K) \cong \mathbb{Z}^{r_K(E)} \oplus E(K)_{tor}.$$

Here, $r_K(E)$ is called the *rank* of E/K and $E(K)_{tor}$ is the (finite) torsion subgroup of $E(K)$.

For a non-archimedean prime v of K let k_v denote the residue field at v. We say that E has good (resp. bad) reduction at v if E has good (resp. bad) reduction over the completion of K at v. For a non-archimedean prime v, we define an integer

$$a_v(E) := q_v + 1 - \#\tilde{E}(k_v)$$

where q_v is the number of elements in the finite field k_v.

Definition 13. The local L-factor of the Hasse-Weil L-function of E at v is the polynomial defined as

$$L_v(E/K,T) = \begin{cases} 1 - a_v(E)T + q_vT^2 & \text{if } E \text{ has good reduction at } v \\ 1 - T & \text{if } E \text{ has split multiplicative reduction at } v \\ 1 + T & \text{if } E \text{ has non-split multiplicative reduction at } v \\ 1 & \text{if } E \text{ has additive reduction at } v. \end{cases}$$

The Hasse-Weil L-function of E over K has the Euler product expansion

$$L(E/K,s) = \prod_v L_v(E/K, q_v^{-s})^{-1} \quad \text{for } Re(s) >> 0$$

where the product varies over all non-archimedean primes of K.

By a theorem of Hasse, if v is a prime of E/K of good reduction and

$$1 - a_v(E)T + q_vT^2 = (1 - \alpha T)(1 - \beta T)$$

then $|\alpha| = |\beta| = \sqrt{q_v}$, where $|\ |$ denotes the complex norm. Thus, $|a_v(E)| \leq 2\sqrt{q_v}$. This, in particular implies that the the above Euler product converges in the right half plane $Re(s) \geq 3/2$.

Conjecture 1. *For an elliptic curve E over a number field K, the Hasse-Weil L-function of E has an analytic continuation to the entire complex plane $\mathbb{C}$.*

As a consequence of the Modularity theorem proved by Wiles *et.al.* and the theory of base-change for automorphic representations of $GL(2)$ (*cf.* [W],[BCDT],[C2]), we have the following deep

Theorem 14. *Let E be an elliptic curve defined over $\mathbb{Q}$ and K be a solvable Galois extension of $\mathbb{Q}$. Then the Hasse-Weil L-function $L(E/K,s)$ has an analytic continuation to the entire complex plane.*

The order of vanishing of $L(E/K, s)$ at $s = 1$ is called the *analytic rank* of E over K.

Periods of elliptic curves. Now, consider an elliptic curve E defined over a number field K. If the class number of K is one, then it is possible to find a Weierstrass equation which is simultaneously minimal at all non-archimedean primes of K. Such an equation is called a *global Weierstrass minimal equation* of E. Suppose that

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x_2 + a_4x + a_6$$

is a global Weierstrass minimal equation of E over K. Then, the *real period* of E is defined as

$$\Omega_{E/K} := \int_{E(\mathbb{R})} \frac{dx}{2y + a_1x + a_3} \in \mathbb{R} \tag{8}$$

Note that $\Omega_{E/K} \in \Lambda_E$ where Λ_E is the period lattice associated to E. If we assume Conjecture 1, then there is a uniformly convergent power series expansion of $L(E/K, s)$ around the point 1 in $\mathbb{C}$. We shall see later that the real period of E associated to a global Weierstrass minimal equation appears in the formula for the leading term of this power series expansion. If K has positive class number, then the global minimal Weierstrass minimal equation may not exist (see [Si, VIII, Corollary 8.3]). In this case, the real period is defined by integrating a suitably chosen differential on the Néron model of E over K called the Néron differential of E. We will not describe the Néron model of E in this exposition and refer the reader to [BLR].

The study of L-functions in a broader context was undertaken by Deligne[De]. Deligne identified certain special values of L-functions in this general setting, the so-called 'critical' values of L-functions (at certain integers) and conjectured that these values are algebraic multiples of determinants of matrices whose entries are 'periods'. We merely point to the part of Deligne's conjecture which predicts that in the specific case of the L-function of an elliptic curve defined over $\mathbb{Q}$, the critical value at the integer 1, when divided by a suitable period gives a rational number. Put

$$\mathcal{L}_E = \varprojlim_{s \to 1} L(E, s)/(s-1)^{r_E}.$$

The Birch and Swinnerton-Dyer conjecture (see Conjecture 2(c) below) predicts an exact formula for the $\mathcal{L}_E/\Omega_E$.

For a nice exposition of the period lattice in the case of elliptic curves, see [De]. For the exact formula in the BSD conjecture, we stress that the choice of the period is important, and will be the real period as in (8). We remark in passing that the theory of periods is profound in itself, a full exposition of which would require delving into cohomology theories and algebraic geometry. For a more in-depth exposition of periods in general, the interested reader is referred to [KZ].

Height function on Elliptic curves. Fix an algebraic closure $\bar{K}$ of a number field K. For a projective n-space $\mathbb{P}^n$, the absolute *logarithmic height* H_n is the function

$$H_n : \mathbb{P}^n(\bar{K}) \longrightarrow [0, \infty)$$

$$H_n([x_0, \cdots, x_n]) = \sum_{all\ places\ v} (max\{\log |x_0|_v, \cdots, \log |x_n|_v\})$$

where $|-|_v$ is the absolute value at v normalized so that $\prod_v |x|_v = 1$ for all $x \neq 0$ in K. It can be easily checked that H is well defined. For an elliptic curve E defined over K by the Weierstrass equation

$$y^2 = x^3 + ax^2 + bx + c,$$

consider the morphism of projective varieties $f : E(\bar{K}) \longrightarrow \mathbb{P}^1(\bar{K})$ given by $f([x : y : 1]) = [x : 1]$ and $f([0 : 1 : 0]) = [0 : 1]$. The *naïve height* on $E(\bar{K})$ is the function defined as

$$h : E(\bar{K}) \longrightarrow [0, \infty)$$

$$h(P) = H_1(f(P)).$$

Finally, the *canonical height* (also called Néron-Tate height) is the function

$$\hat{h} : E(K) \longrightarrow [0, \infty)$$

$$\hat{h}(P) := \lim_{n \to \infty} 4^{-n} h([2^n]P).$$

Theorem 15. *(Néron, Tate) Let E/K be an elliptic curve and $\hat{h}$ be the canonical height on E.*
(a) $\hat{h}(P) = (1/2)h(P) + O(1)$ for all $P \in E(\bar{K})$.
(b) $\hat{h}(P) = 0$ if and only if P is a torsion points.
(c) The canonical height $\hat{h}$ is a quadratic form on $E(\bar{K})$, i.e. $\hat{h}$ is an even function, and the pairing

$$< \cdot, \cdot >: E(\bar{K}) \times E(\bar{K}) \longrightarrow \mathbb{R}$$

$$< P, Q >= \hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q)$$

is bilinear.
(d) For all $P \in E(\bar{K})$ and all $m \in \mathbb{Z}$, we have

$$\hat{h}([m]P) = m^2 \hat{h}(P).$$

(e) For all $P, Q \in E(\bar{K})$, $\hat{h}(P + Q) + \hat{h}(P - Q) = 2\hat{h}(P) + 2\hat{h}(Q)$
(f) The canonical height $\hat{h}$ extends to a positive definite quadratic form on the vector space $E(K) \otimes \mathbb{R}$.
(d) Any function $E(K) \longrightarrow \mathbb{R}$ which satisfy (a) and (d) is equal to the canonical height function $\hat{h}$.

We remark that the above properties of the canonical height function are crucially used in the proof of the Mordell-Weil Theorem (see [Si, Chapter VIII]).

Definition 16. The Néron-Tate height pairing on $E(K)$ is the bilinear form

$$< P, Q >_{NT} = (\hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)) \text{ for all } P, Q \in E(K).$$

The *elliptic regulator* of E over K is defined by

$$Reg_K(E) := det(< P_i, P_j >_{NT})_{1 \leq i,j \leq r_K(E)}$$

where $P_i, \cdots, P_{r_K(E)}$ is a basis for $E(K)/E(K)_{tor}$.

As a consequence of Theorem 15(f) we have the following

Corollary 17. *The elliptic regulator $Reg_K(E) > 0$.*

Selmer group and Tate-Shafarevich group of elliptic curves. In this section, we discuss the Galois cohomology of elliptic curves and define the Tate-Shafarevich group which is an important invariant associated to an elliptic curve defined over a number field. It is conjectured to be finite and its size appears in the exact formula for the leading term of the power series expansion of the associated Hasse-Weil L-function as predicted by BSD conjectures which is discussed below.

For a field K and a discrete module A over the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$ of K, let $H^i(K, A)$ denote the i-th Galois cohomology group of A. Let E be an elliptic curve defined over K. For an abelian group A let A_{tor} denote the torsion subgroup of A. The absolute Galois group of K acts continuously on the discrete group $E(\bar{K})$ (resp. $E(\bar{K})_{tor}$). Now, suppose that K is a number field and let E be an elliptic curve defined over K. We denote by E_{tor} the Galois module $E(\bar{K})_{tor}$. For every prime v of K, we have a natural restriction map from $H^1(K, E_{tor}) \longrightarrow H^1(K_v, E(\bar{K}_v))$ induced by the inclusion $E_{tor} \hookrightarrow E(\bar{K}_v)$. The Selmer group $\mathrm{Sel}(E/K)$ of E over K is defined as

$$\mathrm{Sel}(E/K) := Ker(H^1(K, E_{tor}) \longrightarrow \prod_v H^1(K_v, E(\bar{K}_v))$$

and the Tate-Shafarevich group denoted by $Ш(E/K)$, is defined as

$$Ш(E/K) := Ker(H^1(K, E(\bar{K})) \longrightarrow \prod_v H^1(K_v, E(\bar{K}_v))$$

where v varies over the set of primes of K. Using the fact that $E(\bar{K})$ is divisible, for every positive integer m, we get the exact sequence

$$0 \longrightarrow E(\bar{K})[m] \longrightarrow E(\bar{K}) \xrightarrow{\times m} E(\bar{K}) \longrightarrow 0.$$

From the associated long exact sequence of Galois cohomology for every positive integer m, we have the exact sequence

$$0 \longrightarrow E(K)/mE(K) \longrightarrow H^1(K, E(\bar{K})[m]) \longrightarrow H^1(K, E(\bar{K}).$$

Since

$$\varinjlim_m H^1(K, E(\bar{K})[m]) = H^1(K, E(\bar{K})_{tor}),$$

taking the direct limit over integers m, we obtain the exact sequence,

$$0 \longrightarrow E(K) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H^1(K, E(\bar{K})_{tor}) \longrightarrow H^1(K, E(\bar{K}).$$

Using this exact sequence along with the snake lemma we get that

$$0 \longrightarrow E(K) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Sel}(E/K) \longrightarrow Ш(E/K) \longrightarrow 0.$$

is exact. For a prime p let $\mathrm{Sel}_p(E/K)$ denote the p-primary subgroup of $\mathrm{Sel}(E/K)$.

Proposition 18. *For any number field K, an elliptic curve E defined over K and a prime p, we have an isomorphism*

$$\mathrm{Sel}_p(E/K) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{s_p(E/K)} \oplus \ Fin_p,$$

where Fin_p is a finite group.

The number $s_p(E/K)$ is called the *p-Selmer rank of E over K.*

The conjectures of Birch and Swinnerton-Dyer (BSD). We are now ready to state the Birch and Swinnerton-Dyer conjectures as formulated by Birch and Swinnerton-Dyer following extensive numerical calculations that they made in the 1960's (see [BS], [B]).

Conjecture 2. (Birch, Swinnerton-Dyer) For a number field K and an elliptic curve defined over K,
(a) $order_{s=1}L(E/K,s) = r_K(E)$.
(b) The Tate-Shararevich group $Ш(E/K)$ is finite. In particular, $s_p(E/K) = r_K(E)$ for every prime p.
(c)

$$\lim_{s \longrightarrow 1} \frac{L(E/K,s)}{(s-1)^{r_K(E)}} = \frac{\Omega_{E/K} \times Reg_K(E) \times \#Ш(E/K) \prod_{v<\infty} C_{K_v}(E)}{\sqrt{disc_K} \times (\#E(K)_{tor})^2}$$

where $disc_K$ denotes the discriminant of the field K over $\mathbb{Q}$.

The order of the Hasse-Weil L-series at $s = 1$ is called the *analytic rank* of E. Conjecture 2(a) is usually referred to as the weak BSD conjecture and all assertions together as the strong form of *BSD conjecture*. Conjecture 2(c) is also known as the *BSD exact formula*.

The following quote of Tate neatly summarizes the mystery of the BSD conjecture: *"This remarkable conjecture relates the behavior of a function L where it is not known, to be defined to the order of a group* Ш *not known to be finite".*

The BSD conjectures have been proved in some special cases due to work of Coates-Wiles, Gross-Zagier, Kolyvagin, Rubin and others. We mention that Iwasawa theory and the theory of Euler systems have proved to be effective tools in attacking the BSD conjectures. In 1977, Coates and Wiles proved the following

Theorem 19. *If E is an elliptic curve defined over $\mathbb{Q}$ or a quadratic imaginary extension K of $\mathbb{Q}$, E has complex multiplication by K and $L(E/K,s)$ is non-zero at 1, then $E(K)$ is finite.*

The best result known to date about the weak BSD conjecture is the following.

Theorem 20. *If $order_{s=1}L(E/\mathbb{Q},s) \leq 1$, then $order_{s=1}L(E/\mathbb{Q},s) = r_E$ and $Ш(E/\mathbb{Q})$ is finite.*

There are two key ingredients in the proof :
(i) The so-called 'Heegner points' and the deep results of Gross-Zagier ([GZ]) relating the canonical height of Heegner points to derivatives of the L-functions. The Gross-Zagier formula implies that if an elliptic curve E over $\mathbb{Q}$ has analytic rank of E equal to one, then $r_E \geq 1$.
(ii) The notion of Euler systems as developed by Kolyvagin ([K]). Kolyvagin developed the device of Euler systems, which is a tool that connects the analytic and algebraic sides of the BSD conjecture, and can be used to bound the size of Selmer group. Kolyvagin used it along with Gross-Zagier's formula to prove the finiteness of $Ш(E/\mathbb{Q})$ and the weak BSD conjecture for elliptic curves over $\mathbb{Q}$ with analytic rank at most one (see [Z1] for more detail).

For the theory of Euler systems, the interested reader is referred to W. McCallum ([Mc]) and to the excellent book by Rubin ([R3]). The exact formula of the BSD conjecture is known in special cases and uses very different methods, mainly from Iwasawa theory (see [CW], [R],[R1],[R2]). For more results on the full BSD conjecture using Iwasawa theory, the reader is referred to the work of Kato [Ka], Skinner-Urban [SU]. Even an outline of these results and the methods therein are beyond the scope of this article.

All the above results assumed the truth of Conjecture 1, which itself was proved by Wiles *et.al.* in the late 1990's (see [W], [BCDT]). There is also a related conjecture known as the *parity conjecture*.

Conjecture 3. $order_{s=1}L(E/K,s) = r_K(E) \bmod 2$.

Note that the parity conjecture is a consequence of BSD conjecture. It is still open in general. In recent years, there has been some significant progress towards the proof another related conjecture, namely the it p-parity conjecture due to Greenberg, Nekovar, Dokchitser-Dokschitser, Wei Zhang and others (see [Gr], [N], [DD], [Z]).

Conjecture 4 (p-parity conjecture)**.** $order_{s=1}L(E/K,s) = s_p(E/K) \bmod 2$ for all primes p.

The p-parity conjecture together with finiteness of the p-torsion subgroup of Shafarevich-Tate group implies the parity conjecture. Monsky proved the following

Theorem 21. *(*[Mo]*) (a) The 2-parity conjecture is true for an elliptic curve E defined over $\mathbb{Q}$.*
(b) The p-parity conjecture is true for an elliptic curve E defined over $\mathbb{Q}$ if it has a p-isogeny for a prime p.

Due to work of Nekovar and Dokchitser-Dokschiste, the p-parity conjecture is known in general over $\mathbb{Q}$. Over a more general totally real number field, we have the following theorem by Nekovar.

Theorem 22 ([N])**.** *Let E be an elliptic curve defined over a totally real field F. Then the p-parity conjecture holds for E over F in each of the following situations*
(i) E does not have complex multiplication

(ii) E has complex multiplication and $2 \nmid [F : \mathbb{Q}]$
(iii) E has complex multiplication by a immaginary quadratic field K' and p splits completely in $K'/\mathbb{Q}$.

For more precise technical results on the p-parity conjecture over some other number fields by Dokchitser-Dokchitser see [Do].

We close with a brief mention of what is known about the BSD conjecture when one considers all elliptic curves over number fields. Even though general results on BSD are not known for elliptic curves of algebraic rank > 1, the conjecture is known to hold for a large class of curves. Manjul Bhargava and Arul Shankar introduce and study the 'it average rank' of elliptic curves and show that the average rank is less than one (see [MS], [MS1]). Recent work of Bhargava, Skinner and Zhang implies that 66% of the class of all elliptic curves satisfy the BSD conjecture (see [BSZ]) . In fact, Bhargava and Shankar show that in a statistical sense, a sizeable proportion of elliptic curves defined over $\mathbb{Q}$ has rank zero and another sizeable proportion has rank one (see [BSZ, Theorem 3] for a precise statement).

So far, there is no general algorithm known which can compute the rank of a given elliptic curve defined over a number field. One can compute the rank of an elliptic curve by computing the derivative of the associated Hasse-Weil L-function only if BSD is known. But at present, we do not know a single example of an elliptic curve over $\mathbb{Q}$ of rank ≥ 2 for which $Ш(E/\mathbb{Q})$ is finite.

Numerical examples. The BSD conjectures have been verified extensively for numerous concrete examples. As remarked earlier, the conjecture itself was formulated based on the data obtained by explicit computations. In the last half a century, remarkable progress on the computational side has been made possible, thanks to advances in computing and theoretical knowledge. We refer the reader to the excellent data base compiled by Cremona, Stein, Watkins and others ([C],[WS]). Thus the beauty of BSD conjectures lies in its intricacy combined with the fact that most of the invariants in the exact formula can be explicitly computed.

In this final section, we provide a few illustrative numerical examples using the mathematical software Sage. We shall provide three examples of elliptic curves with analytic rank zero, one and two respectively. Consider the ellipic curve E given by the Weierstrass equation

$$E : y^2 + y = x^3 - x^2 - 7820x - 263580$$

over $\mathbb{Q}$. The discriminant Δ of E over $\mathbb{Q}$ is 11 and the j invariant of E is equal to $-1 \times 2^{12} \times 11^{-1} \times 29^3 \times 809^3$. The curve E has split multiplicative reduction at 11. The BSD invariants of E are as follows:

- analytic rank, $r = 0$,
- regulator, $Reg_{\mathbb{Q}}(E) = 1$,
- real period $\Omega = 0.253841860856\cdots$
- torsion order, $\#E(\mathbb{Q})_{tor} = 1$
- $L(E/\mathbb{Q}, 1) = 0.253841860856\cdots$

- Tamagawa Number at 11, $c_{11}(E) = 1$.

Since $L(E/\mathbb{Q}, 1) \neq 0$, BSD conjectures are true for E over $\mathbb{Q}$. In particular, $r_{\mathbb{Q}}(E) = 0$ and from the above data we get that $\#Ш(E/\mathbb{Q}) = 1$. Next, we consider the following elliptic curve E of positive rank :

$$E : y^2 + y = x^3 + x^2.$$

The curve E has non-split multiplicative reduction at 43. The discriminant of E is -43 and the j invariant is $-1 \times 3^{12} \times 43^{-1}$. The BSD invariants of E are given by

- analytic rank, $r = 1$,
- regulator, $Reg_{\mathbb{Q}}(E) = 0.0628165070875\cdots$,
- real period $\Omega = 5.46868952997\cdots$
- torsion order, $\#E(\mathbb{Q})_{tor} = 1$
- $L(E/\mathbb{Q}, 1) = 0$ and $L'(E/\mathbb{Q}, 1) = 0.343523974618\cdots$
- Tamagawa Number at 43, $c_{43}(E) = 1$.

Since E has analytic rank 1 over $\mathbb{Q}$, BSD conjectures hold and we get that $r_{\mathbb{Q}}(E) = 1$ and $\#Ш(E/\mathbb{Q}) = 1$. We shall end by providing an example of elliptic curve for which we still do not know if the BSD conjectures are true. Consider the elliptic curve

$$E : y^2 + y = x^3 + x^2 - 2x.$$

The curve E has split multiplicative reduction at 389. The discriminant of E is 389 and the j invariant is $2^{12} \times 7^3 \times 389^{-1}$. The BSD invariant of E are given by

- analytic rank, $r = 2$,
- regulator, $Reg_{\mathbb{Q}}(E) = 0.15246017794\cdots$,
- real period $\Omega = 4.98042512171\cdots$
- torsion order, $\#E(\mathbb{Q})_{tor} = 1$
- $L(E/\mathbb{Q}, 1) = 0$ and $L'(E/\mathbb{Q}, 1) = 0$ and $L''(E/\mathbb{Q}, 1) = 0.759316500288\cdots$
- Tamagawa Number at 389, $c_{389}(E) = 1$.

The BSD conjecture for E over $\mathbb{Q}$ predicts that the $r_{\mathbb{Q}}(E) = 2$ and $\#Ш(E/\mathbb{Q}) = 1$. In this case it can be shown (using the method of "2-descent") that $r_{\mathbb{Q}}(E) = 2$ and $\#Ш(E/\mathbb{Q})[2] = 1$ (see [C1]). A set of generators of the Mordell-Weil group of E over $\mathbb{Q}$ is given by $\{(-1, 1), (0, 0)\}$. At present we do not know if the predictions on the size of $\#Ш(E/\mathbb{Q})$ is true.

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