Lectures 2,3: Classical Generation of Gravitational Waves

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1 Linearized GR with source $T_{\mu\nu}$

In Lecture 1, we saw that, in Lorentz gauge, the linearized Einstein equation is

$$\Box \overline{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}. \tag{1}$$

To solve this equation, first find the Green's function $G(x, x')$ which represents the response to a delta-function source at spacetime point x':

$$\Box G(x, x') = \delta^4(x - x'). \tag{2}$$

We can solve this in Fourier space:

$$G(x, x') = \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{e}^{ik_\mu(x^\mu - x'^\mu)}}{-k^2}. \tag{3}$$

To perform the d^4k integral, start with the dk^0 integral. Since $-k^2 = (k^0)^2 - \vec{k}^2$ appears in the denominator, the integrand has two poles at $k^0 = \pm|\vec{k}|$. We have to decide how the k^0 integration contour goes around the poles in the complex k^0 plane: different choices correspond to different Green's functions (*e.g.* the retarded, advanced, or Feynman Green's functions). To obtain the retarded Green's function, proportional to Heaviside function $\theta(t - t')$, we should choose the contour that passes *above* both poles in the complex k^0 plane; then we can evaluate the k^0 integral via the residue theorem to obtain

$$\begin{aligned}
G(x, x') &= i\theta(t - t')\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\mathrm{e}^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{2\omega}[\mathrm{e}^{i\omega(t-t')} - \mathrm{e}^{-i\omega(t-t')}] && \text{(4a)}\\
&= \frac{i\theta(t - t')}{(2\pi)^3}\int_0^\infty d\omega\,\omega^2\int_{-1}^{1} d\mu\int_0^{2\pi} d\varphi \frac{\mathrm{e}^{i\omega|\vec{x}-\vec{x}'|\mu}}{2\omega}[\mathrm{e}^{i\omega(t-t')} - \mathrm{e}^{-i\omega(t-t')}] && \text{(4b)}\\
&= \frac{\theta(t - t')}{2|\vec{x}-\vec{x}'|(2\pi)^2}\int_0^\infty d\omega[\mathrm{e}^{i\omega|\vec{x}-\vec{x}'|} - \mathrm{e}^{-i\omega|\vec{x}-\vec{x}'|}][\mathrm{e}^{i\omega(t-t')} - \mathrm{e}^{-i\omega(t-t')}] && \text{(4c)}\\
&= \frac{-1}{4\pi|\vec{x} - \vec{x}'|}\theta(t - t')\delta[(t - t') - |\vec{x} - \vec{x}'|] && \text{(4d)}
\end{aligned}$$

where from the 1st to 2nd line we have written the $d^3\vec{k}$ integral in spherical coordinates ($\mu = \cos\theta$, where θ is the angle between $\vec{k}$ and $\vec{x}-\vec{x}'$); from the 2nd to 3rd line we have evaluated the $d\varphi$ and $d\mu$ integrals; and from the 3rd to 4th line we have evaluated the $d\omega$ integral.

The retarded Green's function $G(x, x')$ represents the response to a delta function source at spacetime point x'; and the general solution to the wave equation (1) is obtained by convolving the Green's function with the source:

$$\overline{h}_{\mu\nu}(x) = \int d^4x' G(x,x')[-16\pi T_{\mu\nu}(x')] \tag{5a}$$

$$= \int d^3\vec{x}' \frac{4}{|\vec{x}-\vec{x}'|} T_{\mu\nu}(\vec{x}', t - |\vec{x}-\vec{x}'|) \tag{5b}$$

Now, if $T_{\mu\nu}$ is a distant, localized, Newtonian source at distance $|\vec{x}-\vec{x}'| \approx r$ this becomes

$$\overline{h}_{ij}(x) \approx \frac{4}{r}\int d^3\vec{x}' T^{ij}(\vec{x}', t-r) \tag{6a}$$

$$= \frac{2}{r}\int d^3\vec{x}' T^{kl}{}_{,kl}(\vec{x}', t-r)\vec{x}'^i\vec{x}'^j \tag{6b}$$

$$= \frac{2}{r}\int d^3\vec{x}' T^{00}{}_{,00}(\vec{x}', t-r)\vec{x}'^i\vec{x}'^j \tag{6c}$$

$$= \frac{2}{r}\ddot{Q}_{ij}(t-r) \tag{6d}$$

where, from the 1st to 2nd line we have used the identity

$$T^{ij} = (T^{ik}x^j + T^{jk}x^i)_{,k} - \frac{1}{2}(T^{kl}x^ix^j)_{,kl} + \frac{1}{2}T^{kl}{}_{,kl}x^ix^j \tag{7}$$

and thrown away the total divergence terms (since, by Stokes' theorem, they are converted to surface integrals that vanish when the surface is taken outside the source); from the 2nd to 3rd line we have used the identity $T^{kl}{}_{,kl} = T^{00}{}_{,00}$, which follows from conservation of stress energy $T^{\mu\nu}{}_{,\nu} = 0$; and from the 3rd to 4th line we have introduced the quadrupole tensor

$$Q_{ij}(t) = \int d^3\vec{x}\rho(\vec{x})\vec{x}^i\vec{x}^j. \tag{8}$$

and used the notation $\ddot{Q} = d^2Q/dt^2$. Now, taking the TT part of both sides, we finally arrive at the classic formula for the gravitational waveform produced by a localized Newtonian source:

$$h_{ij}^{TT}(t) = \frac{2}{r}\ddot{Q}_{ij}^{TT}(t-r). \tag{9}$$

where the TT part of Q is obtained as

$$Q_{ij}^{TT} = P_{ik}(\hat{n})P_{jl}(\hat{n})[Q_{kl} - \frac{1}{3}\delta_{kl}\mathrm{Tr}\,Q]. \tag{10}$$

Here $\hat{n}$ is the direction to (or from) the source, and $P_{ij}(\hat{n}) = \delta_{ij} - \hat{n}_i\hat{n}_j$ is the transverse projection operator.

Next let us compute the luminosity of the system. First note that gravitational waves have an effective stress-energy tensor

$$T^{gw}_{\mu\nu} = \frac{1}{32\pi}\langle h^{TT}_{ij,\mu} h^{TT}_{ij,\nu}\rangle. \tag{11}$$

So if we integrate $T^{gw}_{00} = (1/8\pi r^2)\langle \dddot{Q}^{TT}_{ij} \dddot{Q}^{TT}_{ij}\rangle$ (the gravitational wave energy density) over a sphere far from the source (where the gravitational waves are propagating radially outward at the speed of light) we find that the total luminosity is

$$L = \frac{1}{5}\langle \dddot{Q}^{T}_{ij} \dddot{Q}^{T}_{ij}\rangle \tag{12}$$

where $Q^T_{ij} = Q_{ij} - \frac{1}{3}\delta_{ij}Q$ is the traceless (but not transverse) part of Q_{ij}.

2 Binary inspiral

Consider a binary with mass M_1 at position $\vec{r}_1$ and mass M_2 at position $\vec{r}_2$. This system has quadrupole moment

$$Q_{ij} = \mu \vec{r}^{\,i}_{21} \vec{r}^{\,j}_{21} \tag{13}$$

where $\vec{r}_{21} \equiv \vec{r}_2 - \vec{r}_1$ is the displacement vector, and $\mu \equiv M_1M_2/(M_1+M_2)$ is the reduced mass. The equation of motion is

$$\frac{d^2\vec{r}_{21}}{dt^2} = -\frac{M}{r^3_{21}}\vec{r}_{21} \tag{14}$$

where $M \equiv M_1 + M_2$ is the total mass. Without loss of generality, let us take the binary to orbit in the xy-plane, with separation a, so

$$\vec{r}_{21} = a(\hat{x}\cos\omega t + \hat{y}\sin\omega t), \qquad \omega = \sqrt{(M_1+M_2)/a^3} \tag{15}$$

and the traceless part of the quadrupole tensor becomes

$$Q^T_{ij} = \mu a^2 \begin{pmatrix} \cos^2\omega t - \frac{1}{3} & \cos\omega t \sin\omega t & 0 \\ \cos\omega t\sin\omega t & \sin^2\omega t - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{16}$$

Now consider a point in the direction $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, at a distance r from the source; as an exercise, you can construct the 3×3 matrix corresponding to the projection operator $P_{ij}(\hat{n}) = \delta_{ij} - \hat{n}_i\hat{n}_j$; then perform the matrix multiplication $Q^{TT} = P(\hat{n})Q^TP(\hat{n})$ to construct the transverse traceless part of Q, and finally take 2 time derivatives to explicitly obtain the $\ddot{Q}^{TT}_{ij}$ and hence the gravitational waveform at an arbitrary distance and direction from the source via Eq. (9). We can also plug Q^T_{ij} into equation (12) to find

$$L_{gw} = \frac{dE_{\rm gw}}{dt} = \frac{32}{5}\frac{\mu^2M^3}{a^5}. \tag{17}$$

The gravitational waves carry away energy from the orbit: $dE_{\rm orbit}/dt = -dE_{\rm gw}/dt$, where $E_{\rm orbit} = -M\mu/2a$. As a result, the orbital separation gradually decreases:

$$\frac{da}{dt} = \frac{dE_{\rm orbit}/dt}{dE_{\rm orbit}/da} = -\frac{64}{5}\frac{\mu M^2}{a^3}. \tag{18}$$

If this system starts at a finite separation a_0, we can integrate this equation to discover that the system inspirals and merges in a finite time:

$$t_{\rm merger} = \frac{5}{256}\frac{a_0^4}{M^2\mu} = \frac{5}{256}\frac{1}{\omega^{8/3}M^{2/3}\mu}. \tag{19}$$

As the system inspirals, the frequency of the orbit (and hence the frequency of the gravitational wave, which is twice the frequency of the orbit) increases by virtue of Kepler's law: $\omega^2 = M/a^3$; and, at the same time, the amplitude of the gravitational wave also increases, since $\ddot{Q} \sim \mu a^2\omega^2 = \mu M/a$. So the combined effect produces a characteristic "chirping" gravitational waveform, which I will draw on the blackboard.