Lecture 1: Gravitational Waves in vacuum

1 Introduction

I am told that most of the students at this school have seen some GR, but that there is a lot of variation in background. For those of you who need more background, I have prepared a set of lecture notes ("Lecture 0") that gives a concise 5 page introduction to Differential Geometry, General Relativity (and, for fun, Classical Yang-Mills theory, too).

In brief, the plan for these lectures is the following:

- Lecture 1: Gravitational waves in vacuum: basic equations, TT gauge, polarizations, physical interpretation.
- Lectures 2,3: Classical generation of gravitational waves (by a classical source of stress energy, $T_{\mu\nu}$). Binary inspiral as the key astrophysical source.
- Lectures 4,5: Quantum generation of gravitational waves (*e.g.* in inflation), and their cosmological evolution to the present. Relationship between primordial gravitational waves, astrophysical gravitational waves, and present-day constraints.

2 Linearized GR

Consider flat spacetime plus small perturbations:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \Rightarrow \quad g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \tag{1}$$

where $g_{\mu\nu}$ is the full spacetime metric, $g^{\mu\nu}$ is its inverse ($g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho}$), $\eta_{\mu\nu} = \text{diag}\{-1, 1, 1, 1\}$ is the unperturbated Minkowski metric, $\eta^{\mu\nu}$ is its inverse, $h_{\mu\nu}$ is the metric perturbation ($|h_{\mu\nu}| \ll 1$), and here and in what follows, we will raise and lower indices on h using the unperturbed metric $\eta_{\mu\nu}$ (*e.g.* $h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$). Under an infinitessimal coordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$, the metric transforms as $g_{\mu\nu} = (\partial x'^{\alpha}/\partial x^{\mu})(\partial x'^{\beta}/\partial x^{\nu})g'_{\alpha\beta}$, from which we find transformation law for $h_{\mu\nu}$:

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}). \tag{2}$$

Using $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta})$, we obtain the first order expression for the Christoffel symbols:

$$\Gamma^\alpha_{\beta\gamma} \approx \frac{1}{2}(h^\alpha{}_{\beta,\gamma} + h^\alpha{}_{\gamma,\beta} - h_{\beta\gamma}{}^{,\alpha}). \tag{3}$$

Then, using $R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\gamma\sigma}\Gamma^\sigma_{\beta\delta} - \Gamma^\alpha_{\delta\sigma}\Gamma^\sigma_{\beta\gamma}$ we find the first order expression for the Riemann tensor

$$R_{\alpha\beta\gamma\delta} \approx \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}). \tag{4}$$

Note that, to first order in h, the expression for $R_{\alpha\beta\gamma\delta}$ is invariant under the gauge transformation (2): *i.e.* the values of the components of $R_{\alpha\beta\gamma\delta}$ are gauge invariant. From here we can calculate the Einstein tensor $G_{\beta\delta} = R_{\beta\delta} - \frac{1}{2} g_{\beta\delta} R$ where $R_{\beta\delta} = g^{\alpha\gamma} R_{\alpha\beta\gamma\delta}$ is the Ricci tensor, and $R = g^{\beta\delta} R_{\beta\delta}$ is the Ricci scalar. We find the first order expression for the Einstein tensor

$$G_{\beta\delta} \approx \frac{1}{2}\left(\overline{h}_{\beta\mu,\delta}{}^{,\mu} + \overline{h}_{\delta\mu,\beta}{}^{,\mu} - \eta_{\beta\delta}\overline{h}_{\mu\nu}{}^{,\mu\nu} - \Box\overline{h}_{\beta\delta} \right) \tag{5}$$

where we have introduced the trace-reversed metric perturbation

$$\overline{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h \qquad (h \equiv h^\alpha_\alpha). \tag{6}$$

Again, to first order in h, the components of $G_{\beta\delta}$ are gauge invariant, but it is convenient to choose "Lorentz gauge"

$$\overline{h}_{\alpha\beta}{}^{,\beta} = 0. \tag{7}$$

Since $\bar{h}_{\mu\nu}{}^{,\nu}$ transforms as

$$\bar{h}'_{\mu\nu}{}^{,\nu} = h_{\mu\nu}{}^{,\nu} - \Box\xi_\mu \qquad (\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu), \tag{8}$$

the coordinate transformation $\xi^\mu(x)$ needed to reach Lorentz gauge is obtained by solving the differential equation

$$\Box\xi_\mu = \bar{h}_{\mu\nu}{}^{,\nu}. \tag{9}$$

The advantage of Lorentz gauge is that the expression for the Einstein tensor simplifies to the form $G_{\beta\delta} = -(1/2)\Box h_{\beta\delta}$, and the Einstein equation $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$ becomes the familiar wave equation

$$\Box\overline{h}_{\beta\delta} = -16\pi G_N T_{\beta\delta}. \tag{10}$$

3 Propagation of gravitational waves in vacuum

In vacuum, the wave equation becomes $\Box\overline{h}_{\alpha\beta} = 0$; the solution is a superposition of plane waves traveling at the speed of light:

$$\overline{h}_{\alpha\beta}(x) = \mathrm{Re} \int \frac{d^3\vec{k}}{(2\pi)^3} A_{\alpha\beta}(\vec{k}) \mathrm{e}^{ik_\mu x^\mu} \tag{11}$$

where $k^\mu = (\omega, \vec{k})$ and $\omega = \sqrt{\delta_{ij}\vec{k}^i\vec{k}^j}$. The Lorentz gauge condition (7) implies that each 4×4 symmetric matrix $A_{\mu\nu}(\vec{k})$ satisfies 4 constraints:

$$k^\alpha A_{\alpha\beta}(\vec{k}) = 0. \tag{12}$$

But the Lorentz gauge condition (7) does not fix the gauge completely; once we are in Lorentz gauge, we can still make a gauge transformation of the form

$$\xi^\mu(x) = \text{Re}\int \frac{d^3\vec{k}}{(2\pi)^3} B^\mu(\vec{k}) \text{e}^{ik_\mu \cdot x^\mu} \tag{13}$$

and, since $\Box\xi^\mu = 0$ for such a transformation, (8) implies Lorentz gauge (7) will be preserved. To fix the 4 $B^\mu(\vec{k})$, we impose 4 more gauge conditions:

$$u^\alpha \overline{h}_{\alpha\beta} = 0 \qquad \text{and} \qquad \overline{h} = 0, \tag{14}$$

where u^α is a constant time-like vector which we can choose to be $(1, 0, 0, 0)$. These conditions imply that each $A_{\mu\nu}(\vec{k})$ satisfies the following constraints:

$$u^\mu A_{\mu\nu}(\vec{k}) \qquad \text{and} \qquad \text{Tr}[A(\vec{k})] = 0. \tag{15}$$

This look like 5 additional gauge conditions, but it is only 4, since one of the conditions is redundant with the Lorentz gauge conditions: the quantity $u^\mu k^\nu A_{\mu\nu}(\vec{k})$ vanishes for two different reasons. We have now completely fixed the gauge: this called "transverse-traceless" or "TT" gauge.

This leaves $A_{\mu\nu}(\vec{k})$ with 10-4-4=2 unconstrained components: these are the 2 polarizations of a gravitational wave. Let us define the unit 3-vector $\hat{n} = \vec{k}/\omega$ and write $k^\mu = \omega(1, \hat{n})$. Now, associated with each $\hat{n}$, pick two other vector $\hat{p}(\hat{n})$ and $\hat{q}(\hat{n})$ to complete an orthonormal triad, and write $p_\mu = \{0, \hat{p}\}$ and $q_\mu = \{0, \hat{q}\}$. We can now form the "plus" and "cross" polarization tensors:

$$P^+_{\mu\nu} = p_\mu p_\nu - q_\mu q_\nu \qquad P^\times_{\mu\nu} = p_\mu q_\nu + q_\mu p_\nu, \tag{16}$$

or the "right" and "left" circularly polarized polarization tensors

$$P^R_{\mu\nu} = \frac{1}{\sqrt{2}}(P^+_{\mu\nu} + iP^\times_{\mu\nu}) \qquad P^L_{\mu\nu} = \frac{1}{\sqrt{2}}(P^+_{\mu\nu} - iP^\times_{\mu\nu}). \tag{17}$$

and then expand $A_{\alpha\beta}(\vec{k})$ in Eq. (11) in the form

$$A_{\alpha\beta}(\vec{k}) = A_+(\vec{k})P^+_{\alpha\beta}(\hat{n}) + A_\times(\vec{k})P^\times_{\alpha\beta}(\hat{n}) \tag{18a}$$

$$= A_R(\vec{k})P^R_{\alpha\beta}(\hat{n}) + A_L(\vec{k})P^L_{\alpha\beta}(\hat{n}) \tag{18b}$$

To understand the physical meaning, consider the geodesic equation for a massive particle bobbing in the gravitational wave metric:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{19}$$

If the particle is initially at rest in TT gauge ($dx^\alpha/d\tau = 0$), then the geodesic equation says it will remain at rest (because Γ^μ_{00} vanishes in this gauge). So if we consider two nearby particles, A and B, at rest at coordinate $\vec{x}_A$ and $\vec{x}_B$, the *coordinate* displacement between them is a constant, but the proper distance between them

$$d_{AB} = \sqrt{(\delta_{ij} + h_{ij})(\vec{x}_B - \vec{x}_A)^i(\vec{x}_B - \vec{x}_A)^j} \tag{20}$$

oscillates with the patterns that I will draw on the board. This oscillation is observable, and various different gravitational wave detection schemes are currently trying to detect it.